Stable Refinable Generators of Shift Invariant Spaces with Certain Regularities and Vanishing Moments

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Abstract. In this paper, we discuss the stable refinable functions that generate shift invariant (SI) spaces and possess the largest possible regularities and required vanishing moments. The stability of the corresponding complementary spaces is also discussed.

§1. Introduction

We start by setting some notation. We define a low-pass filter as

$$m_0(\xi) = 2^{-1} \sum_n h_n e^{-in\xi}. \tag{1}$$

Here, we assume that only finitely many h_n are nonzero. However, some of our results can be extended to infinite sequences that have sufficient decay for $|n| \to \infty$. Next, we define ϕ by

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi). \tag{2}$$

This infinite product converges only if $m_0(0) = 1$, i.e., if $\sum_n h_n = 2$. In this case, the infinite products in (2) converge uniformly and absolutely on compact sets, so that $\hat{\phi}$ is a well-defined C^∞ function. Obviously, $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$, or, equivalently, $\phi(t) = \sum_n h_n \phi(2t-n)$ at least in the sense of distributions. From Lemma 3.1 in [9], ϕ has compact support.

We now consider the simplest possible masks $m_0(\xi)$ with the following form.

Approximation Theory X: Wavelets, Splines, and Applications

Charles K. Chui, Larry L. Schumaker, and Joachim Stöckler (eds.), pp. 305–322.

ISBN 0-8265-1416-2.

Definition 1. *Denote by Φ the set of all functions $\phi(t)$ that have Fourier transform $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$. Here the filter $m_0(\xi) = 2^{-1}\sum_n h_n e^{-in\xi}$ is in the set M that contains all filters of the form*

$$m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi), \tag{3}$$

where

$$F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}. \tag{4}$$

Here, all coefficients of $F(\xi)$ are real, $F(0) = 1$; N and k are positive integers; and $k' \in \mathbb{Z}$. Hence, the corresponding ϕ can be written as

$$\hat{\phi}(\xi) = \left(\frac{1+e^{-i\xi/2}}{2}\right)^N F(\xi/2)\hat{\phi}(\xi/2). \tag{5}$$

Clearly, ϕ is a B-spline of order N if $F(\xi) = 1$. The vanishing moments of ϕ are completely controlled by the exponents of its "spline factor," $\left(\frac{1+e^{-i\xi}}{2}\right)^N$. In addition, the regularity of ϕ is justified by the factors $F(\xi)$, and are independent of their vanishing moments.

A shift invariant (SI) space is a closed subspace of $L_2(\mathbb{R})$ that is invariant under the operator $S_k(f) := f(\cdot - k)$ ($k \in \mathbb{Z}$). For $\phi \in L_2(\mathbb{R})$, we say that $V = S(\phi) := \overline{span}\{\phi(\cdot - k) : \ k \in \mathbb{Z}\}$ is generated by ϕ. In addition, if ϕ is refinable, then ϕ is said to be a refinable generator of $S(\phi)$, and $S(\phi)$ is called a refinable SI space. Each element $\phi \in \Phi$ is a refinable generator of the corresponding SI space $S(\phi)$. A refinable generator is said to be a pseudo-scaling (refinable) generator if it satisfies $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$ and $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$.

In [1,2,11], the following concepts were introduced that are important in our discussion.

Definition 2. *The bracket operator $[\,,\,] : L_2(\mathbb{R}) \times L_2(\mathbb{R}) \to L_1(\mathbb{T})$, $\mathbb{T} = [0, 2\pi)$, is defined by*

$$[f, g] = \sum_{k\in\mathbb{Z}} f(\xi + 2\pi k)\overline{g(\xi + 2\pi k)}. \tag{6}$$

For $f \in L_2(\mathbb{R})$ the function $[f, f] \in L_1(\mathbb{T})$ is called the auto-correlation of f.

If f, g are compactly supported, then $[\hat{f}, \hat{g}]$ is a trigonometric polynomial and has the Fourier expansion

$$[\hat{f}, \hat{g}](\xi) = \sum_{k\in\mathbb{Z}} \langle f(\cdot), g(\cdot + k)\rangle e^{ik\xi}. \tag{7}$$

Definition 3. *Let $S(\phi)$ be a shift invariant space that is generated by ϕ. $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is called a* stable basis *of $S(\phi)$ if there exist constants $0 < A \le B < \infty$ such that for every $\mathbf{c} = \{c_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$*

$$A\,\|\mathbf{c}\|^2_{\ell_2(\mathbb{Z})} \le \left\| \sum_{k\in\mathbb{Z}} c_k \phi(\cdot - k) \right\|^2_{L_2(\mathbb{R})} \le B\,\|\mathbf{c}\|^2_{\ell_2(\mathbb{Z})}. \tag{8}$$

Obviously, a stable basis of $S(\phi)$ is a basis of $S(\phi)$.

Theorem 4. [16] *Let $\phi \in L_2(\mathbb{R})$ and let $0 < A \le B < \infty$. Then (8) and*

$$A \le \left[\hat{\phi}, \hat{\phi}\right] \le B, \text{ a.e.}$$

are equivalent.

In Section 2 we will discuss the conditions for the coefficients $\{a_j\}$ such that the corresponding function ϕ is in $L_2(\mathbb{R})$ and is stable. The stability of the corresponding complementary spaces will be also discussed. Section 3 will give applications of the refinable generators in the construction of the biorthogonal and orthogonal scaling functions (the original generators) and wavelets (the generators of the corresponding complementary spaces) that possess the largest possible regularities and required vanishing moments. Since the biorthogonality and orthogonality imply the stability of the integer translates of the generators, we obtain simpler and sufficient conditions of the stability of a refinable generator with the largest possible regularities and required vanishing moments.

§2. Stable Generators of Shift Invariant Spaces

In [13] we have the following result. For the reader's convenience, we include a simpler alternative proof here.

Lemma 5. *Let $\phi \in \Phi$ be defined as in Definition 1; i.e.,*

$$\hat{\phi} = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) \in M$ is defined by (3) and (4):

$$m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$$

and $F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}$, $N, k \in \mathbb{Z}_+$ and $k' \in \mathbb{Z}$, where $F(0) = 1$. If $F(\pi) \ne -1$ and the coefficients of $F(\xi)$ satisfy

$$(k+1)\sum_{j=0}^{k} a_j^2 < 2^{2N-1}, \tag{9}$$

then ϕ is in $L_2(\mathbb{R})$. In addition, (9) can be replaced by the weaker condition

$$C(\{a_j\},k) < 2^{2N-1}, \tag{9$'$}$$

where $C(\{a_j\},k)$ equals $k\sum_{j=0}^{k} a_j^2$ if $k\geq 1$ and equals a_0^2 if $k=0$.

Proof: We first prove that $\phi \in L_2(\mathbb{R})$; i.e.,

$$C(\{a_j\},k) < 2^{2N-1},$$

implies $\hat{\phi} = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ is in $L^2(\mathbb{R})$. It is sufficient to prove the boundedness of the following integral

$$\begin{aligned}
&\int_{|\xi|\geq\pi} |\hat{\phi}(\xi)|^2 d\xi \\
&= \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \prod_{j=1}^{\infty} \left| \left(\frac{1+e^{-i2^{-j}\xi}}{2}\right)^N F(2^{-j}\xi)\right|^2 d\xi \\
&= \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \left|\frac{1-e^{-i\xi}}{i\xi}\right|^{2N} \prod_{j=1}^{\infty} |F(2^{-j}\xi)|^2 d\xi \\
&\leq C\sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \frac{1}{|\xi|^{2N}} \prod_{j=1}^{\infty} |F(2^{-j}\xi)|^2 d\xi \\
&\leq C\sum_{\ell=1}^{\infty} \frac{1}{2^{2\ell N}} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \prod_{j=1}^{\ell} |F(2^{-j}\xi)|^2 \prod_{j=1}^{\infty} |F(2^{-\ell-j}\xi)|^2 d\xi \\
&\leq C\sum_{\ell=1}^{\infty} 4^{-\ell N} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \prod_{j=1}^{\ell} |F(2^{-j}\xi)|^2 d\xi.
\end{aligned} \tag{10}$$

We now prove the boundedness of the last integral in (10). Denote

$$Tf(\xi) = \left|F\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|F\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right). \tag{11}$$

Hence, for any 2π-periodic continuous function f, we have

$$\begin{aligned}
&\int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} f(2^{-\ell}\xi) \prod_{j=1}^{\ell} \left|F(2^{-j}\xi)\right|^2 d\xi \\
&= \int_{-\pi}^{\pi} T^\ell f(\xi) d\xi \leq \sqrt{2\pi}||T^\ell f||_{L^2} \leq \sqrt{2\pi}||f||_{L^2}||T^\ell||.
\end{aligned} \tag{12}$$

Let $\rho(T)$ be the spectral radius of the operator T. Since $F(0) = 1$ and $F(\pi) \neq -1$, it can be shown as follows that $\rho(T) > 0$ (see also [6]). Considering the Fourier expansion

$$|F(\xi)|^2 = \sum_{\ell=-k}^{k} b_\ell e^{i\ell\xi},$$

where k is a positive integer and $b_t = \sum_{j=0}^{k-|t|} a_{k-|t|-j} a_{k-j}$ $(t = -k, \cdots, k)$, we find that the matrix of T restricted to

$$E_k = \{\sum_{\ell=-k}^{k} c_\ell e^{i\ell\xi} : (c_{-k}, \ldots, c_k) \in \mathbb{C}^{2k+1}\}$$

is

$$M_T = (2b_{i-2j})_{i,j=-k,\ldots,k} = 2\begin{bmatrix} b_k & 0 & 0 & \cdots & 0 \\ b_{k-2} & b_{k-1} & b_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{-k} & b_{-k+1} & b_{-k+2} & \cdots & b_k \\ 0 & 0 & b_{-k} & \cdots & b_{k-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{-k} \end{bmatrix}. \quad (13)$$

Noting that $|F(0)|^2 = \sum_{\ell=-k}^{k} b_\ell = 1$ and $|F(\pi)|^2 = \sum_{\ell=-k}^{k}(-1)^\ell b_\ell = \alpha \neq -1$, it follows that

$$\sum_\ell b_{2\ell} = \sum_\ell b_{2\ell+1} = (\alpha+1)/2,$$

and for the vector $u = (1, \ldots, 1) \in \mathbb{C}^{2k+1}$,

$$Tu = uM = (\alpha+1)u.$$

Thus, T has at least one eigenvalue $\alpha + 1 \neq 0$.

For every $\epsilon > 0$, there is an integer $\ell(\epsilon)$ such that

$$||T^\ell|| \leq (\rho(T)+\epsilon)^\ell, \qquad \ell > \ell(\epsilon).$$

It follows from (10) that

$$\int_{|\xi|\geq\pi} |\hat{\phi}(\xi)|^2 d\xi \leq C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T^\ell|| + C\sum_{\ell=\ell(\epsilon)+1}^{\infty} 4^{-N\ell}\left(\rho(T)+\epsilon\right)^\ell,$$

so $\rho(T)$ must be estimated if we are to choose an $\epsilon > 0$ small enough for the series to converge. Regardless of how small an $\epsilon > 0$ is chosen, the contribution

$$C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T^\ell|| \le C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T||^\ell$$

is finite, although possibly very large.

To evaluate $\rho(T)$, we consider the matrix of T, M_T, which was given in (13). It is clear that $\rho(T) = \rho(M_T)$. We write $M_T = 2H$, where

$$H = (b_{i-2j})_{i,j=-k,\cdots,k}\,.$$

Obviously, b_β can be written as

$$b_\beta = \sum_{j=0}^{k-|\beta|} a_{k-|\beta|-j}a_{k-j}, \quad \beta = -k, \cdots, k.$$

Hence, $b_\beta = b_{-\beta}$, for all $\beta = -k, \cdots, k$. It is also clear that b_k is an eigenvalue of H with multiplicity 2. To estimate bounds of the eigenvalues of H, we establish

$$|b_\beta| \le b_0, \quad \beta = -k, \cdots, k.$$

In fact,

$$\begin{aligned}
|b_\beta| &\le \sum_{j=0}^{k-|\beta|} |a_{k-|\beta|-j}a_{k-j}| \\
&\le \sum_{j=0}^{k-|\beta|} [\frac{1}{2}a^2_{k-|\beta|-j} + \frac{1}{2}a^2_{k-j}] \\
&= \frac{1}{2}\sum_{j=\beta}^{k} a^2_{k-j} + \frac{1}{2}\sum_{j=0}^{k-\beta} a^2_{k-j} \\
&\le \sum_{j=0}^{k} a^2_{k-j} = b_0.
\end{aligned}$$

It is obvious that the spectral radius of H is b_0 if $k = 0$. For $k \ge 1$, the characteristic polynomial of H is $(b_k - \lambda)(b_{-k} - \lambda)$ multiplied by the characteristic polynomial of the core matrix, H_c, which consists of all rows and columns of H except its first and last rows and columns. Hence, the spectral radius of H is

$$\begin{aligned}
\rho(H) &= \max\{b_k, \rho(H_c)\} \le \max\{b_k, \|H_c\|_1\} \\
&= \max\{b_k, \sum_{i=-k+1}^{k-1} |b_{i-2j}| : j = -k+1, \cdots, k-1\} \le kb_0.
\end{aligned}$$

Therefore, $\rho(T) = 2\rho(H) \leq 2C(\{a_j\},k)$. Here, $C(\{a_j\},k)$ equals $k\sum_{j=0}^{k} a_j^2$ if $k \geq 1$ and equals a_0^2 if $k=0$. If $C(\{a_j\},k) < 2^{2N-1}$, then $\rho(T) < 2^{2N}$, so we choose

$$\epsilon = \frac{1}{2}\left(2^{2N} - \rho(T)\right).$$

Thus

$$\rho(T) + \epsilon < 2^{2N},$$

and we obtain the estimate

$$\int_{|\xi|\geq\pi} \left|\hat{\phi}(\xi)\right|^2 d\xi \leq C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T||^{\ell} + \sum_{\ell=\ell(\epsilon)+1}^{\infty}\left(\frac{\rho(T)+\epsilon}{4^N}\right)^{\ell}.$$

The tail of the series is a convergent geometric series, thus completing the proof of $\phi \in L_2(\mathbb{R})$ if condition (9)', $C(\{a_j\},k) < 2^{2N-1}$, holds. Since $C(\{a_j\},k) \leq (k+1)\sum_{j=0}^{k} a_j^2$, the proof is complete. □

We now discuss the stability of ϕ. From [15], we obtain a necessary and sufficient condition for a refinable function defined as in Definition 1 to be stable.

Lemma 6. *The function ϕ defined in Definition 1 is stable if and only if $F(\xi)$ satisfies the following two conditions.*

(i) *$F(\xi)$ does not have any symmetric zeroes on $\mathbb{T} = [0, 2\pi)$;*

(ii) *For any odd integer $m > 1$ and a primitive mth root $\omega = e^{-i2n\pi/m}$ of unity (i.e., n is an integer relatively prime to m), there exists an integer d, $0 \leq d < ord_m\, 2$, such that $F\left(-2^{d+1}n\pi/m\right) \neq 0$, where $p = ord_m\, 2$ is the smallest positive integer with $2^p \equiv 1(mod\ m)$.*

In addition, if ϕ is stable, then $|\hat{\phi}(\xi)|^2$ and

$$\begin{aligned}\bar{F}(\xi) &= \prod_{j=1}^{\infty}\left|F\left(2^{-j}(\xi+2\pi u)\right)\right|^2 / \prod_{j=1}^{\infty}\left|F\left(2^{-j}\xi\right)\right|^2 \\ &= \frac{\prod_{j=1}^{\infty}\left|\sum_{\ell=0}^{k} a_\ell e^{-i\ell 2^{-j}(\xi+2\pi u)}\right|^2}{\prod_{j=1}^{\infty}\left|\sum_{\ell=0}^{k} a_\ell e^{-i\ell 2^{-j}\xi}\right|^2}\end{aligned} \tag{14}$$

have no roots in $\mathbb{T}$.

Proof: By using Lemma 6.6 of [5] and Theorem 1 of [15], we obtain that ϕ is stable. In addition, since

$$\left|\hat{\phi}(\xi)\right|^2 = \left|\frac{1-e^{-i\xi}}{i\xi}\right|^{2N}\prod_{j=1}^{\infty}\left|F\left(2^{-j}\xi\right)\right|^2, \tag{15}$$

we have

$$\left|\hat{\phi}(\xi+2\pi u)\right|^2 = \bar{F}(\xi)\left|\hat{\phi}(\xi)\right|^2 \frac{\xi^{2N}}{(\xi+2\pi u)^{2N}},$$

where $\bar{F}(\xi)$ is defined as (14). Therefore,

$$\begin{aligned}\left[\hat{\phi},\hat{\phi}\right] &= \sum_{u\in\mathbb{Z}}\left|\hat{\phi}(\xi+2\pi u)\right|^2 \\ &= \left|\hat{\phi}(\xi)\right|^2 \bar{F}(\xi)(\xi)^{2N}\sum_{u\in\mathbb{Z}}\frac{1}{(\xi+2\pi u)^{2N}}.\end{aligned}$$

By using formula (4.2.7) in [6], we can rewrite the last expression as

$$\left[\hat{\phi},\hat{\phi}\right] = \left|\hat{\phi}(\xi)\right|^2 \frac{\bar{F}(\xi)(\xi/2)^{2N}}{\sin^{2N}(\xi/2)} \sum_{u=-\infty}^{\infty}\left|\hat{B}_N(\xi+2\pi u)\right|^2 \tag{16}$$

where $\hat{B}_N(\xi)$ is the Fourier transform of the B-spline of order N. In addition, the sum on the right-hand side of (16) can be evaluated by using formula (4.2.10) in [6]:

$$\sum_{u=-\infty}^{\infty}\left|\hat{B}_N(2\xi+2\pi u)\right|^2 = \frac{-\sin^{2N}\xi}{(2N-1)!}\frac{d^{2N-1}}{d\xi^{2N-1}}\cot\xi.$$

Therefore, noting that ϕ is stable and applying Theorem 4 to the above $\left[\hat{\phi},\hat{\phi}\right]$, we immediately know that $|\hat{\phi}(\xi)|^2, \bar{F}(\xi) \neq 0$ for all $\xi \in \mathbb{T}$. This completes the proof. $\square$

Combining Lemmas 5 and 6, we obtain the following result.

Theorem 7. *Let $\phi \in \Phi$ be defined as in Definition 1. If $F(\xi)$ satisfies $F(\pi) \neq -1$, conditions (i) and (ii) in Lemma 6, and (9) or (9)', then the corresponding ϕ is in $L_2(\mathbb{R})$ and is stable.*

Remark. Obviously, the stability condition described in Lemma 6 is not easy to check. We will give a simpler sufficient condition in the next section.

Let $V_j := \overline{span}\{\phi(2^j t - k) : k \in \mathbb{Z}\}$. Following [11], for any $\phi \in L_2(\mathbb{R})$, we define the (natural) dual $\tilde{\phi}$ by its Fourier transform

$$\hat{\tilde{\phi}} := \frac{\hat{\phi}}{\left[\hat{\phi},\hat{\phi}\right]},$$

where we interpret $0/0 = 0$. Thus, for ϕ given in Theorem 7, from (16), the dual function's Fourier transform is

$$\hat{\tilde{\phi}} = \frac{\hat{\phi}}{\left[\hat{\phi},\hat{\phi}\right]} = \frac{\sin^{2N}(\xi/2)}{(\xi/2)^{2N}\overline{\hat{\phi}}\bar{F}(\xi)\sum_{u=-\infty}^{\infty}\left|\hat{B}_N(\xi+2\pi u)\right|^2}.$$

The properties of the dual function $\tilde{\phi}$ can be found in [6,12].

It is clear that $S(\phi) \subset V_1 = \overline{span}\{\phi(2\cdot -k) : k \in \mathbb{Z}\}$. We now consider the complementary space of $S(\phi)$ in V_1, which is generated by a function $\psi \in V_1$. We say that a function $f \in L_2(\mathbb{T})$ is in $\mathbf{W}$, the Wiener Algebra, if its Fourier series $\sum_{k\in\mathbb{Z}} f_k e^{-ik\xi}$ satisfies $\{f_k\} \in \ell_1(\mathbb{Z})$. From [11], we can establish the following theorem.

Theorem 8. *Let $\phi \in \Phi$ satisfy all conditions of Theorem 7, where Φ is defined in Definition 1, and let $\psi \in V_1 := \overline{span}\{\phi(2\cdot -k) : k \in \mathbb{Z}\}$ have the symbol $m_1(\xi) \in \mathbf{W}$, the Wiener Algebra, such that*

$$|m_1(\xi)|^2 + |m_1(\xi+\pi)|^2 > 0, \quad \xi \in \mathbb{T}. \tag{17}$$

Then ψ is stable (i.e., a stable generator for $S(\psi)$).

Proof: Since $\psi \in V_1$, using the two-scale relation of ψ yields

$$\begin{aligned}
\left[\hat{\psi},\hat{\psi}\right](\xi) &= \sum_{i\in\mathbb{Z}}\left|m_1\left(\frac{\xi}{2}+\pi i\right)\right|^2\left|\hat{\phi}\left(\frac{\xi}{2}+\pi i\right)\right|^2 \\
&= \sum_{i\in\mathbb{Z}}\left|m_1\left(\frac{\xi}{2}+2\pi i\right)\right|^2\left|\hat{\phi}\left(\frac{\xi}{2}+2\pi i\right)\right|^2 \\
&\quad + \sum_{i\in\mathbb{Z}}\left|m_1\left(\frac{\xi}{2}+\pi+2\pi i\right)\right|^2\left|\hat{\phi}\left(\frac{\xi}{2}+\pi+2\pi i\right)\right|^2 \\
&= \left|m_1\left(\frac{\xi}{2}\right)\right|^2\left[\hat{\phi}\left(\frac{\xi}{2}\right),\hat{\phi}\left(\frac{\xi}{2}\right)\right] + \left|m_1\left(\frac{\xi}{2}+\pi\right)\right|^2 \times \\
&\qquad \left[\hat{\phi}\left(\frac{\xi}{2}+\pi\right),\hat{\phi}\left(\frac{\xi}{2}+\pi\right)\right].
\end{aligned}$$

From Theorem 7, ϕ is in $L_2(\mathbb{R})$ and is stable. Thus, Theorem 4 shows there exist $0 < A \le B < \infty$ such that

$$A \le \left[\hat{\phi},\hat{\phi}\right] \le B \quad a.e.$$

Thus we can bound the auto-correlation of ψ by

$$AM_1(\xi) \le \left[\hat{\psi},\hat{\psi}\right](\xi) \le BM_1(\xi),$$

where

$$M_1(\xi) := \left| m_1\left(\frac{\xi}{2}\right)\right|^2 + \left| m_1\left(\frac{\xi}{2}+\pi\right)\right|^2.$$

Since $m_1(\xi) \in (\mathbb{T})$, from condition (17) we have

$$\bar{A} := \min_{\xi\in\mathbb{T}} M_1(\xi) > 0.$$

On the other hand, $\bar{B} := \|m_1(\xi)\|_{C(\mathbb{T})} < \infty$. Therefore,

$$0 < \bar{A}A \le \left[\hat{\psi}, \hat{\psi}\right](\xi) \le \bar{B}B < \infty, \quad a.e.$$

By using Theorem 4, we have proved that ψ is a stable generator for $S(\psi)$. □

§3. Applications of Stable Refinable Generators

Obviously, the stability condition given in Lemma 6 is not easy to apply. Recalling Cohen's following result (see [7]), we can give a simpler and sufficient condition for the stability of the integer translates of a refinable generator. In [7], Cohen showed that a refinable generator ϕ with $\hat{\phi}(0) = 1$ has orthonormal integer translates (i.e., $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal set) if and only if ϕ is a stable and pseudo-scaling generator. Hence, by extension of the results given in the author's paper [13] on the biorthogonal refinable generators, we can find stable refinable generators that possess the largest possible regularities and required vanishing moments.

Biorthogonal refinable generators defined by Definition 1 were discussed in [13]. Similarly, from Cohen and Daubechies' result in [8], the biorthogonality of the integer translates implies the stability of the translate system. The biorthogonal system associated with the integer translates of $\phi \in \Phi$ is the set of the integer translates of another refinable generator $\tilde{\phi} \in \Phi$. Here, $\tilde{\phi}(t) = \sum_n \tilde{h}_n \tilde{\phi}(2t-n)$ or equivalently, $\hat{\tilde{\phi}}(\xi) = \tilde{m}_0(\xi/2)\hat{\tilde{\phi}}(\xi/2)$ with $\tilde{m}_0(\xi) = 2^{-1}\sum_n \tilde{h}_n e^{-in\xi} \in M$, which is defined in Definition 1. Therefore, we can write

$$\tilde{m}_0(\xi) = \tilde{m}_0^{\tilde{N}}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}} \tilde{F}(\xi), \tag{18}$$

where

$$\tilde{F}(\xi) = e^{-i\tilde{k}'\xi}\sum_{j=0}^{\tilde{k}} \tilde{a}_j e^{-ij\xi}.$$

Here, all coefficients of $\tilde{F}(\xi)$ are real, $\tilde{F}(0) = 1$; $\tilde{N}$ and $\tilde{k}$ are positive integers; and $\tilde{k}' \in \mathbb{Z}$. Hence, the corresponding ϕ and $\tilde{\phi}$ can be written as

$$\hat{\phi}(\xi) = \left(\frac{1+e^{-i\xi/2}}{2}\right)^N F(\xi/2)\hat{\phi}(\xi/2),$$

$$\hat{\tilde{\phi}}(\xi) = \left(\frac{1+e^{-i\xi/2}}{2}\right)^{\tilde{N}} \tilde{F}(\xi/2)\hat{\tilde{\phi}}(\xi/2). \tag{19}$$

We also define the corresponding ψ and $\tilde{\psi}$ by

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{\tilde{m}_0(\xi/2+\pi)}\hat{\phi}(\xi/2) \quad \hat{\tilde{\psi}}(\xi) = e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\hat{\tilde{\phi}}(\xi/2), \tag{20}$$

or, equivalently,

$$\psi(x) = \sum_n (-1)^{n-1}\tilde{h}_{-n-1}\phi(2x-n)$$

$$\tilde{\psi}(x) = \sum_n (-1)^{n-1}h_{-n-1}\phi(2x-n). \tag{21}$$

Similar to ϕ, $\tilde{\phi}$ is also a B-spline of order $\tilde{N}$ if $\tilde{F}(\xi) = 1$. Since vanishing moment conditions $\int x^\ell \psi(x)dx = 0$, $\ell = 0, 1, \cdots, L$, are equivalent to $\frac{d^\ell}{d\xi^\ell}\hat{\psi}|_{\xi=0} = 0$, $\ell = 0, 1, \cdots, L$, we immediately know that the maximum orders of vanishing moment for ψ and $\tilde{\psi}$ are $N-1$ and $\tilde{N}-1$, respectively. Therefore, the vanishing moments of ϕ and $\tilde{\phi}$ are completely controlled by the exponents of their respective "spline factors," $\left(\frac{1+e^{-i\xi}}{2}\right)^N$ and $\left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}}$. In addition, as we pointed out at the beginning of this paper, the regularities of ϕ and $\tilde{\phi}$ are justified by the factors $F(\xi)$ and $\tilde{F}(\xi)$, respectively, and are independent of their vanishing moments.

From Lemma 5, if $\tilde{\phi} \in \Phi$ and the coefficients of the corresponding $\tilde{F}(\xi)$ satisfy

$$(\tilde{k}+1)\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2 < 2^{2\tilde{N}-1} \tag{22}$$

and $\tilde{F}(\pi) \neq -1$, then $\tilde{\phi} \in L_2(\mathbb{R})$.

In addition, from [8], the stability of ϕ, $\tilde{\phi}$, ψ, and $\tilde{\psi}$ are implied by their biorthogonality. In fact, [8] gave the following results.

Lemma 9. [8] *If $\phi, \tilde{\phi} \in L_2(\mathbb{R})$ satisfy $\langle \phi(t), \tilde{\phi}(t-n)\rangle = \delta_{n0}$, then $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ and $\{\tilde{\phi}(t-k)\}_{k\in\mathbb{Z}}$ are stable; i.e., they are stable bases (Riesz bases) in the subspace that they generate. In addition, the corresponding biorthogonal wavelet functions ψ and $\tilde{\psi}$ are also stable; i.e., they are stable bases (Riesz bases) in the subspace that they generate.*

Hence, from the following result established in [13], we can construct biorthogonal (stable) refinable generators with the largest possible regularities and required vanishing moments by using the algorithm shown in [13].

Theorem 10. [13] *Let $\phi, \tilde{\phi} \in \Phi$ be defined in Definition 1; that is $\phi = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ and $\tilde{\phi} = \prod_{j=1}^{\infty} \tilde{m}_0^{\tilde{N}}(2^{-j}\xi)$, where $m_0^N(\xi)$ and $\tilde{m}_0^{\tilde{N}}(\xi)$ are as in (3) and (18). Suppose $F(\pi) \neq -1$, $\tilde{F}(\pi) \neq -1$; the coefficients of $F(\xi)$ and $\tilde{F}(\xi)$ satisfy respectively conditions (9) and (22); and*

$$\sum_{j=\mu}^{\nu} \sum_{\ell=0}^{k} \sum_{\tilde{\ell}=0}^{\tilde{k}} \binom{\tilde{N}}{j-\tilde{\ell}-\tilde{k}'} \binom{N}{j+2n-\ell-k'} \tilde{a}_{\tilde{\ell}} a_{\ell} = 2^{N+\tilde{N}-1}\delta_{n0}, \tag{23}$$

where $\mu = \min\{k', \tilde{k}'\}$; $\nu = \max\{N+k+k', \tilde{N}+\tilde{k}+\tilde{k}'\}$; δ_{n0} is the Kronecker symbol; and $n = 0, \pm 1, \pm 2, \cdots$; then $\phi, \tilde{\phi} \in L_2(\mathbb{R})$ are stable and $\langle \phi(t), \tilde{\phi}(t-i) \rangle = \delta_{i,0}$ for all $i \in \mathbb{Z}$. The corresponding biorthogonal wavelets ψ and $\tilde{\psi}$ defined as (20) or (21) are in C^{α} and $C^{\tilde{\alpha}}$, respectively. Here α and $\tilde{\alpha}$ satisfy

$$\alpha > N - \frac{1}{2}\log_2\left((k+1)\sum_{j=0}^{k} a_j^2\right),$$

and

$$\tilde{\alpha} > \tilde{N} - \frac{1}{2}\log_2\left((\tilde{k}+1)\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2\right), \tag{24}$$

respectively. Equivalently, the Sobolev exponents

$$\alpha := \sup\{s \geq 0 : \int_R (1+|\xi|^2)^s \left|\hat{\phi}(\xi)\right|^2 d\xi < \infty\}$$

and

$$\tilde{\alpha} := \sup\{s \geq 0 : \int_R (1+|\xi|^2)^s \left|\hat{\tilde{\phi}}(\xi)\right|^2 d\xi < \infty\}$$

satisfy (24).

If we consider the case of $\tilde{\phi} = \phi$ in Theorem 10, then we obtain the following result.

Theorem 11. *Let $\phi \in \Phi$ be defined by Definition 1. If F also satisfies $F(\pi) \neq -1$ and its coefficients satisfy condition (9) and*

$$\sum_{j=k'}^{N+k+k'} \sum_{\ell=0}^{k} \sum_{\tilde{\ell}=0}^{k} \binom{N}{j-\tilde{\ell}-k'} \binom{N}{j+2n-\ell-k'} a_{\tilde{\ell}} a_{\ell} = 2^{2N-1}\delta_{n0}, \tag{25}$$

where δ_{n0} is the Kronecker symbol and $n = 0, \pm 1, \pm 2, \cdots$, then $\phi \in L_2(\mathbb{R})$ and is a stable pseudo-scaling generator. In addition, The corresponding ψ defined as (20) or (21) is an orthogonal wavelet in C^α. Here α is more than

$$N - \frac{1}{2}\log_2\left((k+1)\sum_{j=0}^{k} a_j^2\right).$$

The regularity of ϕ (i.e., the Sobolev exponent of ϕ),

$$\alpha := sup\{s \geq 0 : \int_R (1+|\xi|^2)^s \left|\hat{\phi}(\xi)\right|^2 d\xi < \infty\},$$

satisfies

$$\alpha > N - \frac{1}{2}\log_2\left((k+1)\sum_{j=0}^{k} a_j^2\right).$$

Here, the condition (9) can be replaced by the following weaker condition:

$$C(\{a_j\}, k) < 2^{2N-1}, \tag{9'}$$

where $C(\{a_j\}, k)$ equals $k\sum_{j=0}^{k} a_j^2$ if $k \geq 1$ and equals a_0^2 if $k = 0$. Hence, the corresponding regularities of ψ is determined by $\psi \in C^{\alpha'}$, where α' is more than $N - \frac{1}{2}\log_2(2C(\{a_j\}, k))$. And the regularity of ϕ is $\alpha > N - \frac{1}{2}\log_2(2C(\{a_j\}, k))$.

Proof: Let $\phi \in \Phi$ be the function defined by Definition 1 that satisfies $F(\pi) \neq -1$ and (9). Then, from Lemma 5, ϕ is in $L_2(\mathbb{R})$. Noting Theorem 10, we have that (9) and (25) imply

$$\langle \phi(t), \phi(t-n) \rangle = \delta_{n0},$$

which is equivalent to ϕ being a stable refinable pseudo-scaling generator (see [7]). To prove that the generator ψ of the complementary space of $S(\phi)$ is an orthogonal wavelet, it is sufficient (see [14]) to prove that it satisfies

$$\sum_{j \in \mathbb{Z}} \left|\hat{\psi}\left(2^j \xi\right)\right|^2 = 1 \qquad a.e. \tag{26}$$

and

$$\sum_{j \geq 0} \hat{\psi}\left(2^j \xi\right) \overline{\hat{\psi}\left(2^j(\xi + 2q\pi)\right)} = 0 \quad a.e., \tag{27}$$

for all $q \in 2\mathbb{Z}+1$; i.e., for all odd integers, q. First,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(2^j\xi\right)\right|^2 &= \sum_{j\in\mathbb{Z}} \left|m_0\left(2^{j-1}\xi+\pi\right)\right|^2 \left|\hat{\phi}\left(2^{j-1}\xi\right)\right|^2 \\
&= \lim_{n\to\infty}\sum_{j=-n}^{n} \left|m_0\left(2^{j-1}\xi+\pi\right)\right|^2 \left|\hat{\phi}\left(2^{j-1}\xi\right)\right|^2 \\
&= \lim_{n\to\infty}\sum_{j=-n}^{n} \left[1-\left|m_0\left(2^{j-1}\xi\right)\right|^2\right] \left|\hat{\phi}\left(2^{j-1}\xi\right)\right|^2 \\
&= \lim_{n\to\infty}\left[\left|\hat{\phi}\left(2^{-n-1}\xi\right)\right|^2 - \left|\hat{\phi}\left(2^{n}\xi\right)\right|^2\right].
\end{aligned}$$

Since $\phi \in L_2(\mathbb{R})$, $\lim_{n\to\infty}\left|\hat{\phi}\left(2^n\xi\right)\right|^2 = 0$ for $a.e.\ \xi$. From the condition $F(0)=1$, we have $\lim_{n\to\infty}\left|m_0\left(2^{-n}\xi\right)\right| = 1$. Hence, taking limit $n\to\infty$ on both sides of equation $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$ and noting that

$$\lim_{n\to\infty}\left|\hat{\phi}(\xi)\right| = \lim_{n\to\infty}\prod_{j=1}^{\infty}\left|m_0(2^{-j}\xi)\right| \neq 0,$$

we obtain

$$\lim_{n\to\infty}\left|\hat{\phi}\left(2^{-n}\xi\right)\right| = 1,$$

which shows $\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(2^j\xi\right)\right|^2 = 1$. Therefore, (26) holds for our ψ. Secondly, for any odd integer q,

$$\begin{aligned}
&\sum_{j\geq 0}\hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j\left(\xi+2q\pi\right)\right)} \\
&\quad= \sum_{j>0} e^{i2^{j-1}\xi}\overline{m_0\left(2^{j-1}\xi+\pi\right)}\hat{\phi}\left(2^{j-1}\xi\right)\times \\
&\quad\quad \overline{e^{i2^{j-1}\xi}\overline{m_0\left(2^{j-1}\xi+\pi\right)}\hat{\phi}\left(2^{j-1}\xi+2^jq\pi\right)} \\
&\quad\quad + e^{i2^{-1}\xi}\overline{m_0\left(2^{-1}\xi+\pi\right)}\hat{\phi}\left(2^{-1}\xi\right)\overline{e^{i2^{-1}\xi+\pi}\overline{m_0\left(2^{-1}\xi\right)}\hat{\phi}\left(2^{-1}\xi+q\pi\right)} \\
&\quad= \sum_{j>0}\left|m_0\left(2^{j-1}\xi+\pi\right)\right|^2\hat{\phi}\left(2^{j-1}\xi\right)\overline{\hat{\phi}\left(2^{j-1}\xi+2^j\pi\right)} \\
&\quad\quad - \overline{m_0\left(2^{-1}\xi+\pi\right)}\hat{\phi}\left(2^{-1}\xi\right)m_0\left(2^{-1}\xi\right)\overline{\hat{\phi}\left(2^{-1}\xi+\pi\right)} \\
&\quad= \sum_{j>0}\left[1-\left|m_0\left(2^{j-1}\xi\right)\right|^2\right]\hat{\phi}\left(2^{j-1}\xi\right)\overline{\hat{\phi}\left(2^{j-1}\xi+2^j\pi\right)} - \hat{\phi}(\xi)\overline{\hat{\phi}(\xi+2\pi)}
\end{aligned}$$

$$= \sum_{j>0} \left[\hat{\phi}\left(2^{j-1}\xi\right) \overline{\hat{\phi}\left(2^{j-1}\xi + 2^j\pi\right)} - \hat{\phi}\left(2^j\xi\right) \overline{\hat{\phi}\left(2^j\xi + 2^{j+1}\pi\right)} \right]$$
$$- \hat{\phi}(\xi)\overline{\hat{\phi}(\xi + 2\pi)}$$
$$= - \lim_{n\to\infty} \hat{\phi}\left(2^n\xi\right) \overline{\hat{\phi}\left(2^n\xi + 2^{n+1}\pi\right)} = 0,$$

where $\hat{\phi}\left(2^{j-1}\xi + 2^j q\pi\right) = \hat{\phi}\left(2^{j-1}\xi + 2^j\pi\right)$ for all $q \in 2\mathbb{Z} + 1$ because $m_0(\xi)$ is $2\pi-$ periodic. Therefore, (27) also holds, and the proof of Theorem 11 is complete. $\square$

Remark. From Theorem 11, we immediately know that a refinable generator $\phi \in \Phi$ defined by Definition 1 is a stable pseudo-scaling generator if its corresponding $F(\xi)$ satisfies $F(\pi) \neq -1$ and equations (9) and (25).

We now give a general algorithm to construct the pseudo-scaling generator ϕ such that it possesses the largest possible regularity and the required vanishing moments. In fact, this method can be described as an optimization problem of finding suitable $F(\xi)$, or, equivalently, a suitable coefficient set $\mathbf{a} = \{a_0, \cdots, a_k\}$, of $F(\xi)$, such that $\sum_{j=0}^{k} a_j^2$ is the minimum under all conditions shown in Theorem 11. Thus, the above optimization problem can be written as follows:

$$\min_{\mathbf{a}} \qquad \sum_{j=0}^{k} a_j^2, \tag{28}$$

$$\text{subject to} \quad \left(\sum_{j=0}^{k} (-1)^j a_j + 1 \right)^2 > 0, \tag{29}$$

$$\sum_{j=0}^{k} a_j^2 < 2^{2N-1}/(k+1), \tag{30}$$

$$\sum_{j=k'}^{N+k+k'} \left(\sum_{\ell=0}^{k} \binom{N}{j + 2n - \ell - k'} a_\ell \right) \left(\sum_{\tilde{\ell}=0}^{k} \binom{N}{j - \tilde{\ell} - k'} a_{\tilde{\ell}} \right)$$
$$= 2^{2N-1}\delta_{n0},$$
$$n = 0, \pm 1, \pm 2, \cdots, \tag{31}$$

where the objective function (28) gives the largest possible regularity, condition (29) is from the definition of $F(\pi) \neq -1$, and conditions (30) and (31) come from conditions (9) and (25) of Theorem 11.

Problem (28)–(31) can be written in a form without the inequality conditions by defining parameters $s, t \neq 0$ as $s^2 = 2^{2N-1}/(k+1) - \sum_{j=0}^{k} a_j^2$

and $t^2 = \left(\sum_{j=0}^k (-1)^j a_j + 1\right)^2$, respectively. Hence, the optimization problem becomes

$$\min_{\mathbf{a}} \quad \sum_{j=0}^k a_j^2 + \frac{1}{s^2} + \frac{1}{t^2},$$

$$\text{subject to } t^2 = \left(\sum_{j=0}^k (-1)^j a_j + 1\right)^2$$

$$s^2 + \sum_{j=0}^k a_j^2 = 2^{2N-1}/(k+1)$$

$$\sum_{j=k'}^{N+k+k'} \left(\sum_{\ell=0}^k \binom{N}{j+2n-\ell-k'} a_\ell\right) \left(\sum_{\tilde{\ell}=0}^k \binom{N}{j-\tilde{\ell}-k'} a_{\tilde{\ell}}\right)$$
$$= 2^{2N-1}\delta_{n0},$$
$$n = 0, \pm 1, \pm 2, \cdots,$$

As examples, we choose $N = 1$ and $k = k' = 0$. Then the solution of problem (28)–(31) is $a_0 = 1$, and we obtain the Haar function. If we choose $N = 2$, $k = 1$, and $k' = 0$, then the solutions of the problem are $a_0 = \frac{1 \pm \sqrt{3}}{2}$ and $a_1 = \frac{1 \mp \sqrt{3}}{2}$. Hence, the corresponding ϕ is defined by $\hat{\phi}(\xi) = \prod_{j=1}^\infty \tilde{m}_0(2^{-j}\xi)$, where

$$m_0(\xi) = \frac{1 \pm \sqrt{3}}{2}\left(1 - e^{-i\xi}\right)\left(\frac{1 + e^{-i\xi}}{2}\right)^2.$$

In addition, the regularity of ϕ is more than $2 - \log_2(4)/2 = 1$ and $\psi \in C^1$.

Finally, we discuss the error of L_2 approximation from $S(\phi)$. Denote

$$E(f, S(\phi))_{L_2} := \inf_{g \in S(\phi)} \|f - g\|_{L_2}$$

where ϕ is a stable function. Since ϕ defined in Theorem 7 is compactly supported and has N vanishing moments (i.e., accuracy N), it has approximation order N. In addition, from [4], the approximation coefficient is

$$C_\phi^N = \frac{1}{m!}\sqrt{\sum_{u \neq 0} \left|\hat{\phi}^{(m)}(2\pi u)\right|^2}.$$

Therefore, for any function $f \in W_2^{N+1}(\mathbb{R})$, the Sobolev space,

$$E\left(f, S(\phi)^h\right)_2 = C_\phi^N \|f\|_{W_2^N(\mathbb{R})} + O\left(h^{N+1}\right),$$

where $S(\phi)^h := \{f(\cdot/h) | f \in S(\phi)\}$.

Acknowledgments. I would like to thank the referees and the editor for their comments that helped improve the presentation of this paper.

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