Short Time Fourier Transform, Integral Wavelet Transform, and Wavelet Functions Associated With Splines
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Abstract
In this paper, we will discuss Short Time Fourier transforms, integral wavelets transforms, and wavelet series expansions associated with spline functions in shift-invariant spaces of B-splines. A recurrence relation formula and the corresponding algorithm about the B-wavelets will also be given.

1. Introduction
Let $N_m(x)$ be the cardinal B-spline function of order $m$, and $S(N_m(x))$ be the shift-invariant space of $N_m(x)$ that is defined as the span of the integer translations of $N_m(x)$. Thus, $s_m \in S(N_m(x))$ can be written as
$$s_m(x) = \sum_j c_j N_m(x - j), \quad x \in \mathbb{R}, j \in \mathbb{Z}. \quad (1)$$
Throughout, we will always assume $\{c_j\} \in \ell_2$.

We will discuss Short Time Fourier transforms (STFT), integral wavelet transforms (IWT), and wavelet series expansions associated with $s_m(x)$ because of the following reasons. First, the basic function of a cardinal B-spline interpolation from $S(N_m(x))$ can be expressed as $s_m(x)$; the wavelet function associated with $s_m(x)$ can then be used for numerical analysis. Second, by choosing a suitable $\{c_j\}$, we can obtain an $s_m(x)$ as a basic wavelet function although an $N_m(x)$ could not be attained that has this property. Third, we may use $s_m(x)$ as a scaling function to construct smooth and symmetric mother wavelet functions. Finally, orthogonalization of wavelet functions associated with $s_m(x)$ can be implemented by adjusting $\{c_j\}$.

In Section 2, we will give the size of time-frequency window of the STFT associated with the window function $s_m(x)$. The size of the window can be justified by its coefficient set $\{c_j\}$. In Section 3, we will give conditions under which an $s_m(x)$ can be a basic wavelet function for IWT. In Section 4, we will discuss wavelet functions associated with $s_m(x)$ that were derived in [5]. This type of wavelets can be considered as an extension of B-wavelets (cf. [1]). In Section 5, we will give two different approaches for orthogonalization of the wavelet functions associated with $s_m(x)$. The last section will give a recurrence algorithm of $s_m(x)$ based on the recurrence relation of B-wavelets.
2. Short Time Fourier Transforms

Let \( \hat{s}_m(\omega) \) and \( \hat{N}_m(\omega) \) be the Fourier transform of \( s_m(x) \) and \( N_m(x) \) respectively. Thus,

\[
\hat{s}_m(\omega) = C(z^2)\hat{N}_m(\omega), \tag{2}
\]

where

\[
\hat{N}_m(\omega) = \left( \frac{1 - e^{-i\omega}}{i\omega} \right)^m, \tag{3}
\]

\[ z = e^{-i\omega/2}, \text{ and } C(z^2) = \sum_j c_j z^{2j} = \sum_j c_j e^{-i\omega} \text{ is the symbol of } \{c_j\}. \]

It is obvious, if \( m \geq 2 \), both \( x s_m(x) \) and \( \omega \hat{s}_m(\omega) \) \( \in L^2(\mathbb{R}) \). Thus, for \( m \geq 2 \), \( s_m(x) \) defined by (1) can be considered as a window function for STFT

\[
(G_b f)(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t) s_m(t-b) dt. \tag{4}
\]

We will find the center \( t^* \) and the radius \( \Delta s_m \) of \( s_m \) and the center \( \omega^* \) and the radius \( \Delta \hat{s}_m \) of \( \hat{s}_m \), the fourier transform of \( s_m \). Thus, the corresponding time-frequency window can be determined. Since the set of the arbitrary coefficient \( \{c_j\} \) are involved, the size of window can be adjusted for certain purposes.

In order to find the centers and radiuses, we need the following lemmas.

**Lemma 1.** Let \( \hat{N}_m(\omega) \) be the Fourier transform of the B-spline of order \( m \), \( N_m(x) \), shown in equation (3). Then

\[
\hat{N}_m(\omega) = \hat{N}_m(\omega) e^{im\omega}, \tag{5}
\]

and

\[
\hat{N}_m'(\omega) = m\hat{N}_{m-1}(\omega)\hat{N}_1'(\omega). \tag{6}
\]

**Proof.** From equation (3), we obtain

\[
\hat{N}_m(\omega) = e^{-im\omega/2} \left( \frac{\sin \omega/2}{\omega/2} \right)^m.
\]

Hence, equation (5) is obvious. Equation (6) can be derived immediately from the Fourier transform of the B-spline of order \( m \): \( \hat{N}_m(\omega) = \left( \frac{1-e^{-i\omega}}{i\omega} \right)^m \).

**Lemma 2.** Let \( N_m(x) \) be the B-spline of order \( m \). Then

\[
\int_{-\infty}^{\infty} N_m(x)N_m(x+\ell) dx = N_{2m}(m-\ell), \tag{7}
\]

\[
\int_{-\infty}^{\infty} xN_m(x)N_m(x+\ell) dx = \frac{m-\ell}{2}N_{2m}(m-\ell), \tag{8}
\]

and
Lemma 1, we can obtain expressions (7) and (8).

To prove the equation (9), we need an equation about the integral of \( x^2 N_m(x)N_m(x + \ell) \).

\[
\int_{-\infty}^{\infty} x^2 N_m(x)N_m(x + \ell)dx = \frac{(m - \ell)^2}{2} N_{2m}(m - \ell). \tag{9}
\]

**Proof.** By using the Parseval Identity, the recurrence relation \( N_{n+1} = N_m \ast N_1 \), and Lemma 1, we can obtain expressions (7) and (8).

To prove the equation (9), we need an equation about the integral of \( x^2 N_m(x)N_m(x + \ell) \).

\[
\int_{-\infty}^{\infty} x^2 N_m(x)N_m(x + \ell)dx
\]

\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \tilde{N}_m(\omega) \right)'' \tilde{N}_m(\omega)e^{i(m-\ell)\omega}d\omega
\]

\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} [m(m-1)\tilde{N}_{2m-2}(\omega) \left( \tilde{N}_1'(\omega) \right)^2 + m\tilde{N}_{2m-1}(\omega)\tilde{N}_1''(\omega)]e^{i(m-\ell)\omega}d\omega
\]

\[
= -\frac{1}{4\pi} \int_{-\infty}^{\infty} \tilde{N}''_{2m}(\omega)e^{i(m-\ell)\omega}d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} [m\tilde{N}_{m-1}(\omega)\tilde{N}_1'(\omega)]^2 e^{i(m-\ell)\omega}d\omega
\]

\[
= \frac{1}{2} (m - \ell)^2 N_{2m}(m - \ell) + \int_{-\infty}^{\infty} (m - \ell - x)xN_m(x)N_m(m - \ell - x)dx
\]

\[
= \frac{1}{2} (m - \ell)^2 N_{2m}(m - \ell) + \int_{-\infty}^{\infty} (m - \ell)xN_m(x)N_m(x + \ell)dx
\]

\[
- \int_{-\infty}^{\infty} x^2 N_m(x)N_m(x + \ell)
\]

Hence, by solving the last equation for \( \int_{-\infty}^{\infty} x^2 N_m(x)N_m(x + \ell)dx \) and using equation (8), we obtain equation (9).

By using Lemma 2, we can find the center \( t^* \) and the radius \( \Delta_{s_m} \) of \( s_m \) as follows.

\[
t^* = \frac{1}{\|s_m\|^2} \int_{-\infty}^{\infty} x[s_m(t)]^2 dt
\]

\[
= \frac{1}{\|s_m\|^2} \sum_j \sum_{j'} c_jc_{j'} \int_{-\infty}^{\infty} N_m(t - j)N_m(t - j')dt
\]

\[
= \frac{1}{\|s_m\|^2} \sum_j \sum_{j'} c_jc_{j'} \frac{m + j + j'}{2} N_{2m}(m + j' - j), \tag{10}
\]

and

\[
\Delta_{s_m} = \frac{1}{\|s_m\|^2} \left\{ \int_{-\infty}^{\infty} (t - t^*)^2 [s_m(t)]^2 dt \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{\|s_m\|^2} \left\{ \sum_j \sum_{j'} c_jc_{j'} \int_{-\infty}^{\infty} (t^2 - 2tt^* + (t^*)^2)N_m(t - j)N_m(t - j')dt \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{\|s_m\|^2} \left\{ \sum_j \sum_{j'} \frac{1}{2} c_jc_{j'} [(m + j' - t^*)^2 + (j - t^*)^2] N_{2m}(m + j' - j) \right\}^{\frac{1}{2}}, \tag{11}
\]
where \( \| s_m \|^2 = \sum_j \sum_{j'} c_j c_{j'} N_{2m}(m + j' - j) \).

To obtain \( \omega^* \) and \( \Delta \hat{s}_m \), the center and the radius of \( \hat{s}_m \), respectively, we need the following \( \| \hat{s}_m(\omega) \|^2 \) and Lemma 1.

\[
\| \hat{s}_m(\omega) \|^2 = 2\pi \| s_m \|^2 = 2\pi \sum_j \sum_{j'} c_j c_{j'} N_{2m}(m + j' - j).
\]

Thus,

\[
\omega^* = \frac{1}{\| \hat{s}_m(\omega) \|^2} \int_{-\infty}^{\infty} \omega |\hat{s}_m(\omega)|^2 d\omega
\]

\[
= \sum_j \sum_{j'} c_j c_{j'} \int_{-\infty}^{\infty} \omega \tilde{N}_m(\omega) e^{-ij\omega} \bar{N}_m(\omega) e^{ij'\omega} d\omega
\]

\[
= \sum_j \sum_{j'} c_j c_{j'} (2\pi)(-i) N_{2m}'(m + j' - j)
\]

\[
= 0
\]

and

\[
\Delta \hat{s}_m = \frac{1}{\| \hat{s}_m(\omega) \|^2} \left\{ \int_{-\infty}^{\infty} \omega^2 |\hat{s}_m(\omega)|^2 d\omega \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{\| \hat{s}_m(\omega) \|^2} \left\{ \sum_j \sum_{j'} c_j c_{j'} \int_{-\infty}^{\infty} \omega^2 \left( \tilde{N}_m(\omega) \right)^2 e^{i(m+j'-j)\omega} d\omega \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{\| \hat{s}_m(\omega) \|^2} \left\{ \sum_j \sum_{j'} c_j c_{j'} (-2\pi) N'_{2m}(m + j' - j) \right\}^{\frac{1}{2}}
\]

\[
= \sqrt{\frac{2\pi}{\| \hat{s}_m(\omega) \|^2}} \left\{ \sum_j \sum_{j'} c_j c_{j'} [2N_{2m-2}(m + j' - j) - N_{2m-2}(m + j' - j - 2) - N_{2m-2}(m + j' - j)] \right\}^{\frac{1}{2}}.
\]

Hence, we obtain the following theorem.

**Theorem 1.** \( s_m(x) \), \( m \geq 2 \), defines by (1) is a window function for Short Time Fourier Transform shown in (4), which possesses a time-frequency window

\[
[t^* + b - \Delta s_m, t^* + b + \Delta s_m] \times [\omega^* + \omega - \Delta \hat{s}_m, \omega^* + \omega + \Delta \hat{s}_m]
\]

with width \( 2\Delta s_m \) and window area \( 4\Delta s_m \delta \hat{s}_m \). Here \( t^* \), \( \Delta s_m \), \( \omega^* \), and \( \Delta \hat{s}_m \) are defined in (10), (11), (12), and (13), respectively.
3. Wavelet Transforms

From the well-known admissibility condition of basic wavelet functions for wavelet transforms, we can verify the following theorem.

**Theorem 2.** Let $C(z) = C(e^{-i\omega})$ be the symbol of $\{C_j\}$, the coefficient set of the summation in (1). If both $C(z^2)/\omega^{1/2} = C(e^{-i\omega})/\omega^{1/2}$ and $C(z^{-2})/\omega^{1/2} = C(e^{i\omega})/\omega^{1/2}$ are in $L^2(0,2\pi)$, then $s_m(x)$ defined in (1) is a basic wavelet. Relative $s_m(x)$, the integral wavelet transform (IWT) on $L^2(R)$ is defined by

$$ (W_{s_m}f)(b,a) := |a|^{-1/2} \int_{-\infty}^{\infty} f(t) s_m\left(\frac{t-b}{a}\right) dt, $$

$(14)$

$f \in L^2(R)$, where $a,b \in R$ with $a \neq 0$.

**Proof.** It is sufficient to prove that $C_{s_m} = \int_{-\infty}^{\infty} |\hat{s}_m(\omega)|^2 / |\omega| d\omega < \infty$. In fact, noting that $\sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2k\pi)|^2 \leq 1$, we have

$$ C_{s_m} = \int_{-\infty}^{\infty} \frac{|C(z^2)|^2}{|\omega|} |\hat{N}_m(\omega)|^2 d\omega $$

$$ = \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} |\hat{N}_m(\omega)|^2 d\omega $$

$$ = \sum_{k=-\infty}^{\infty} \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega + 2k\pi|} |\hat{N}_m(\omega + 2k\pi)|^2 d\omega $$

$$ \leq \sum_{k=-1}^{\infty} \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} |\hat{N}_m(\omega + 2k\pi)|^2 d\omega + \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega - 2\pi|} |\hat{N}_m(\omega - 2\pi)|^2 d\omega $$

$$ = \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} \sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2k\pi)|^2 d\omega $$

$$ + \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{2\pi - \omega} |\hat{N}_m(\omega - 2\pi)|^2 d\omega - \int_{0}^{2\pi} \frac{|C(e^{i\omega})|^2}{\omega - 2\pi} |\hat{N}_m(\omega - 2\pi)|^2 d\omega $$

$$ \leq \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} d\omega + \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{2\pi - \omega} d\omega + \int_{0}^{2\pi} \frac{|C(e^{i\omega})|^2}{|\omega|} d\omega $$

$$ = 2 \int_{0}^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} d\omega + \int_{0}^{2\pi} \frac{|C(e^{i\omega})|^2}{|\omega|} d\omega $$

In the last step of the above process, we use the integral substitution $\omega' = 2\pi - \omega$ in the second integral. Thus, the proof is complete.

**Corollary.** If $s_m(x)$ defined by (1) is a basic function with the corresponding IWT on $L^2(R)$ shown in (14), then $C(1) = 0$, i.e., $\sum_j c_j = 0$. 

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4. Wavelet Series Expansions

Since \( N_m(x) \) possesses the finite two-scale relation (cf. Chui's [1])

\[
N_m(x) = \sum_{k=0}^{N} p_{m,k} N_m(2x - k),
\]

where

\[
p_{m,k} = \begin{cases} 
2^{-m+1} \binom{m}{k}, & \text{for } 0 \leq k \leq m; \\
0, & \text{otherwise},
\end{cases}
\]

\( s_m(x) \) also possesses two-scale relation

\[
s_m(x) = \sum_k \hat{p}_{m,k} s_m(2x - k).
\]

In order to give the coefficients \( \{\hat{p}_{m,k}\}_k \), we make Fourier transform on both sides of (15) and (16). Thus,

\[
\hat{N}_m(\omega) = P_m(z) \hat{N}_m(\frac{\omega}{2})
\]

and

\[
\hat{s}_m(\omega) = \hat{P}_m(z) \hat{s}_m(\frac{\omega}{2}),
\]

where \( P_m(z) = \frac{1}{2} \sum_k p_{m,k} z^k = \left(\frac{1+z}{2}\right)^2 \) and \( \hat{P}_m(z) = \frac{1}{2} \sum_k \hat{p}_{m,k} z^k \) are two-scale symbols of \( \{p_{m,k}\}_k \) and \( \{\hat{p}_{m,k}\}_k \), respectively. Substituting (17) into expression (2), we have \( \hat{s}_m(\omega) = C(z^2) P_m(z) \hat{N}_m(\frac{\omega}{2}) \). On the other hand, from (2) we also have \( \hat{s}_m(\frac{\omega}{2}) = C(z) \hat{N}_m(\frac{\omega}{2}) \). Thus, we have the following relation between \( \hat{s}_m(\omega) \) and \( \hat{s}_m(\omega/2) \):

\[
\hat{s}_m(\omega) = \frac{C(z^2) P_m(z)}{C(z)} \hat{s}_m(\frac{\omega}{2}).
\]

Comparing (18) and (19), we obtain

\[
\hat{P}_m(z) = \frac{C(z^2) P_m(z)}{C(z)} = \frac{C(z^2)}{C(z)} \left(\frac{1+z}{2}\right)^m.
\]

Obviously, if \( C(z) \) is a finite symbol and \( C(z)|C(z^2) \), the two-scale relation (15) is also of finite terms.

Next, we discuss the corresponding wavelet function \( \psi_{s_m}(x) \) associated with the scaling function \( s_m(x) \). First, we need

\[
\psi_{s_m}(x) = \sum_k \hat{q}_{m,k} \hat{s}_m(2x - k),
\]

or equivalently,

\[
\hat{\psi}_{s_m}(\omega) = \hat{Q}_m(z) \hat{s}_m(\frac{\omega}{2}),
\]
where \( z = e^{-i\omega/2} \), and \( \tilde{Q}_m(z) \) is the two-scale symbol of \( \{ \tilde{q}_{m,k} \} \). In order to find \( \{ \tilde{q}_{m,k} \} \) or \( \tilde{Q}_m(z) \), we consider the B-wavelet (cf. [1]) \( \psi_m(x) \) associated with B-spline \( N_m(x) \),

\[
\psi_m(x) = \sum_k q_{m,k} N_m(2x - k),
\]

where

\[
q_{m,k} = \begin{cases} 
\frac{(-1)^k}{2m} \sum_{\ell=0}^m \binom{m}{\ell} N_{2m}(k + 1 - \ell), & 0 \leq k \leq 3m - 2, \\
0, & \text{otherwise.}
\end{cases}
\]

The Fourier transform of (23) is (cf. [1])

\[
\hat{\psi}_m(\omega) = Q_m(z) \hat{N}_m(\omega/2),
\]

where \( z = e^{-i\omega/2} \), and \( Q_m(z) \) is the two-scale symbol of \( \{ q_{m,k} \} \) with the following expression (cf. equation (6.2.3) in [1]).

\[
Q_m(z) = -z^{-2} \left( \frac{1 - z}{2} \right)^m \frac{K(z^2)}{E_{2m-1}(z^2)} E_{2m-1}(-z),
\]

where \( K \) is in Wiener’s class \( W \) with \( K(z) \neq 0 \) for \( |z| = 1 \) and \( E_{2m-1}(z) \) is \( 2m-1 \) order Euler-Frobenius polynomial

\[
E_{2m-1}(z) = (2m - 1)!z^{-m+1} \sum_{-m+1}^{m-1} N_{2m}(m+k)z^k.
\]

We now derive the expression of \( \tilde{Q}_m(z) \). From equations (6.2.2) in [1], we obtain

\[
\tilde{Q}_m(z) = z^{-1} E_{sm}(-z) \frac{K(z^2)}{E_{sm}(z^2)}, \quad |z| = 1,
\]

where \( K \) is also in Wiener’s class \( W \) with \( K(z) \neq 0 \) for \( |z| = 1 \), \( z = e^{-i\omega/2} \), and

\[
E_{sm}(z) = \sum_k \left\{ \int_{-\infty}^{\infty} s_m(k+y)s_m(y) dy \right\} z^k = \sum_{k=-\infty}^{\infty} |\hat{s}_m(\omega/2 + 2\pi k)|^2.
\]

\( E_{sm}(z) \) can be expressed in terms of \( E_{2m-1}(z) \). In fact,

\[
E_{sm}(z) = \sum_{k=-\infty}^{\infty} |\hat{\phi}_m(\omega/2 + 2\pi k)|^2
= \sum_{k=-\infty}^{\infty} |C(z)|^2 |\hat{N}_m(\omega/2 + 2\pi k)|^2
= |C(z)|^2 \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k
= |C(z)|^2 \frac{E_{2m-1}(z)}{(2m-1)!z^{2m-1}}.
\]
Substituting the above expression of $E_{s_m}(z)$ and equation (20) into equation (27), we obtain

$$
\tilde{Q}_m(z) = (-1)^{m-1}z^{m-2} \frac{C(-z)}{C(z^2)} \left( \frac{1 - \bar{z}}{2} \right)^m \frac{K(z^2)}{E_{2m-1}(z^2)} E_{2m-1}(-z)
$$

$$
= -z^{-2} \frac{C(-z)}{C(z^2)} \left( \frac{1 - \bar{z}}{2} \right)^m \frac{K(z^2)}{E_{2m-1}(z^2)} E_{2m-1}(-z).
$$

Thus, comparing the above expression of $\tilde{Q}_m(z)$ and expression (26) of $Q_m(z)$, we have

$$
\tilde{Q}_m(z) = \frac{1}{(2m-1)!} \frac{C(-z)}{C(z^2)} \left( \frac{1 - \bar{z}}{2} \right)^m E_{2m-1}(-z) = \frac{C(-z)}{C(z^2)} Q_m(z). \quad (28)
$$

Secondly, we will prove $\psi_{s_m}(x)$ defined by equations (21), (22), and (28) is indeed the wavelet function associated with $s_m(x)$. Therefore, we need to consider the following matrix.

$$
M_{\tilde{P}_m Q_m} = \begin{pmatrix}
\tilde{P}_m(z) & \tilde{Q}_m(z) \\
\tilde{P}_m(-z) & \tilde{Q}_m(-z)
\end{pmatrix}.
$$

Obviously, $\det M_{\tilde{P}_m Q_m} = P_m(z)Q_m(-z) - Q_m(z)P_m(-z) = \det M_{\tilde{P}_m Q_m}$, where $\det M_{\tilde{P}_m Q_m}$ is the determinant of the matrix $M_{\tilde{P}_m Q_m}$ associated with the B-wavelet $\psi_m$. Thus, $\det M_{\tilde{P}_m Q_m} \neq 0$ on the unit circle $|z| = 1$ because $\det M_{\tilde{P}_m Q_m} \neq 0$ on $|z| = 1$. It follows that $\psi_{s_m}$ is the wavelet function associated with $s_m$. That is, the family $\{\psi_{s_m}(\cdot - k) : k \in \mathbb{Z}\}$, which is governed by $\tilde{Q}_m(z)$ shown in equation (28), is a Riesz basis of $W_0$.

To derive the decomposition relation of $s_m(2x - \ell)$, we define

$$
\tilde{G}(z) = \frac{\tilde{Q}_m(-z)}{\det M_{\tilde{P}_m Q_m}} = \frac{C(z)}{C(z^2)} \frac{Q_m(-z)}{\det M_{\tilde{P}_m Q_m}} = \frac{C(z)}{C(z^2)} G(z)
$$

and

$$
\tilde{H}(z) = -\frac{\tilde{P}_m(-z)}{\det M_{\tilde{P}_m Q_m}} = \frac{C(z^2)}{C(-z)} \left( -\frac{P_m(-z)}{\det M_{\tilde{P}_m Q_m}} \right) = \frac{C(z^2)}{C(-z)} H(z),
$$

where $G(z) = Q_m(-z)/\det M_{\tilde{P}_m Q_m}$ and $H(z) = -P_m(-z)/\det M_{\tilde{P}_m Q_m}$ are defined as equation (5.3.11) in [1].

It is also easy to prove that

$$
M_{\tilde{G} \tilde{H}}^T \cdot M_{\tilde{P}_m \tilde{Q}_m} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = M_{\tilde{P}_m \tilde{Q}_m} \cdot M_{\tilde{G} \tilde{H}}^T.
$$

Also, $\tilde{Q}_m(z)$ and $\tilde{P}_m(z)$ satisfy the following conditions.

$$
\tilde{P}_m(1) = P_m(1) = 1, \quad \tilde{P}_m(-1) = P_m(-1) = 0, \quad \text{and} \quad \tilde{Q}_m(1) = Q_m(1) = 0.
$$

We write the expansions of $\tilde{G}(z)$ and $\tilde{H}(z)$ as follows:

$$
\tilde{G}(z) = \frac{1}{2} \sum_{n=0}^{\infty} \tilde{g}_n z^n
$$

and
\[ \tilde{H}(z) = \frac{1}{2} \sum_{n} \tilde{h}_n z^n. \]

Thus, the following decomposition relation holds for all \( x \in \mathbb{R} \).

\[ s_m(2x - \ell) = \frac{1}{2} \sum_{-\infty}^{\infty} \left[ \tilde{g}_{2k-\ell} s_m(x - k) + \tilde{h}_{2k-\ell} \psi_{s_m}(x - k) \right], \quad \ell \in \mathbb{Z}. \]

Next, we will discuss the duals of \( \psi_{s_m} \) and the algorithms of decomposition and reconstruction. Define

\[ \tilde{G}^*(z) := \overline{G(z)} = \bar{G} \left( \frac{1}{z} \right), \quad |z| = 1, \]

\[ \tilde{H}^*(z) := \overline{H(z)} = \bar{H} \left( \frac{1}{z} \right), \quad |z| = 1, \]

and

\[ \hat{s}_m(\omega) := \prod_{k=1}^{\infty} \tilde{G}^*(e^{-i\omega/2^k}). \]

Thus, \( \hat{s}_m(x) \) is a dual scaling function of \( s_m(x) \), and

\[ \hat{\psi}_m(\omega) := \tilde{H}^*(z) \hat{s}_m(\omega) \]
gives the dual wavelet function \( \hat{\psi}_{s_m}(\omega)(x) \). Hence, by using a similar argument as that in [1], we have the following theorem.

**Theorem 3.** Let \( \{V_j\} \) be the MRA generated by \( s_m(x) \). If \( f_j(x) \in V_j \) and \( g_j(x) \in W_j \) with

\[ f_j(x) = \sum_k c_k^j s_m(2^j x - k), \quad g_j(x) = \sum_k d_k^j \psi_{s_m}(2^j x - k). \]

Then we have the following decomposition algorithm

\[ c_k^{j-1} = \frac{1}{2} \sum_{\ell} \tilde{g}_{2k-\ell} c_{k}^j, \quad d_k^{j-1} = \frac{1}{2} \sum_{\ell} \tilde{h}_{2k-\ell} c_{k}^j, \]

and reconstruction algorithm

\[ c_k^j = \sum_{\ell} \left[ \tilde{p}_{m,k-2\ell} c_{k}^{j-1} + \tilde{q}_{m,k-2\ell} d_{k}^{j-1} \right]. \]
5. Orthogonal MRA Generated By $s_m(x)$

In this section, we discuss the orthogonal wavelets associated with certain $s_m(x)$. We call a scaling function $\phi$ an orthogonal scaling function if $\phi$ yields an orthogonal MRA; i.e., its corresponding mother wavelet function $\psi$ gives a complete orthogonal system $\{\psi_{j,k} = 2^{j/2}\psi(2^j x - k)\}$ in $L^2(R)$. In the following we will give two approaches to construct orthogonal scaling functions with the form defined in (1) by using the similar argument shown in [10].

**Theorem 4.** Let $C(z)$ be the symbol of $\{c_j\}$, the set of coefficients shown in (1). If $|C(z^2)|^2 = 1/\sum_k |\hat{N}_m(\omega + 2k\pi)|^2$, then the corresponding scaling function $s_m(x) = \sum_{j,k} c_j N_m(x)$ is an orthogonal scaling function and its Fourier transform is

$$\hat{s}_m(\omega) = \frac{\hat{N}_m(\omega)}{\sum_{-m+1}^{m-1} N_{2m}(m+k)e^{-ik\omega}} = \frac{\hat{N}_m(\omega)}{\sum_{-\infty}^\infty (\hat{N}_m(\cdot + m))(\omega + 2k\pi)},$$

where $\hat{N}_{2m}(\cdot + m)$ denotes the Fourier transform of $N_{2m}(x + m)$ and this Fourier transform is evaluated at $\omega + 2k\pi$.

**Proof.** If

$$\sum_k |\hat{s}_m(\omega + 2k\pi)|^2 = |C(z^2)|^2 \sum_k |\hat{N}_m(\omega + 2k\pi)|^2 = 1,$$

then $s_m(x)$ generates an orthogonal MRA. Thus, we obtain that $|C(z^2)|^2 = 1/\sum_k |\hat{N}_m(\omega + 2k\pi)|^2$. By using the Theorem 2.28, and identities (4.2.14) and (4.6.8) in [1], we may prove Theorem 4.

**Theorem 5.** Suppose that $s_m(x)$ defined in (1) satisfies $s_m(x) = 0$ for $|x| \geq a$. If $C(1) = \frac{3a}{\pi}$; i.e., $\sum_j c_j = \frac{3a}{\pi}$, then the corresponding $s_m(x)$ yields an orthogonal scaling function $\hat{\phi}_{s_m}(x)$ with its Fourier transform

$$\hat{\phi}_{s_m}(\omega) = \left(\int_{\omega - \pi}^{\omega + \pi} s_m\left(\frac{3ax}{\pi}\right) dx\right)^{1/2}.$$

**Proof.** Suppose supp $s_m(x) \subset [-a,a]$. Thus, supp $s_m(\frac{3ax}{\pi}) \subset [-\frac{\pi}{3}, \frac{\pi}{3}]$. It is obvious that $\hat{\phi}_{s_m}(x)$ generates an orthogonal MRA if $\sum_{k \in Z} |\hat{\phi}_{s_m}(\omega + 2k\pi)|^2 = 1. \sum_{k \in Z} |\hat{\phi}_{s_m}(\omega + 2k\pi)|^2 = 1$ can be simplified as follows:

$$\sum_{k \in Z} |\hat{\phi}_{s_m}(\omega + 2k\pi)|^2 = \sum_{k \in Z} \int_{\omega + \pi(2k-1)}^{\omega + \pi(2k+1)} s_m\left(\frac{3ax}{\pi}\right) dx = \int_{-\infty}^{\infty} s_m\left(\frac{3ax}{\pi}\right) dx = \int_{-\pi/3}^{\pi/3} s_m\left(\frac{3ax}{\pi}\right) dx.$$
Thus, $\phi_{s_m}(x)$ generates an orthogonal MRA if

$$\int_{-\pi/3}^{\pi/3} s_m \left( \frac{3ax}{\pi} \right) dx = 1.$$ 

On the other hand, from the definition of $s_m$, equation (1), we obtain

$$\int_{-\pi/3}^{\pi/3} s_m \left( \frac{3ax}{\pi} \right) dx = \sum_j c_j \int_{-\pi/3}^{\pi/3} N_m \left( \frac{3ax}{\pi} - j \right) dx
= \sum_j \frac{3a}{\pi} \int_{-a}^{a} N_m(x - j) dx
= \frac{3a}{\pi} \sum_j c_j.$$ 

Thus, if $\sum_j c_j = \frac{3a}{\pi}$, the corresponding $\phi_{s_m}$ generates MRA.

Obviously, $\hat{\phi}_{s_m}^2(\omega)$ is a $C^{m-1}$ continuous function that satisfies

$$\hat{\phi}_{s_m}^2 = \begin{cases} 1 & |\omega| < \frac{2\pi}{3}, \\ g(|\omega|) & \frac{2\pi}{3} \leq |\omega| \leq \frac{4\pi}{3}, \\ 0 & |\omega| > \frac{4\pi}{3}, \end{cases}$$

where $g(\omega)$ and $g(-\omega)$ are symmetric about the origin and are defined on $\frac{2\pi}{3} \leq \omega \leq \frac{4\pi}{3}$ and $-\frac{4\pi}{3} \leq \omega \leq -\frac{2\pi}{3}$, respectively. For instance, if $s_1(x) = c_0N_1(x) + c_{-1}N_1(x+1)$, $c_0 + c_1 = \frac{3}{\pi}$, then $g(|\omega|) = \frac{3}{2\pi} \left( \frac{4\pi}{3} - |\omega| \right)$. If $s_2(x) = c_0N_2(x) + c_{-1}N_2(x+1) + c_{-2}N_2(x+2)$, $c_0 + c_{-1} + c_{-2} = \frac{6}{\pi}$, then the Bernstein-Bézier expression of $g(\omega)$ is

$$g(\omega) = \sum_{i+j=2} a_{ij} \frac{2!}{i!j!} u^i v^j,$$

where $u$ and $v$ are the corresponding barycentric coordinates of $\omega$ when $\omega \in \left[ \frac{2\pi}{3}, \pi \right]$ and $\left[ \pi, \frac{4\pi}{3} \right]$, respectively. The corresponding Bézier coefficients, $[a_{2,0}, a_{1,1}, a_{0,2}]$, of $g(\omega)$, are $[1, 1, \frac{1}{2}]$ and $[\frac{1}{2}, 0, 0]$ when $\omega \in \left[ \frac{2\pi}{3}, \pi \right]$ and $\omega \in \left[ \pi, \frac{4\pi}{3} \right]$, respectively. $g(-\omega)$ can be found by symmetry.

The $\phi_{s_m}$ is a Meyer type scaling function with dilation condition

$$\hat{\phi}_{s_m}(\omega) = m_0 \left( \frac{\omega}{2} \right) \hat{\phi}_{s_m} \left( \frac{\omega}{2} \right),$$

where $m_0 \left( \frac{\omega}{2} \right)$ is defined on $[-2\pi, 2\pi]$ as

$$m_0 \left( \frac{\omega}{2} \right) = \begin{cases} \hat{\phi}_{s_m}(\omega) & |\omega| \leq \frac{4\pi}{3}, \\ 0 & \frac{4\pi}{3} < |\omega| \leq 2\pi \end{cases}$$

and is extended $4\pi$ periodically to all $\omega \in R$. Hence, the corresponding wavelet $\psi_{s_m}$ satisfies

$$\hat{\psi}_{s_m}(\omega) = e^{-i\omega/2} m_0 \left( \frac{\omega}{2} + \pi \right) \hat{\phi} \left( \frac{\omega}{2} \right).$$
6. Recurrence Algorithm of B-wavelets

In this section, we will give a recurrence relation of B-wavelets in terms of their orders and the corresponding algorithm. Hence, a recurrence algorithm for construction of wavelets derived in Section 3 can be given similarly.

**Theorem 6.** Let \( \psi_m(x) \) be the B-wavelet associated with the B-spline of order \( m \), \( N_m(x) \). Then there exists the following recurrence relation formula between \( \psi_m(x) \) and \( \psi_{m+1}(x) \), \( m = 1, 2, \ldots, \)

\[
\psi_{m+1}(x) = \sum_{k=\max\{0, \ell-4m+1\}}^{\ell+1} b_{m+1,k} \int_{x-(k+1)/2}^{x-k/2} \psi_m(t)dt,
\]

or, equivalently,

\[
\psi'_{m+1}(x) = \sum_{k=\max\{0, \ell-4m+1\}}^{\ell+1} b_{m+1,k} \left[ \psi_m \left( x - \frac{k}{2} \right) - \psi_m \left( x - \frac{k+1}{2} \right) \right],
\]

where \( x \in \left[ \frac{\ell}{2}, \frac{\ell+1}{2} \right], \ell = 0, 1, \ldots, 4m+1, \) and \( \{b_{m+1,k}\} \) is the set of coefficients of the expansion of \( 2Q_{m+1}(z) \) in terms of \( z \), which can be determined by the following formulas.

\[
b_{m+1,0} = \tilde{N}_{m+1}(0)/\tilde{N}_m(0), \tag{31}
\]

\[
b_{m+1,j} = \left( \tilde{N}_{m+1}(j) - \sum_{\ell=0}^{j-1} (-1)^\ell b_{m+1,\ell} \tilde{N}_m(3m-2-\ell) \right) /\tilde{N}_m(0), \tag{32}
\]

for \( j = 0, 1, \ldots, 3m-2, \) and

\[
b_{m+1,j} = \left( \tilde{N}_{m+1}(j) - \sum_{\ell=j-3m+2}^{j-1} (-1)^\ell b_{m+1,\ell} \tilde{N}_m(j-\ell) \right) /\tilde{N}_{m+1}(0), \tag{33}
\]

for \( j = 3m-1, 3m, \ldots, 4m+2, \) where

\[
\tilde{N}_m(k) = \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k+1-\ell). \tag{34}
\]

**Proof.** From equation (25), we have

\[
\hat{\psi}_{m+1}(\omega) = Q_{m+1}(z)\hat{N}_{m+1}\left(\frac{\omega}{2}\right).
\]

Dividing the above equation by equation (25) side by side, we obtain

\[
\frac{\hat{\psi}_{m+1}(\omega)}{\hat{\psi}_m(\omega)} = \frac{Q_{m+1}(z)}{Q_m(z)} \frac{\tilde{N}_{m+1}\left(\frac{\omega}{2}\right)}{\tilde{N}_m\left(\frac{\omega}{2}\right)}. \tag{35}
\]

Noting that \( \tilde{N}_{m+1}\left(\frac{\omega}{2}\right)/\tilde{N}_m\left(\frac{\omega}{2}\right)\tilde{N}_1\left(\frac{\omega}{2}\right) \), we have

\[
\hat{\psi}_{m+1}(\omega) = \frac{Q_{m+1}(z)}{Q_m(z)} \hat{\psi}_m(\omega)\hat{N}_1\left(\frac{\omega}{2}\right). \tag{36}
\]
In order to express \( \psi_{m+1}(x) \) in terms of \( \psi_m(x) \), we need to find the inverse Fourier transformation of \( \hat{\psi}_m(\omega)\hat{N}_1(\omega/2) \) and the expression of \( Q_{m+1}(z)/Q_m(z) \). Define \( \hat{\psi}(\omega) = \hat{\psi}_m(\omega)\hat{N}_1(\omega/2) \). It follows that \( \bar{\psi}(x) = 2\psi_m(x) * N_1(2x) \) or, equivalently,

\[
\bar{\psi}(x) = 2\int_{-\infty}^{\infty} N_1(2t)\psi_m(x-t)dt
\]

\[
= 2\int_0^{1/2} \psi_m(x-t)dt.
\]

Hence, if we write \( 2Q_{m+1}(z)/Q_m(z) = \sum_{k=0}^{\infty} b_{m+1,k}z^k \) formally, from equation (36) we obtain

\[
\psi_{m+1}(x) = \frac{1}{2} \sum_{k=0}^{\infty} b_{m+1,k} \bar{\psi}(x - \frac{k}{2})
\]

\[
= \sum_{k=0}^{\infty} b_{m+1,k} \int_0^{1/2} \psi_m(x - \frac{k}{2} - t)dt
\]

\[
= \sum_{k=0}^{\infty} b_{m+1,k} \int_{x - \frac{k+1}{2}}^{x - \frac{k}{2}} \psi_m(t)dt.
\]

(37)

We will now determine the range of the summation in expression (37). Since \( \text{supp} \psi_m = [0, 2m - 1] \), we need that \( x - \frac{k}{2} \geq 0 \) and \( x - \frac{k+1}{2} \leq 2m - 1 \); i.e., \( k \leq 2x \) and \( k \geq 2x - 4m + 1 \). Hence, if \( x \in \left[ \frac{\ell}{2}, \frac{\ell+1}{2} \right] \), then \( k \leq \ell + 1 \) and \( k \geq \ell - 4m + 1 \). Where \( \ell = 0, 1, \ldots, 4m + 2 \) because \( \left[ \frac{\ell}{2}, \frac{\ell+1}{2} \right] \subset \text{supp} \psi_m = [0, 2m - 1] \). Therefore, we obtain equation (29). Equation (29) can be written in a more general form as follows:

\[
\psi_{m+1}(x) = \sum_{k=0}^{4m+2} b_{m+1,k} \int_{x - \frac{k+1}{2}}^{x - \frac{k}{2}} \psi_m(t)dt
\]

or, equivalently,

\[
\psi_{m+1}(x) = \sum_{k=0}^{4m+2} b_{m+1,k} [\psi_m(x - \frac{k}{2}) - \psi_m(x - \frac{k+1}{2})],
\]

(38)

where \( x \in [0, 2m + 1] \).

In order to complete the proof of the theorem, we only need to prove that the following expansion of \( 2Q_{m+1}(z)/Q_m(z) \) exists and to give the expression of \( \{b_{m+1,k}\} \).

\[
2\frac{Q_{m+1}(z)}{Q_m(z)} = \sum_{k=0}^{\infty} b_{m+1,k}z^k.
\]

(39)

In fact, from \([1]\), \( Q_m(z) = -zE_m(-z)P_m(-z) \), where \( E_m(-z) \) is the Euler-Frobenius Laurent polynomial with respect to \( N_m \) and \( P_m(z) = \left( \frac{1+z}{2} \right)^m \). Hence, \( Q_{m+1}(z)/Q_m(z) = \)

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\( \left( \frac{1}{2} - 2 \right) E_{m+1}(-z)/E_m(-z) \) is zero-free and pole-free on \(|z| = 1\). It follows that expansion (34) exists on \(|z| = 1\). To find \( \{b_{m+1,k}\} \), we write

\[
Q_m(z) = \sum_{k=0}^{3m-2} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k + 1 - \ell)z^k
\]

and

\[
Q_{m+1}(z) = \sum_{k=0}^{3m+1} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2m+2}(k + 1 - \ell)z^k.
\]

It follows from equation (39) that

\[
\sum_{j=0}^{\infty} b_{m+1,j} z^j \sum_{k=0}^{3m-2} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k + 1 - \ell)z^k
\]

\[
= \sum_{k=0}^{3m+1} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2(m+1)}(k + 1 - \ell)z^k.
\] (40)

On the left hand side of equation (40), we exchange the last two summations, then take transform \( k + j = k' \). Noting that \( \text{supp} \psi_m = [0, 2m - 1] \), we finally obtain

\[
\sum_{k'=0}^{3m+1} \sum_{j=0}^{4m+2} b_{m+1,j} \frac{(-1)^{k'-j}}{2^{m-1}} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k' - j + 1 - \ell)z^{k'}
\]

\[
= \sum_{k=0}^{3m+4} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2(m+1)}(k + 1 - \ell)z^k.
\] (41)

Hence, for \( k = 0, 1, \ldots, 3m + 1 \), we have

\[
\sum_{j=0}^{4m+2} (-1)^j b_{m+1,j} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k - j + 1 - \ell)
\]

\[
= \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2(m+1)}(k + 1 - \ell).
\] (42)

System (43) can be written as the following matrix form:

\[
A_m b_{m+1} = n_{m+1},
\] (43)

where \( b_{m+1} = (b_{m+1,0}, -b_{m+1,1}, b_{m+1,2}, \ldots, -b_{m+1,4m+1}, b_{m+1,4m+2}) \), \( A_m = [a_{k,j}]_{k,j=0}^{4m+2} = [\tilde{N}_m(k - j)]_{k,j=0}^{4m+2} \), \( n_{m+1} = (\tilde{N}_m(0), \tilde{N}_m(1), \ldots, \tilde{N}_m(4m + 2)) \), and \( \tilde{N}_m(k) \) is defined as equation (34). It is easy to have \( \tilde{N}_m(0) = N_{2m}(1) = 1/(2m - 1)! \), \( \tilde{N}_m(1) = \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(2 - \ell) = N_{2m}(2) + m N_{2m}(1) \), ..., and \( \tilde{N}_m(3m - 2) = \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(3m - 2) \).
\[1 - \ell = N_{2m}(2m - 1) = 1/(2m - 1)!\]. Note that \(\bar{N}_m(i) = 0\) if \(i < 0\) or \(i > 3m - 2\). Matrix \(A_m\) in (43) is actually

\[
A_m = \begin{bmatrix}
\bar{N}_m(0) & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\bar{N}_m(1) & \bar{N}_m(0) & \cdots & 0 & 0 & \cdots & 0 \\
0 & \bar{N}_m(3m - 2) & \bar{N}_m(3m - 3) & \bar{N}_m(0) & 0 & \cdots & 0 \\
0 & 0 & \bar{N}_m(3m - 2) & \bar{N}_m(1) & \bar{N}_m(0) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \bar{N}_m(3m - 2) & \bar{N}_m(3m - 3) & \cdots & \bar{N}_m(0)
\end{bmatrix}
\]

Thus, \(b_{m+1}\) can be solved and can be expressed as formulas (31)-(33).

If we express \(\psi_m(x)\) using their Bézier coefficients, from equation (38) we obtain the following recurrence algorithm for constructing \(\psi_m(x)\) by using the Bézier coefficients in their Bernstein-Bézier expressions.

\[
a_{k+1}^{m+1}(\ell) = a_k^{m+1}(\ell) + \frac{1}{2m} \sum_{j=0}^{4m+2} b_{m+1,j} [a_k^m(\ell + j) - a_k^m(\ell + j - 1)],
\]

(44)

where \(a_p^r(q)\) is the \(p^{th}\) Bézier coefficient of \(\psi_r(x)\) over the interval \([\ell, \ell + 1]\). Here the order of arrangement for the Bézier coefficients is in terms of the increase in powers of the second coordinate of the barycentric coordinates in the Bernstein-Bézier polynomial expression of the wavelets.

References