Short Time Fourier Transform, Integral Wavelet Transform, and Wavelet Functions Associated With Splines

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Abstract

In this paper, we will discuss Short Time Fourier transforms, integral wavelets transforms, and wavelet series expansions associated with spline functions in shift-invariant spaces of B-splines. A recurrence relation formula and the corresponding algorithm about the B-wavelets will also be given.

1. Introduction

Let $N_m(x)$ be the cardinal B-spline function of order m, and $S(N_m(x))$ be the shift-invariant space of $N_m(x)$ that is defined as the span of the integer translations of $N_m(x)$. Thus, $s_m \in S(N_m(x))$ can be written as

$$s_m(x) = \sum_j c_j N_m(x-j), \quad x \in R, j \in Z. \tag{1}$$

Throughout, we will always assume $\{c_j\} \in \ell_2$.

We will discuss Short Time Fourier transforms (STFT), integral wavelet transforms (IWT), and wavelet series expansions associated with $s_m(x)$ because of the following reasons. First, the basic function of a cardinal B-spline interpolation from $S(N_m(x))$ can be expressed as $s_m(x)$; the wavelet function associated with $s_m(x)$ can then be used for numerical analysis. Second, by choosing a suitable $\{c_j\}$, we can obtain an $s_m(x)$ as a basic wavelet function although an $N_m(x)$ could not be attained that has this property. Third, we may use $s_m(x)$ as a scaling function to construct smooth and symmetric mother wavelet functions. Finally, orthogonalization of wavelet functions associated with $s_m(x)$ can be implemented by adjusting $\{c_j\}$.

In Section 2, we will give the size of time-frequency window of the STFT associated with the window function $s_m(x)$. The size of the window can be justified by its coefficient set $\{c_j\}$. In Section 3, we will give conditions under which an $s_m(x)$ can be a basic wavelet function for IWT. In Section 4, we will discuss wavelet functions associated with $s_m(x)$ that were derived in [5]. This type of wavelets can be considered as an extension of B-wavelets (cf. [1]). In Section 5, we will give two different approaches for orthogonalization of the wavelet functions associated with $s_m(x)$. The last section will give a recurrence algorithm of $s_m(x)$ based on the recurrence relation of B-wavelets.

2. Short Time Fourier Transforms

Let $\hat{s}_m(\omega)$ and $\hat{N}_m(\omega)$ be the Fourier transform of $s_m(x)$ and $N_m(x)$ respectively. Thus,

$$\hat{s}_m(\omega) = C(z^2)\hat{N}_m(\omega), \tag{2}$$

where

$$\hat{N}_m(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m, \tag{3}$$

$z = e^{-i\omega/2}$, and $C(z^2) = \sum_j c_j z^{2j} = \sum_j c_j e^{-i\omega}$ is the symbol of $\{c_j\}$. It is obvious, if $m \geq 2$, both $xs_m(x)$ and $\omega\hat{s}_m(\omega) \in L^2(R)$. Thus, for $m \geq 2$, $s_m(x)$ defined by (1) can be considered as a window function for STFT

$$(G_b f)(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\overline{s_m(t-b)}dt. \tag{4}$$

We will find the center t^* and the radius Δ_{s_m} of s_m and the center ω^* and the radius $\Delta_{\hat{s}_m}$ of $\hat{s}_m$, the fourier transform of s_m. Thus, the corresponding time-frequency window can be determined. Since the set of the arbitrary coefficient $\{c_j\}$ are involved, the size of window can be adjusted for certain purposes.

In order to find the centers and radiuses, we need the following lemmas.

Lemma 1. Let $\hat{N}_m(\omega)$ be the Fourier transform of the B-spline of order m, $N_m(x)$, shown in equation (3). Then

$$\overline{\hat{N}_m(\omega)} = \hat{N}_m(\omega)e^{im\omega} \tag{5}$$

and

$$\hat{N}'_m(\omega) = m\hat{N}_{m-1}(\omega)\hat{N}'_1(\omega). \tag{6}$$

Proof. From equation (3), we obtain

$$\hat{N}_m(\omega) = e^{-im\omega/2}\left(\frac{\sin\omega/2}{\omega/2}\right)^m.$$

Hence, equation (5) is obvious. Equation (6) can be derived immediately from the Fourier transform of the B-spline of order m: $\hat{N}_m(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m$.

Lemma 2. Let $N_m(x)$ be the B-spline of order m. Then

$$\int_{-\infty}^{\infty} N_m(x)N_m(x+\ell)dx = N_{2m}(m-\ell), \tag{7}$$

$$\int_{-\infty}^{\infty} xN_m(x)N_m(x+\ell)dx = \frac{m-\ell}{2}N_{2m}(m-\ell), \tag{8}$$

and

$$\int_{-\infty}^{\infty} x^2 N_m(x) N_m(x+\ell)dx = \frac{(m-\ell)^2}{2} N_{2m}(m-\ell). \tag{9}$$

Proof. By using the Parseval Identity, the recurrence relation $N_{n+1} = N_m * N_1$, and Lemma 1, we can obtain expressions (7) and (8).

To prove the equation (9), we need an equation about the integral of $x^2 N_m(x) N_m(x+\ell)$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^2 N_m(x) N_m(x+\ell)dx \\
&= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\left(\hat{N}_m(\omega)\right)'' \hat{N}_m(\omega) e^{i(m-\ell)\omega} d\omega \\
&= \frac{-1}{2\pi}\int_{-\infty}^{\infty} [m(m-1)\hat{N}_{2m-2}(\omega)\left(\hat{N}_1'(\omega)\right)^2 + m\hat{N}_{2m-1}(\omega)\hat{N}_1''(\omega)] e^{i(m-\ell)\omega} d\omega \\
&= \frac{-1}{4\pi}\int_{-\infty}^{\infty} \hat{N}_{2m}''(\omega) e^{i(m-\ell)\omega} d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} [m\hat{N}_{m-1}(\omega)\hat{N}_1'(\omega)]^2 e^{i(m-\ell)\omega} d\omega \\
&= \frac{1}{2}(m-\ell)^2 N_{2m}(m-\ell) + \int_{-\infty}^{\infty}(m-\ell-x)x N_m(x) N_m(m-\ell-x)dx \\
&= \frac{1}{2}(m-\ell)^2 N_{2m}(m-\ell) + \int_{-\infty}^{\infty}(m-\ell)x N_m(x) N_m(x+\ell)dx \\
&-\int_{-\infty}^{\infty} x^2 N_m(x) N_m(x+\ell)
\end{aligned}$$

Hence, by solving the last equation for $\int_{-\infty}^{\infty} x^2 N_m(x) N_m(x+\ell)dx$ and using equation (8), we obtain equation (9).

By using Lemma 2, we can find the center t^* and the radius Δ_{s_m} of s_m as follows.

$$\begin{aligned}
t^* &= \frac{1}{\|s_m\|^2}\int_{-\infty}^{\infty} x[s_m(t)]^2 dt \\
&= \frac{1}{\|s_m\|^2}\sum_j\sum_{j'} c_j c_{j'}\int_{-\infty}^{\infty} N_m(t-j) N_m(t-j')dt \\
&= \frac{1}{\|s_m\|^2}\sum_j\sum_{j'} c_j c_{j'}\frac{m+j+j'}{2} N_{2m}(m+j'-j),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\Delta_{s_m} &= \frac{1}{\|s_m\|^2}\{\int_{-\infty}^{\infty}(t-t^*)^2[s_m(t)]^2 dt\}^{\frac{1}{2}} \\
&= \frac{1}{\|s_m\|^2}\{\sum_j\sum_{j'} c_j c_{j'}\int_{-\infty}^{\infty}(t^2 - 2tt^* + (t^*)^2) N_m(t-j) N_m(t-j')dt\}^{\frac{1}{2}} \\
&= \frac{1}{\|s_m\|^2}\{\sum_j\sum_{j'}\frac{1}{2} c_j c_{j'}[(m+j'-t^*)^2 + (j-t^*)^2] N_{2m}(m+j'-j)\}^{\frac{1}{2}},
\end{aligned} \tag{11}$$

where $\|s_m\|^2 = \sum_j \sum_{j'} c_j c_{j'} N_{2m}(m + j' - j)$.

To obtain ω^* and $\Delta_{\hat{s}_m}$, the center and the radius of $\hat{s}_m$, respectively, we need the following $\|\hat{s}_m(\omega)\|^2$ and Lemma 1.

$$\|\hat{s}_m(\omega)\|^2 = 2\pi\|s_m\|^2 = 2\pi \sum_j \sum_{j'} c_j c_{j'} N_{2m}(m + j' - j).$$

Thus,

$$\begin{aligned}
\omega^* &= \frac{1}{\|\hat{s}_m(\omega)\|^2} \int_{-\infty}^{\infty} \omega |\hat{s}_m(\omega)|^2 d\omega \\
&= \sum_j \sum_{j'} c_j c_{j'} \int_{-\infty}^{\infty} \omega \hat{N}_m(\omega) e^{-ij\omega} \overline{\hat{N}_m(\omega)} e^{ij'\omega} d\omega \\
&= \sum_j \sum_{j'} c_j c_{j'} (2\pi)(-i) N'_{2m}(m + j' - j) \\
&= 0 \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\hat{s}_m} &= \frac{1}{\|\hat{s}_m(\omega)\|^2} \{\int_{-\infty}^{\infty} \omega^2 |\hat{s}_m(\omega)|^2 d\omega\}^{\frac{1}{2}} \\
&= \frac{1}{\|\hat{s}_m(\omega)\|^2} \{\sum_j \sum_{j'} c_j c_{j'} \int_{-\infty}^{\infty} \omega^2 \left(\hat{N}_m(\omega)\right)^2 e^{i(m+j'-j)\omega} d\omega\}^{\frac{1}{2}} \\
&= \frac{1}{\|\hat{s}_m(\omega)\|^2} \{\sum_j \sum_{j'} c_j c_{j'} (-2\pi) N''_{2m}(m + j' - j)\}^{\frac{1}{2}} \\
&= \frac{\sqrt{2\pi}}{\|\hat{s}_m(\omega)\|^2} \{\sum_j \sum_{j'} c_j c_{j'} [2N_{2m-2}(m + j' - j - 1) - N_{2m-2}(m + j' - j - 2) \\
&\quad - N_{2m-2}(m + j' - j)]\}^{\frac{1}{2}}. \qquad (13)
\end{aligned}$$

Hence, we obtain the following theorem.

Theorem 1. $s_m(x)$, $m \geq 2$, defines by (1) is a window function for Short Time Fourier Transform shown in (4), which possesses a time-frequency window

$$[t^* + b - \Delta_{s_m}, t^* + b + \Delta_{s_m}] \times [\omega^* + \omega - \Delta_{\hat{s}_m}, \omega^* + \omega + \Delta_{\hat{s}_m}]$$

with width $2\Delta_{s_m}$ and window area $4\Delta_{s_m}\delta_{\hat{s}_m}$. Here t^*, $\Delta_{s_m}*$, ω^*, and $\Delta_{\hat{s}_m}$ are defined in (10), (11), (12), and (13), respectively.

3. Wavelet Transforms

From the well-known admissibility condition of basic wavelet functions for wavelet transforms, we can verify the following theorem.

Theorem 2. Let $C(z) = C(e^{-i\omega})$ be the symbol of $\{C_j\}$, the coefficient set of the summation in (1). If both $C(z^2)/\omega^{1/2} = C(e^{-i\omega})/\omega^{1/2}$ and $C(z^{-2})/\omega^{1/2} = C(e^{i\omega})/\omega^{1/2}$ are in $L^2(0, 2\pi)$, then $s_m(x)$ defined in (1) is a basic wavelet. Relative $s_m(x)$, the integral wavelet transform (IWT) on $L^2(R)$ is defined by

$$(W_{s_m} f)(b, a) := |a|^{-1/2} \int_{-\infty}^{\infty} f(t) \overline{s_m\left(\frac{t-b}{a}\right)} dt, \tag{14}$$

$f \in L^2(R)$, where $a, b \in R$ with $a \neq 0$.

Proof. It is sufficient to prove that $C_{s_m} = \int_{-\infty}^{\infty} |\hat{s}_m(\omega)|^2/|\omega| d\omega < \infty$. In fact, noting that $\sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2k\pi)|^2 \leq 1$, we have

$$\begin{aligned}
C_{s_m} &= \int_{-\infty}^{\infty} \frac{|C(z^2)|^2}{|\omega|} |\hat{N}_m(\omega)|^2 d\omega \\
&= \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} |\hat{N}_m(\omega)|^2 d\omega \\
&= \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega + 2k\pi|} |\hat{N}_m(\omega + 2k\pi)|^2 d\omega \\
&\leq \sum_{k \neq -1} \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} |\hat{N}_m(\omega + 2k\pi)|^2 d\omega + \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega - 2\pi|} |\hat{N}_m(\omega - 2\pi)|^2 d\omega \\
&= \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} \sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2k\pi)|^2 d\omega \\
&\quad + \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{2\pi - \omega} |\hat{N}_m(\omega - 2\pi)|^2 d\omega - \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{\omega} |\hat{N}_m(\omega - 2\pi)|^2 d\omega \\
&\leq \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} d\omega + \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{2\pi - \omega} d\omega + \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} d\omega \\
&= 2 \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} d\omega + \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{2\pi - \omega} d\omega \\
&= 2 \int_0^{2\pi} \frac{|C(e^{-i\omega})|^2}{|\omega|} d\omega + \int_0^{2\pi} \frac{|C(e^{i\omega})|^2}{|\omega|} d\omega.
\end{aligned}$$

In the last step of the above process, we use the integral substitution $\omega' = 2\pi - \omega$ in the second integral. Thus, the proof is complete.

Corollary. If $s_m(x)$ defined by (1) is a basic function with the corresponding IWT on $L^2(R)$ shown in (14), then $C(1) = 0$, i.e., $\sum_j c_j = 0$.

4. Wavelet Series Expansions

Since $N_m(x)$ possesses the finite two-scale relation (cf. Chui's [1])

$$N_m(x) = \sum_{k=0}^{N} p_{m,k} N_m(2x-k), \tag{15}$$

where

$$p_{m,k} = \begin{cases} 2^{-m+1}\binom{m}{k}, & \text{for } 0 \le k \le m; \\ 0, & \text{otherwise,} \end{cases}$$

$s_m(x)$ also possesses two-scale relation

$$s_m(x) = \sum_k \tilde{p}_{m,k} s_m(2x-k). \tag{16}$$

In order to give the coefficients $\{\tilde{p}_{m,k}\}_k$, we make Fourier transform on both sides of (15) and (16). Thus,

$$\hat{N}_m(\omega) = P_m(z)\hat{N}_m(\frac{\omega}{2}) \tag{17}$$

and

$$\hat{s}_m(\omega) = \tilde{P}_m(z)\hat{s}_m(\frac{\omega}{2}), \tag{18}$$

where $P_m(z) = \frac{1}{2}\sum_k p_{m,k} z^k = \left(\frac{1+z}{2}\right)^2$ and $\tilde{P}_m(z) = \frac{1}{2}\sum_k \tilde{p}_{m,k} z^k$ are two-scale symbols of $\{p_{m,k}\}_k$ and $\{\tilde{p}_{m,k}\}_k$, respectively. Substituting (17) into expression (2), we have $\hat{s}_m(\omega) = C(z^2)P_m(z)\hat{N}_m(\frac{\omega}{2})$. On the other hand, from (2) we also have $\hat{s}_m(\frac{\omega}{2}) = C(z)\hat{N}_m(\frac{\omega}{2})$. Thus, we have the following the relation between $\hat{s}_m(\omega)$ and $\hat{s}_m(\omega/2)$:

$$\hat{s}_m(\omega) = \frac{C(z^2)P_m(z)}{C(z)}\hat{s}_m(\frac{\omega}{2}). \tag{19}$$

Comparing (18) and (19), we obtain

$$\tilde{P}_m(z) = \frac{C(z^2)P_m(z)}{C(z)} = \frac{C(z^2)}{C(z)}\left(\frac{1+z}{2}\right)^m. \tag{20}$$

Obviously, if C(z) is a finite symbol and $C(z)|C(z^2)$, the two-scale relation (15) is also of finite terms.

Next, we discuss the corresponding wavelet function $\psi_{s_m}(x)$ associated with the scaling function $s_m(x)$. First, we need

$$\psi_{s_m}(x) = \sum_k \tilde{q}_{m,k} s_m(2x-k), \tag{21}$$

or equivalently,

$$\hat{\psi}_{s_m}(\omega) = \tilde{Q}_m(z)\hat{s}_m(\frac{\omega}{2}), \tag{22}$$

where $z = e^{-i\omega/2}$, and $\tilde{Q}_m(z)$ is the two-scale symbol of $\{\tilde{q}_{m,k}\}_k$. In order to find $\{\tilde{q}_{m,k}\}_k$ or $\tilde{Q}_m(z)$, we consider the B-wavelet (cf. [1]) $\psi_m(x)$ associated with B-spline $N_m(x)$,

$$\psi_m(x) = \sum_k q_{m,k} N_m(2x-k), \tag{23}$$

where

$$q_{m,k} = \begin{cases} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^m \binom{m}{\ell} N_{2m}(k+1-\ell), & 0 \le k \le 3m-2, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

The Fourier transform of (23) is (cf. [1])

$$\hat{\psi}_m(\omega) = Q_m(z)\hat{N}_m(\frac{\omega}{2}), \tag{25}$$

where $z = e^{-i\omega/2}$, and $Q_m(z)$ is the two-scale symbol of $\{q_{m,k}\}_k$ with the following expression (cf. equation (6.2.3) in [1]).

$$Q_m(z) = -z^{-2}\left(\frac{1-z}{2}\right)^m \frac{K(z^2)}{E_{2m-1}(z^2)} E_{2m-1}(-z), \tag{26}$$

where K is in Wiener's class W with $K(z) \neq 0$ for $|z| = 1$ and $E_{2m-1}(z)$ is 2m-1 order Euler-Frobenius polynomial

$$E_{2m-1}(z) = (2m-1)! z^{-m+1} \sum_{-m+1}^{m-1} N_{2m}(m+k)z^k.$$

We now derive the expression of $\tilde{Q}_m(z)$. From equations (6.2.2) in [1], we obtain

$$\tilde{Q}_m(z) = z^{-1} E_{s_m}(-z)\overline{\tilde{P}_m(-z)} \frac{K(z^2)}{E_{s_m}(z^2)}, \quad |z| = 1, \tag{27}$$

where K is also in Wiener's class W with $K(z) \neq 0$ for $|z| = 1$, $z = e^{-i\omega/2}$, and

$$\begin{aligned} E_{s_m}(z) &= \sum_k \{\int_{-\infty}^{\infty} s_m(k+y)\overline{s_m(y)}dy\} z^k \\ &= \sum_{k=-\infty}^{\infty} |\hat{s}_m(\frac{\omega}{2} + 2\pi k)|^2. \end{aligned}$$

$E_{s_m}(z)$ can be expressed in terms of $E_{2m-1}(z)$. In fact,

$$\begin{aligned} E_{s_m}(z) &= \sum_{k=-\infty}^{\infty} |\hat{\phi}_m(\frac{\omega}{2} + 2\pi k)|^2 \\ &= \sum_{k=-\infty}^{\infty} |C(z)|^2 |\hat{N}_m(\frac{\omega}{2} + 2\pi k)|^2 \\ &= |C(z)|^2 \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k \\ &= |C(z)|^2 \frac{E_{2m-1}(z)}{(2m-1)! z^{m-1}}. \end{aligned}$$

Substituting the above expression of $E_{s_m}(z)$ and equation (20) into equation (27), we obtain

$$\begin{aligned}\tilde{Q}_m(z) &= (-1)^{m-1} z^{m-2} \frac{C(-z)}{C(z^2)} \left(\frac{1-\bar{z}}{2}\right)^m \frac{K(z^2)}{E_{2m-1}(z^2)} E_{2m-1}(-z) \\ &= -z^{-2} \frac{C(-z)}{C(z^2)} \left(\frac{1-z}{2}\right)^m \frac{K(z^2)}{E_{2m-1}(z^2)} E_{2m-1}(-z).\end{aligned}$$

Thus, comparing the above expression of $\tilde{Q}_m(z)$ and expression (26) of $Q_m(z)$, we have

$$\tilde{Q}_m(z) = \frac{1}{(2m-1)!} \frac{C(-z)}{C(z^2)} \left(\frac{1-z}{2}\right)^m E_{2m-1}(-z) = \frac{C(-z)}{C(z^2)} Q_m(z). \tag{28}$$

Secondly, we will prove $\psi_{s_m}(x)$ defined by equations (21), (22), and (28) is indeed the wavelet function associated with $s_m(x)$. Therefore, we need to consider the following matrix.

$$M_{\tilde{P}_m \tilde{Q}_m} = \begin{pmatrix} \tilde{P}_m(z) & \tilde{Q}_m(z) \\ \tilde{P}_m(-z) & \tilde{Q}_m(-z) \end{pmatrix}.$$

Obviously, $det\, M_{\tilde{P}_m \tilde{Q}_m} = P_m(z)Q_m(-z) - Q_m(z)P_m(-z) = det\, M_{P_m Q_m}$, where $det\, M_{P_m Q_m}$ is the determinant of the matrix $M_{P_m Q_m}$ associated with the B-wavelet ψ_m. Thus, $det\, M_{\tilde{P}_m \tilde{Q}_m} \neq 0$ on the unit circle $|z| = 1$ because $det\, M_{P_m Q_m} \neq 0$ on $|z| = 1$. It follows that ψ_{s_m} is the wavelet function associated with s_m. That is, the family $\{\psi_{s_m}(\cdot - k) : k \in Z\}$, which is governed by $\tilde{Q}_m(z)$ shown in equation (28), is a Riesz basis of W_0.

To derive the decomposition relation of $s_m(2x - \ell)$, we define

$$\tilde{G}(z) = \frac{\tilde{Q}_m(-z)}{det\, M_{\tilde{P}_m \tilde{Q}_m}} = \frac{C(z)}{C(z^2)} \frac{Q_m(-z)}{det\, M_{P_m Q_m}} = \frac{C(z)}{C(z^2)} G(z)$$

and

$$\tilde{H}(z) = -\frac{\tilde{P}_m(-z)}{det\, M_{\tilde{P}_m \tilde{Q}_m}} = \frac{C(z^2)}{C(-z)} \left(-\frac{P_m(-z)}{det\, M_{P_m Q_m}}\right) = \frac{C(z^2)}{C(-z)} H(z),$$

where $G(z) = Q_m(-z)/det\, M_{P_m Q_m}$ and $H(z) = -P_m(-z)/det\, M_{P_m Q_m}$ are defined as equation (5.3.11) in [1].

It is also easy to prove that

$$M^T_{\tilde{G}\tilde{H}} \cdot M_{\tilde{P}_m \tilde{Q}_m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M_{\tilde{P}_m \tilde{Q}_m} \cdot M^T_{\tilde{G}\tilde{H}}.$$

Also, $\tilde{Q}_m(z)$ and $\tilde{P}_m(z)$ satisfy the following conditions.

$$\tilde{P}_m(1) = P_m(1) = 1, \quad \tilde{P}_m(-1) = P_m(-1) = 0, \quad and \quad \tilde{Q}_m(1) = Q_m(1) = 0.$$

We write the expansions of $\tilde{G}(z)$ and $\tilde{H}(z)$ as follows:

$$\tilde{G}(z) = \frac{1}{2} \sum_n \tilde{g}_n z^n$$

and

$$\tilde{H}(z) = \frac{1}{2}\sum_{n} \tilde{h}_n z^n.$$

Thus, the following decomposition relation holds for all $x \in R$.

$$s_m(2x-\ell) = \frac{1}{2}\sum_{-\infty}^{\infty}\left[\tilde{g}_{2k-\ell}s_m(x-k) + \tilde{h}_{2k-\ell}\psi_{s_m}(x-k)\right], \quad \ell \in Z.$$

Next, we will discuss the duals of ψ_{s_m} and the algorithms of decomposition and reconstruction. Define

$$\tilde{G}^*(z) := \overline{\tilde{G}(z)} = \tilde{G}\left(\frac{1}{z}\right), \quad |z| = 1,$$

$$\tilde{H}^*(z) := \overline{\tilde{H}(z)} = \tilde{H}\left(\frac{1}{z}\right), \quad |z| = 1,$$

and

$$\hat{\tilde{s}}_m(\omega) := \Pi_{k=1}^{\infty}\tilde{G}^*(e^{-i\omega/2^k}).$$

Thus, $\tilde{s}_m(x)$ is a dual scaling function of $s_m(x)$, and

$$\hat{\tilde{\psi}}_m(\omega) := \tilde{H}^*(z)\hat{\tilde{s}}_m(\omega)$$

gives the dual wavelet function $\tilde{\psi}_{s_m}(\omega)(x)$. Hence, by using a similar argument as that in [1], we have the following theorem.

Theorem 3. Let $\{V_j\}$ be the MRA generated by $s_m(x)$. If $f_j(x) \in V_j$ and $g_j(x) \in W_j$ with

$$f_j(x) = \sum_k c_k^j s_m(2^j x - k), \quad g_j(x) = \sum_k d_k^j \psi_{s_m}(2^j x - k).$$

Then we have the following decomposition algorithm

$$c_k^{j-1} = \frac{1}{2}\sum_{\ell} \tilde{g}_{2k-\ell}c_\ell^j, \quad d_k^{j-1} = \frac{1}{2}\sum_{\ell}\tilde{h}_{2k-\ell}c_\ell^j,$$

and reconstruction algorithm

$$c_k^j = \sum_{\ell}\left[\tilde{p}_{m,k-2\ell}c_\ell^{j-1} + \tilde{q}_{m,k-2\ell}d_\ell^{j-1}\right].$$

5. Orthogonal MRA Generated By $s_m(x)$

In this section, we discuss the orthogonal wavelets associated with certain $s_m(x)$. We call a scaling function ϕ an orthogonal scaling function if ϕ yields an orthogonal MRA; i.e., its corresponding mother wavelet function ψ gives a complete orthogonal system $\{\psi_{j,k} = 2^{j/2}\psi(2^j x - k)\}$ in $L^2(R)$. In the following we will give two approaches to construct orthogonal scaling functions with the form defined in (1) by using the similar argument shown in [10].

Theorem 4. Let $C(z)$ be the symbol of $\{c_j\}$, the set of coefficients shown in (1). If $|C(z^2)|^2 = 1/\sum_k |\hat{N}_m(\omega + 2k\pi)|^2$, then the corresponding scaling function $s_m(x) = \sum_j c_j N_m(x)$ is an orthogonal scaling function and its Fourier transform is

$$\begin{aligned} \hat{s}_m(\omega) &= \hat{N}_m(\omega)/\sum_{-m+1}^{m-1} N_{2m}(m+k)e^{-ik\omega} \\ &= \hat{N}_m(\omega)/\sum_{-\infty}^{\infty}\left(\hat{N}_m(\cdot + m)\right)(\omega + 2k\pi), \end{aligned}$$

where $\hat{N}_{2m}(\cdot + m)$ denotes the Fourier transform of $N_{2m}(x+m)$ and this Fourier transform is evaluated at $\omega + 2k\pi$.

Proof. If

$$\sum_k |\hat{s}_m(\omega + 2k\pi)|^2 = |C(z^2)|^2 \sum_k |\hat{N}_m(\omega + 2k\pi)|^2 = 1,$$

then $s_m(x)$ generates an orthogonal MRA. Thus, we obtain that $|C(z^2)|^2 = 1/\sum_k |\hat{N}_m(\omega + 2k\pi)|^2$. By using the Theorem 2.28, and identities (4.2.14) and (4.6.8) in [1], we may prove Theorem 4.

Theorem 5. Suppose that $s_m(x)$ defined in (1) satisfies $s_m(x) = 0$ for $|x| \geq a$. If $C(1) = \frac{3a}{\pi}$; i.e., $\sum_j c_j = \frac{3a}{\pi}$, then the corresponding $s_m(x)$ yields an orthogonal scaling function $\phi_{s_m}(x)$ with its Fourier transform

$$\hat{\phi}_{s_m}(\omega) = \left(\int_{\omega-\pi}^{\omega+\pi} s_m\left(\frac{3ax}{\pi}\right)dx\right)^{\frac{1}{2}}.$$

Proof. Suppose $supp\, s_m(x) \subset [-a, a]$. Thus, $supp\, s_m(\frac{3ax}{\pi}) \subset [-\frac{\pi}{3}, \frac{\pi}{3}]$. It is obvious that $\phi_{s_m}(x)$ generates an orthogonal MRA if $\sum_{k\in Z} |\hat{\phi}_{s_m}(\omega + 2k\pi)|^2 = 1$. $\sum_{k\in Z} |\hat{\phi}_{s_m}(\omega + 2k\pi)|^2 = 1$ can be simplified as follows:

$$\begin{aligned} \sum_{k\in Z} |\hat{\phi}_{s_m}(\omega + 2k\pi)|^2 &= \sum_{k\in Z} \int_{\omega+\pi(2k-1)}^{\omega+\pi(2k+1)} s_m\left(\frac{3ax}{\pi}\right)dx \\ &= \int_{-\infty}^{\infty} s_m\left(\frac{3ax}{\pi}\right)dx \\ &= \int_{-\pi/3}^{\pi/3} s_m\left(\frac{3ax}{\pi}\right)dx. \end{aligned}$$

Thus, $\phi_{s_m}(x)$ generates an orthogonal MRA if

$$\int_{-\pi/3}^{\pi/3} s_m\left(\frac{3ax}{\pi}\right)dx = 1.$$

On the other hand, from the definition of s_m, equation (1), we obtain

$$\begin{aligned}\int_{-\pi/3}^{\pi/3} s_m\left(\frac{3ax}{\pi}\right)dx =& \sum_j c_j \int_{-\pi/3}^{\pi/3} N_m\left(\frac{3ax}{\pi}-j\right)dx\\ =& \sum_j c_j \frac{\pi}{3a}\int_{-a}^{a} N_m(x-j)dx\\ =& \frac{\pi}{3a}\sum_j c_j.\end{aligned}$$

Thus, if $\sum_j c_j = \frac{3a}{\pi}$, the corresponding ϕ_{s_m} generates MRA.

Obviously, $\hat{\phi}^2_{s_m}(\omega)$ is a C^{m-1} continuous function that satisfies

$$\hat{\phi}^2_{s_m} = \begin{cases} 1 & |\omega| < \frac{2\pi}{3},\\ g(|\omega|) & \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3},\\ 0 & |\omega| > \frac{4\pi}{3},\end{cases}$$

where $g(\omega)$ and $g(-\omega)$ are symmetric about the origin and are defined on $\frac{2\pi}{3} \le \omega \le \frac{4\pi}{3}$ and $-\frac{4\pi}{3} \le \omega \le -\frac{2\pi}{3}$, respectively. For instance, if $s_1(x) = c_0N_1(x) + c_{-1}N_1(x+1)$, $c_0+c_1 = \frac{3}{\pi}$, then $g(|\omega|) = \frac{3}{2\pi}\left(\frac{4\pi}{3} - |\omega|\right)$. If $s_2(x) = c_0N_2(x)+c_{-1}N_2(x+1)+c_{-2}N_2(x+2)$, $c_0 + c_{-1} + c_{-2} = \frac{6}{\pi}$, then the Bernstein-Bézier expression of $g(\omega)$ is

$$g(\omega) = \sum_{i+j=2} a_{ij}\frac{2!}{i!j!}u^iv^j,$$

where u and v are the corresponding barycentric coordinates of ω when $\omega \in [\frac{2\pi}{3}, \pi]$ and $[\pi, \frac{4\pi}{3}]$, respectively. The corresponding Bézier coefficients, $[a_{2,0}, a_{1,1}, a_{0,2}]$, of $g(\omega)$, are $[1, 1, \frac{1}{2}]$ and $[\frac{1}{2}, 0, 0]$ when $\omega \in [\frac{2\pi}{3}, \pi]$ and $\omega \in [\pi, \frac{4\pi}{3}]$, respectively. $g(-\omega)$ can be found by symmetry.

The ϕ_{s_m} is a Meyer type scaling function with dilation condition

$$\hat{\phi}_{s_m}(\omega) = m_0\left(\frac{\omega}{2}\right)\hat{\phi}_{s_m}\left(\frac{\omega}{2}\right),$$

where $m_0\left(\frac{\omega}{2}\right)$ is defined on $[-2\pi, 2\pi]$ as

$$m_0\left(\frac{\omega}{2}\right) = \begin{cases} \hat{\phi}_{s_m}(\omega) & |\omega| \le \frac{4\pi}{3},\\ 0 & \frac{4\pi}{3} < |\omega| \le 2\pi\end{cases}$$

and is extended 4π periodically to all $\omega \in R$. Hence, the corresponding wavelet ψ_{s_m} satisfies

$$\hat{\psi}_{s_m}(\omega) = e^{-i\omega/2}\overline{m_0\left(\frac{\omega}{2}+\pi\right)}\hat{\phi}\left(\frac{\omega}{2}\right).$$

6. Recurrence Algorithm of B-wavelets

In this section, we will give a recurrence relation of B-wavelets in terms of their orders and the corresponding algorithm. Hence, a recurrence algorithm for construction of wavelets derived in Section 3 can be given similarly.

Theorem 6. Let $\psi_m(x)$ be the B-wavelet associated with the B-spline of order m, $N_m(x)$. Then there exists the following recurrence relation formula between $\psi_m(x)$ and $\psi_{m+1}(x)$, $m = 1, 2, \cdots$,

$$\psi_{m+1}(x) = \sum_{k=\max\{0,\ell-4m+1\}}^{\ell+1} b_{m+1,k} \int_{x-(k+1)/2}^{x-k/2} \psi_m(t)dt, \tag{29}$$

or, equivalently,

$$\psi'_{m+1}(x) = \sum_{k=\max\{0,\ell-4m+1\}}^{\ell+1} b_{m+1,k} \left[\psi_m\left(x-\frac{k}{2}\right) - \psi_m\left(x - \frac{k+1}{2}\right)\right], \tag{30}$$

where $x \in [\frac{\ell}{2}, \frac{\ell+1}{2}]$, $\ell = 0, 1, \cdots, 4m+1$, and $\{b_{m+1,k}\}$ is the set of coefficients of the expansion of $2\frac{Q_{m+1}(z)}{Q_m(z)}$ in terms of z, which can be determined by the following formulas.

$$b_{m+1,0} = \bar{N}_{m+1}(0)/\bar{N}_m(0), \tag{31}$$

$$b_{m+1,j} = \left(\bar{N}_{m+1}(j) - \sum_{\ell=0}^{j-1}(-1)^\ell b_{m+1,\ell}\bar{N}_m(3m-2-\ell)\right)/\bar{N}_m(0), \tag{32}$$

for $j = 0, 1, \cdots, 3m-2$, and

$$b_{m+1,j} = \left(\bar{N}_{m+1}(j) - \sum_{\ell=j-3m+2}^{j-1}(-1)^\ell b_{m+1,\ell}\bar{N}_m(j-\ell)\right)/\bar{N}_{m+1}(0), \tag{33}$$

for $j = 3m-1, 3m, \cdots, 4m+2$, where

$$\bar{N}_m(k) = \sum_{\ell=0}^{m}\binom{m}{\ell}N_{2m}(k+1-\ell). \tag{34}$$

Proof. From equation (25), we have

$$\hat{\psi}_{m+1}(\omega) = Q_{m+1}(z)\hat{N}_{m+1}(\frac{\omega}{2}).$$

Dividing the above equation by equation (25) side by side, we obtain

$$\frac{\hat{\psi}_{m+1}(\omega)}{\hat{\psi}_m(\omega)} = \frac{Q_{m+1}(z)}{Q_m(z)}\frac{\hat{N}_{m+1}(\frac{\omega}{2})}{\hat{N}_m(\frac{\omega}{2})}. \tag{35}$$

Noting that $\hat{N}_{m+1}(\frac{\omega}{2})/\hat{N}_m(\frac{\omega}{2})\hat{N}_1(\frac{\omega}{2})$, we have

$$\hat{\psi}_{m+1}(\omega) = \frac{Q_{m+1}(z)}{Q_m(z)}\hat{\psi}_m(\omega)\hat{N}_1(\frac{\omega}{2}). \tag{36}$$

In order to express $\psi_{m+1}(x)$ in terms of $\psi_m(x)$, we need to find the inverse Fourier transformation of $\hat{\psi}_m(\omega)\hat{N}_1(\omega/2)$ and the expression of $Q_{m+1}(z)/Q_m(z)$. Define $\hat{\bar{\psi}}(\omega) = \hat{\psi}_m(\omega)\hat{N}_1(\frac{\omega}{2})$. It follows that $\bar{\psi}_n(x) = 2\psi_m(x) * N_1(2x)$ or, equivalently,

$$\begin{aligned}\bar{\psi}_m(x) &= 2\int_{-\infty}^{\infty} N_1(2t)\psi_m(x-t)dt \\ &= 2\int_0^{\frac{1}{2}} \psi_m(x-t)dt.\end{aligned}$$

Hence, if we write $2Q_{m+1}(z)/Q_m(z) = \sum_{k=0}^{\infty} b_{m+1,k}z^k$ formally, from equation (36) we obtain

$$\begin{aligned}\psi_{m+1}(x) &= \frac{1}{2}\sum_{k=0}^{\infty} b_{m+1,k}\bar{\psi}_m(x-\frac{k}{2}) \\ &= \sum_{k=0}^{\infty} b_{m+1,k}\int_0^{\frac{1}{2}} \psi_m(x-\frac{k}{2}-t)dt \\ &= \sum_{k=0}^{\infty} b_{m+1,k}\int_{x-\frac{k+1}{2}}^{x-\frac{k}{2}} \psi_m(t)dt. \end{aligned} \tag{37}$$

We will now determine the range of the summation in expression (37). Since *supp* $\psi_m = [0, 2m-1]$, we need that $x - \frac{k}{2} \geq 0$ and $x - \frac{k+1}{2} \leq 2m-1$; i.e., $k \leq 2x$ and $k \geq 2x - 4m + 1$. Hence, if $x \in [\frac{\ell}{2}, \frac{\ell+1}{2}]$, then $k \leq \ell + 1$ and $k \geq \ell - 4m + 1$. Where $\ell = 0, 1, \cdots, 4m+2$ because $[\frac{\ell}{2}, \frac{\ell+1}{2}] \subset \mathit{supp}\,\psi_m = [0, 2m-1]$. Therefore, we obtain equation (29). Equation (29) can be written in a more general form as follows:

$$\psi_{m+1}(x) = \sum_{k=0}^{4m+2} b_{m+1,k}\int_{x-\frac{k+1}{2}}^{x-\frac{k}{2}} \psi_m(t)dt$$

or, equivalently,

$$\psi'_{m+1}(x) = \sum_{k=0}^{4m+2} b_{m+1,k}[\psi_m(x-\frac{k}{2}) - \psi_m(x-\frac{k+1}{2})], \tag{38}$$

where $x \in [0, 2m+1]$.

In order to complete the proof of the theorem, we only need to prove that the following expansion of $2Q_{m+1}(z)/Q_m(z)$ exists and to give the expression of $\{b_{m+1,k}\}$.

$$2\frac{Q_{m+1}(z)}{Q_m(z)} = \sum_{k=0}^{\infty} b_{m+1,k}z^k. \tag{39}$$

In fact, from [1], $Q_m(z) = -zE_m(-z)\overline{P_m(-z)}$, where $E_m(-z)$ is the Euler-Frobenius Laurent polynomial with respect to N_m and $P_m(z) = \left(\frac{1+z}{2}\right)^m$. Hence, $Q_{m+1}(z)/Q_m(z) =$

$\left(\frac{1-\bar{z}}{2}\right) E_{m+1}(-z)/E_m(-z)$ is zero-free and pole-free on $|z|=1$. It follows that expansion (34) exists on $|z|=1$. To find $\{b_{m+1,k}\}$, we write

$$Q_m(z) = \sum_{k=0}^{3m-2} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k+1-\ell) z^k$$

and

$$Q_{m+1}(z) = \sum_{k=0}^{3m+1} \frac{(-1)^k}{2^{m}} \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2m+2}(k+1-\ell) z^k.$$

It follows from equation (39) that

$$\sum_{j=0}^{\infty} b_{m+1,j} z^j \sum_{k=0}^{3m-2} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k+1-\ell) z^k$$
$$= \sum_{k=0}^{3m+1} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2(m+1)}(k+1-\ell) z^k. \tag{40}$$

On the left hand side of equation (40), we exchange the last two summations, then take transform $k+j=k'$. Noting that $supp\, \psi_m = [0, 2m-1]$, we finally obtain

$$\sum_{k'=0}^{3m+1} \sum_{j=0}^{4m+2} b_{m+1,j} \frac{(-1)^{k'-j}}{2^{m-1}} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k'-j+1-\ell) z^{k'}$$
$$= \sum_{k=0}^{3m+4} \frac{(-1)^k}{2^{m-1}} \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2(m+1)}(k+1-\ell) z^k. \tag{41}$$

Hence, for $k = 0, 1, \cdots, 3m+1$, we have

$$\sum_{j=0}^{4m+2} (-1)^j b_{m+1,j} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k-j+1-\ell)$$
$$= \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} N_{2(m+1)}(k+1-\ell). \tag{42}$$

System (43) can be written as the following matrix form:

$$A_m \mathbf{b}_{m+1} = \mathbf{n}_{m+1}, \tag{43}$$

where $\mathbf{b}_{m+1} = (b_{m+1,0}, -b_{m+1,1}, b_{m+1,2}, \cdots, -b_{m+1,4m+1}, b_{m+1,4m+2})$, $A_m = [a_{k,j}]_{k,j=0}^{4m+2} = [\bar{N}_m(k-j)]_{k,j=0}^{4m+2}$, $\mathbf{n}_{m+1} = (\bar{N}_{m+1}(0), \bar{N}_{m+1}(1), \cdots, \bar{N}_{m+1}(4m+2))$, and $\bar{N}_m(k)$ is defined as equation (34). It is easy to have $\bar{N}_m(0) = N_{2m}(1) = 1/(2m-1)!$, $\bar{N}_m(1) = \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(2-\ell) = N_{2m}(2) + mN_{2m}(1)$, ..., and $\bar{N}_m(3m-2) = \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(3m-$

$1-\ell) = N_{2m}(2m-1) = 1/(2m-1)!$. Note that $\bar{N}_m(i) = 0$ if $i < 0$ or $i > 3m-2$. Matrix A_m in (43) is actually

$$A_m = \begin{bmatrix} \bar{N}_m(0) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \bar{N}_m(1) & \bar{N}_m(0) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \bar{N}_m(3m-2) & \bar{N}_m(3m-3) & \cdots & \bar{N}_m(0) & 0 & \cdots & 0 \\ 0 & \bar{N}_m(3m-2) & \cdots & \bar{N}_m(1) & \bar{N}_m(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{N}_m(3m-2) & \bar{N}_m(3m-3) & \cdots & \bar{N}_m(0) \end{bmatrix}.$$

Thus, $\mathbf{b}_{m+1}$ can be solved and can be expressed as formulas (31)-(33).

If we express $\psi_m(x)$ using their Bézier coefficients, from equation (38) we obtain the following recurrence algorithm for constructing $\psi_m(x)$ by using the Bézier coefficients in their Bernstein-Bézier expressions.

$$a_{k+1}^{m+1}(\ell) = a_k^{m+1}(\ell) + \frac{1}{2m} \sum_{j=0}^{4m+2} b_{m+1,j} \left[a_k^m(\ell+j) - a_k^m(\ell+j-1) \right], \tag{44}$$

where $a_p^r(q)$ is the p^{th} Bézier coefficient of $\psi_r(x)$ over the interval $[\frac{\ell}{2}, \frac{\ell+1}{2}]$. Here the order of arrangement for the Bézier coefficients is in terms of the increase in powers of the second coordinate of the barycentric coordinates in the Bernstein-Bézier polynomial expression of the wavelets.

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