Q-Analogues of Symbolic Operators

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Abstract

Here presented are q-extensions of several linear operators including a novel q-analogue of the the derivative operator D. Some q-analogues of the symbolic substitution rules given in [4] are obtained. As sample applications, we show how these q-substitution rules may be used to construct symbolic summation and series transformation formulas, including q-analogues of the classical Euler transformations for accelerating the convergence of alternating series.

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1 Definitions, Basic Identities

Unless otherwise stated, we consider all operators to act on formal power series in the single variable t, with coefficients possibly depending on q. We assume $0 < |q| < 1$. Issues of convergence will be addressed in a later paper.

We will use 1 to denote the identity operator, and define the following operators:

1. $E_q f(t) = f(tq)$, (forward multiplicative shift),

2. $\Delta_q f(t) = f(tq) - f(t)$, (forward q-difference),

3. $L_q f(t) = t(\log q) f'(t)$, (forward logarithmic shift).

The first two of these can be regarded as q-analogues of the ordinary (additive) shift and forward difference operators, respectively. L_q will play a role similar to that of the derivative D.

The operator inverse of E_q (which we denote as E_q^{-1}) clearly exists and is equal to $E_{q^{-1}}$. We define the central q-difference operator δ_q by

$$\delta_q = f(tq^{1/2}) - f(tq^{-1/2}) \tag{1.1}$$

and note that $\delta_q = \Delta_q E_q^{-1/2} = \Delta / E_q^{1/2}, \quad \delta_q^{2k} = \Delta_q^{2k} E_q^{-k}$.

The q-operators above are linear and satisfy some familiar identities, for example, $E_q = 1 + \Delta_q$. The binomial identity

$$\Delta_q^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E_q^k \tag{1.2}$$

can be established by induction, or by considering the operator expansion of $(1 - E_q)^n$.

Treating these operators formally, we need only consider their effect on nonnegative integer powers of t. E_q, Δ_q, and L_q are "diagonal" in the sense that each maps $t^k \mapsto M(q,k)t^k$, with the function M depending on the particular operator. For example, $\Delta_q[t^k] = (q^k - 1)t^k$ for $k > 0$, and $\Delta_q[1] = 0$. Similarly, $L_q[t^k] = t^k \log(q^k)$.

With this observation, it is easy to verify many additional identities. For example, consider the alternating geometric series $\sum\limits_{n=0}^{\infty} (-1)^n \Delta_q^n$ applied to t^k. We have

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\Delta_q^n[t^k] &= t^k\sum_{n=0}^{\infty}(-1)^n(q^k-1)^n \\ &= t^k\frac{1}{1-(1-q^k)} \\ &= t^kq^{-k}.\end{aligned}$$

In other words, this formal power series gives the operator $E_{q^{-1}}$. Stated differently,

$$(1+\Delta_q)^{-1} = (E_q)^{-1} = E_{q^{-1}} = \sum_{n=0}^{\infty}(-1)^n\Delta_q^n, \tag{1.3}$$

which is exactly the result we should expect. We may establish the following identities in similar fasion:

$$(1-\Delta_q)^{-1} = \sum_{n=0}^{\infty}\Delta_q^n. \tag{1.4}$$

$$\log(1+\Delta_q) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Delta_q^n = L_q. \tag{1.5}$$

$$e^{L_q} = \sum_{n=0}^{\infty}\frac{1}{n!}L_q^n = E_q. \tag{1.6}$$

In addition to these last two identities, L_q obeys the product rule

$$L_q\left[f(t)g(t)\right] = L_q[f(t)]g(t) + f(t)L_q[g(t)], \tag{1.7}$$

so that L_q is a q-analogue of the ordinary derivative operator D.

2 Main Results

We begin with some q-analogues of the symbolic substitution rules in [4] (specifically, equations (2.4) and (2.5)):

Proposition 2.1. *Let $F(t)$ have the formal power series expansion $F(t) = \sum_{k\geq 0} f_kt^k$, with coefficients possibly dependent on q. We may obtain operational formulas according to the following rules:*

1. The substitution $t \mapsto E_q$ leads to the symbolic formula

$$F(E_q) = \sum_{k=0}^{\infty} f_k E_q^k. \tag{2.1}$$

2. If $F(t) = G(t, e^t)$, the substitution $t \mapsto L_q$ leads to

$$G(L_q, E_q) = \sum_{k=0}^{\infty} f_k L_q^k. \tag{2.2}$$

3. If $F(t) = G(t, \log(1+t))$, the substitution $t \mapsto \Delta_q$ leads to

$$G(\Delta_q, L_q) = \sum_{k=0}^{\infty} f_k \Delta_q^k. \tag{2.3}$$

Note that each of the identities in equations (1.4)-(1.6) can be obtained from elementary Maclaurin series by applying one of these substitution rules. We now present a less trivial example.

For k a positive integer, let $\alpha_k(x)$ denote the Eulerian fraction (cf. [1] pg. 245). It is well-known that

$$\sum_{j=0}^{\infty} j^k x^j = \frac{A_k(x)}{(1-x)^{k+1}} = \alpha_k(x), \quad (|x| < 1), \tag{2.4}$$

where $A_k(x)$ is the kth Eulerian polynomial. Additionally, ([6], pg. 24) gives the formula

$$(1 - xe^t)^{-1} = \sum_{k=0}^{\infty} \alpha_k(x) \frac{t^k}{k!}. \tag{2.5}$$

Substituting $t \mapsto L_q$ leads to the formal identity

$$(1 - xE_q)^{-1} = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} L_q^k. \tag{2.6}$$

We can obtain additional identities in this fashion from other expansions of $(1 - xe^t)^{-1}$. For example, if $x \neq 0$ and $x \neq 1$, we have the following analogues of (3.1)-(3.4) in [3]):

$$
\begin{aligned}
(1-xE_q)^{-1} &= \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta_q^k & (2.7)\\
(1-xE_q)^{-1} &= \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(x^{-1}\frac{\Delta_q^{2k}}{E_q^k} - \frac{\Delta^{2k}}{E_q^{k+1}}\right) & (2.8)\\
(1-xE_q)^{-1} &= 1+\sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(\frac{\Delta_q^{2k}}{E_q^{k-1}} - x\frac{\Delta^{2k}}{E_q^{k}}\right) & (2.9)
\end{aligned}
$$

Direct proofs of (2.6)-(2.9) are given in §5 below.

Proposition 2.2. *For a given function* $f(t)$*, define* $F_q(x) = \sum_{k\geq 0} f(q^k)x^k$*. If* $x \neq 0$ *and* $x \neq 1$*,*

$$
\begin{aligned}
F_q(x) &= \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} L_q^k f(1) & (2.10)\\
F_q(x) &= \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta_q^k f(1) & (2.11)\\
F_q(x) &= \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(x^{-1}\delta_q^{2k} f(1) - \delta_q^{2k} f(q^{-1})\right) & (2.12)\\
F_q(x) &= 1+\sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(\delta_q^{2k} f(q) - x\delta_q^{2k} f(1)\right) & (2.13)
\end{aligned}
$$

Proof. Clearly, these follow by applying the operators in equations (2.6)-(2.9) to the function $f(t)$ and then evaluating at $t = 1$. $\square$

3 Some Applications

As an application, taking $f(t) = \frac{1}{\log_q(t+1)}$, $x = -1$ in (2.11) leads to

$$\begin{aligned}
\sum_{k\geq 0}(-1)^k \frac{1}{k+1} &= \sum_{k\geq 0} \frac{(-1)^k}{2^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{j+1} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)2^{k+1}} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)2^{k+1}} = \sum_{k=1}^{\infty} \frac{1}{k2^k},
\end{aligned}$$

which gives

$$\ln 2 = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \cdots .$$

The rate of convergence of this series is $O(1/2^n)$, much faster than

$$\ln 2 = \sum_{k\geq 0}(-1)^k \frac{1}{k+1},$$

whose convergence rate is $O(1/n)$.

As for a second application, we may substitute $x = -1$ in Proposition 2.2, obtaining the following series transformation formulas:

$$\sum_{k\geq 0}(-1)^k f(q^k) = \sum_{k=0}^{\infty} \frac{\alpha_k(-1)}{k!} L_q^k f(1) \tag{3.14}$$

$$\sum_{k\geq 0}(-1)^k f(q^k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \Delta_q^k f(1) \tag{3.15}$$

$$\sum_{k\geq 0}(-1)^k f(q^k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} (\delta_q^{2k} f(1) + \delta_q^{2k} f(q^{-1})) \tag{3.16}$$

$$\sum_{k\geq 1}(-1)^k f(q^k) = 1 + \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^{k+1} (\delta_q^{2k} f(q) + \delta_q^{2k} f(1)) \tag{3.17}$$

Thes four identities appear to be novel, and could be used to accelerate slowly convergent alternating series $\sum_{k=1}^{\infty}(-1)^k f(q^k)$. We consider them as q-analogues of the ordinary Euler transformations.

4 Extensions of the Main Results

All operational formulas presented in Proposition 2.1 can be extended and the corresponding symbolic substitution formulas established accordingly with an analogous form of 2.2. For example, we may consider a generating function of the form

$$\sum_{k\geq 0} f_k t^k = F(t, e^t, e^{\alpha t}).$$

Letting $t \mapsto L_q$ gives

$$\sum_{k\geq 0} f_k L_q^k = F(L_q, E_q, E_q^{\alpha}).$$

Applying this to the well-known identity

$$\sum_{k\geq 0} \frac{4^k}{(2k)!} B_{2k} t^{2k} = t \coth t = t \frac{e^t + e^{-t}}{e^t - e^{-t}},$$

with B_n being the nth Bernoulli number, we obtain

$$\sum_{k\geq 0} \frac{4^k}{(2k)!} B_{2k} L_q^{2k} = L_q \frac{E_q + E_q^{-1}}{E_q - E_q^{-1}}.$$

Hence, we obtain a symbolic formula

$$\sum_{k\geq 0} \frac{4^k}{(2k)!} B_{2k} L_q^{2k-1} (E_q - E_q^{-1}) = E_q + E_q^{-1}. \tag{4.18}$$

Applying this to an infinitely differentiable function $f(t)$ at $t = 1$ yields

$$\sum_{k\geq 0} \frac{4^k}{(2k)!} B_{2k} L_q^{2k-1} (E_q - E_q^{-1}) f(1) = (E_q + E_q^{-1}) f(1). \tag{4.19}$$

Similarly, using the symbolic relation

$$L_q \frac{E_q + E_q^{-1}}{E_q - E_q^{-1}} = L_q \left(1 + \Delta_q^{-1} - (E_q + 1)^{-1}\right),$$

we obtain another operational formula

$$-1+\sum_{k\geq 0}\frac{4^k}{(2k)!}B_{2k}L_q^{2n-1}+(E_q^{-1}+1)^{-1}=\Delta_q^{-1},$$

from which one may construct a series transformation formula.

Another extension is a q-analogue of the symbolic formulas presented in [2], which is actually a Newton series type extension of the symbolic expansions given in [4]. Consider

$$(1+E_q)^x f(1)=\sum_{k\geq 0}\binom{x}{k}E_q^k f(1)=\sum_{k\geq 0}f(q^k)\frac{(x)_k}{k!},$$

where $(x)_k=x(x-1)\cdots(x-k+1)$. We have

$$(1+E_q)^x = 2^x\sum_{k=0}^{\infty}\frac{(x)_k}{2^k k!}\Delta_q^k, \tag{4.20}$$

$$\begin{aligned}(1+E_q)^x &= \sum_{k=0}^{\infty}\binom{x}{k}\Big[\,{}_2F_1\left(k-x,2k+1;k+1;-1\right)\\ &\quad+\frac{x-k}{k+1}\,{}_2F_1\left(k+1-x,2k+2;k+2;-1\right)E_q^{-1}\Delta_q\Big]\delta_q^{2k},\end{aligned} \tag{4.21}$$

$$\begin{aligned}(1+E_q)^x &= 1+\sum_{k=0}^{\infty}\binom{x}{k+1}\Big[\,{}_2F_1\left(k+1-x,2k+2;k+2;-1\right)E_q\\ &\quad-\frac{x-k-1}{k+2}\,{}_2F_1\left(k+2-x,2k+2;k+3;-1\right)\Big]\delta_q^{2k}.\end{aligned} \tag{4.22}$$

Finally, we present an extension of (2.6) using Bell polynomials (see, for example, pg. 134 in [1]) as follows.

$$(1+E_q)^x=2^x\sum_{k=0}^{\infty}P_k^{(x)}\left(\frac{1}{2},\frac{1}{2},\cdots,\right)\frac{L_q^k}{k!}, \tag{4.23}$$

where the values of potential Bell polynomials at $(1/2,1/2,\ldots)$ are defined by

$$P_k^{(x)}\left(\frac{1}{2},\frac{1}{2},\cdots,\right)=\sum_{\ell=0}^{\infty}\ell^k\frac{(x)_\ell}{\ell!}. \tag{4.24}$$

For a given function $f(t)$, define $F_q(x) = \sum_{k\geq 0} f(q^k)(x)_k/k!$, from (4.20)-(4.23) we obtain series transformation formulas by simply applying (4.20)-(4.22) to f.

$$\begin{aligned}
F_q(x) &= 2^x \sum_{k=0}^{\infty} \frac{(x)_k}{2^k k!} \Delta_q^k f(1), && (4.25)\\
F_q(x) &= \sum_{k=0}^{\infty} \binom{x}{k} \Big[{}_2F_1\left(k-x, 2k+1; k+1; -1\right) \\
&\quad + \frac{x-k}{k+1}\, {}_2F_1\left(k+1-x, 2k+2; k+2; -1\right) E_q^{-1} \Delta_q \Big] \delta_q^{2k} f(1), && (4.26)\\
F_q(x) &= 1 + \sum_{k=0}^{\infty} \binom{x}{k+1} \Big[{}_2F_1\left(k+1-x, 2k+2; k+2; -1\right) E_q \\
&\quad - \frac{x-k-1}{k+2}\, {}_2F_1\left(k+2-x, 2k+2; k+3; -1\right) \Big] \delta_q^{2k} f(1). && (4.27)\\
F_q(x) &= 2^x \sum_{k=0}^{\infty} P_k^{(x)}\left(\frac{1}{2}, \frac{1}{2}, \cdots, \right) \frac{L_q^k}{k!}. && (4.28)
\end{aligned}$$

As an example, substituting $f(t) = t^n$ into (4.28) yields the series transformation formula

$$\sum_{k=0}^{\infty} t^k \frac{(x)_k}{k!} = 2^x \sum_{j=0}^{\infty} P_j^{(x)}\left(\frac{1}{2}, \frac{1}{2}, \cdots\right) \frac{n^j}{j!}.$$

5 Selected Proofs

Here we present the proofs of (2.6)-(2.9) in the sense of symbolic calculus, viz., every series expansion is considered as a formal series.

Equation (2.7) may be derived as follows:

$$
\begin{aligned}
(1-xE_q)^{-1} &= (1-x(1+\Delta_q))^{-1} \\
&= (1-x)^{-1}\left(1-\frac{x\Delta_q}{1-x}\right) \\
&= \sum_{k=0}^{\infty}\frac{x^k\Delta_q^k}{(1-x)^{k+1}}.
\end{aligned}
$$

For proving (2.6), it suffices to make use of $E = e^{L_q}$ and (2.4). Indeed we have

$$
\begin{aligned}
(1-xE_q)^{-1} = (1-xe^{L_q})^{-1} = \sum_{k=0}^{\infty} x^k e^{kL_q} \\
= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty}\frac{(kL_q)^j}{j!} = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} x^k k^j\right)\frac{L_q^j}{j!} = \sum_{j=0}^{\infty}\alpha_j(x)\frac{L_q^j}{j!}.
\end{aligned}
$$

To prove (2.8) and (2.9), we first establish the following lemma.

Lemma 5.1. *Let $\beta = 1+\alpha$ with $0<\alpha<1$, and let x be any real number. We have symbolic identities involving the first Gauss series:*

$$
\beta^x = \sum_{k=0}^{\infty}\left[\binom{x+k}{2k}\frac{\alpha^{2k}}{\beta^k} + \binom{x+k}{2k+1}\frac{\alpha^{2k+1}}{\beta^k}\right] \tag{5.29}
$$

and a modified q-form of Gauss's first symbolic expression (cf §127 of [5]):

$$
\begin{aligned}
E_q^x &= \sum_{k=0}^{\infty}\left[\binom{x+k}{2k}\frac{\Delta_q^{2k}}{E_q^k} + \binom{x+k}{2k+1}\frac{\Delta_q^{2k+1}}{E_q^k}\right] \\
&= \sum_{k=0}^{\infty}\left[\binom{x+k}{2k}\delta_q^{2k} + \binom{x+k}{2k+1}\Delta_q\delta_q^{2k}\right].
\end{aligned} \tag{5.30}
$$

Proof. Starting from Newton's formula:

$$
\beta^x = \sum_{k=0}^{\infty}\binom{x}{k}\alpha^k.
$$

We multiply $\frac{\alpha+1}{\beta}=1$ to the summation from the term α up and obtain

$$\begin{aligned}
\beta^x &= 1+\binom{x}{1}\frac{\alpha}{\beta}+\binom{x}{1}\frac{\alpha^2}{\beta}+\sum_{k=2}^{\infty}\binom{x}{k}\frac{\alpha^k(1+\alpha)}{\beta} \\
&= 1+\binom{x}{1}\frac{\alpha}{\beta}+\binom{x+1}{2}\frac{\alpha^2}{\beta}+\sum_{k=2}^{\infty}\frac{\alpha^{k+1}}{\beta}\left[\binom{x}{k}+\binom{x}{k+1}\right] \\
&= 1+\binom{x}{1}\frac{\alpha}{\beta}+\binom{x+1}{2}\frac{\alpha^2}{\beta}+\sum_{k=2}^{\infty}\binom{x+1}{k+1}\frac{\alpha^{k+1}}{\beta} \\
&= 1+\binom{x}{1}\frac{\alpha}{\beta}+\binom{x+1}{2}\frac{\alpha^2}{\beta}+\binom{x+1}{3}\frac{\alpha^3}{\beta}+\sum_{k=3}^{\infty}\binom{x+1}{k+1}\frac{\alpha^{k+1}}{\beta}.
\end{aligned}$$

Repeating the operation on the series from the term α^3 up yields

$$\begin{aligned}
\beta^x &= 1+\binom{x}{1}\frac{\alpha}{\beta}+\binom{x+1}{2}\frac{\alpha^2}{\beta}+\binom{x+1}{3}\frac{\alpha^3}{\beta^2}+\binom{x+1}{3}\frac{\alpha^4}{\beta^2} \\
&\quad+\sum_{k=3}^{\infty}\binom{x+1}{k+1}\frac{\alpha^{k+1}(1+\alpha)}{\beta^2} \\
&= 1+\binom{x}{1}\frac{\alpha}{\beta}+\binom{x+1}{2}\frac{\alpha^2}{\beta}+\binom{x+1}{3}\frac{\alpha^3}{\beta^2}+\binom{x+2}{4}\frac{\alpha^4}{\beta^2} \\
&\quad+\sum_{k=3}^{\infty}\binom{x+2}{k+2}\frac{\alpha^{k+2}}{\beta^2}.
\end{aligned}$$

The above operation is repeated from α^5 up, and so on. We obtain

$$\beta^x=\sum_{k=0}^{\infty}\left[\binom{x+k}{2k}\frac{\alpha^{2k}}{\beta^k}+\binom{x+k}{2k+1}\frac{\alpha^{2k+1}}{\beta^k}\right]. \tag{5.31}$$

Substituting $\beta = E_q$ and $\alpha = \Delta_q$ into the above identity, we obtain the desired result. □

(2.7) and (2.8) can be proved using the first Gauss symbolic expression (5.30) and the following q-form of the Everett's symbolic expression (cf [5], §129), respectively.

$$
\begin{aligned}
E^x &= \sum_{k=0}^{\infty}\left[\binom{x+k}{2k+1}\frac{\Delta_q^{2k}}{E_q^{k-1}}-\binom{x+k-1}{2k+1}\frac{\Delta_q^{2k}}{E_q^{k}}\right]\\
&= \sum_{k=0}^{\infty}\left[\binom{x+k}{2k+1}E_q\delta_q^{2k}-\binom{x+k-1}{2k+1}\delta_q^{2k}\right]. \qquad (5.32)
\end{aligned}
$$

Indeed, using (5.30) and noting the identity

$$
\sum_{m=k}^{\infty}\binom{m}{k}x^m=\frac{x^k}{(1-x)^{k+1}}, \quad (|x|<1).
$$

one may derive (2.7) as follows:

$$
(1-xE_q)^{-1}=\sum_{j=0}^{\infty}(xE_q)^j
$$

$$
=\sum_{k=0}^{\infty}\left\{\left(\sum_{j=0}^{\infty}\binom{j+k}{2k}x^j\right)\frac{\Delta_q^{2k}}{E_q^k}+\left(\sum_{j=0}^{\infty}\binom{j+k}{2k+1}x^j\right)\frac{\Delta_q^{2k+1}}{E_q^{k+1}}\right\}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\left\{\frac{x^k}{(1-x)^{2k+1}}\frac{\Delta_q^{2k}}{E_q^k}+\frac{x^{k+1}}{(1-x)^{2k+2}}\frac{\Delta_q^{2k+1}}{E_q^{k+1}}\right\}\\
&= \sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}\left(\frac{1-x}{x}\frac{\Delta_q^{2k}}{E_q^k}+\frac{\Delta_q^{2k+1}}{E_q^{k+1}}\right)\\
&= \sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}\left(x^{-1}\frac{\Delta_q^{2k}}{E_q^k}-\frac{\Delta_q^{2k}}{E_q^{k+1}}\right).
\end{aligned}
$$

(2.8) can be proved similarly using (5.32). However, it can also be verified by a direct symbolic computations. In fact we have

$$
\begin{aligned}
&RHS\ of\ (2.8) = 1 + \frac{x}{(1-x)^2}\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^k\left(\frac{\Delta_q^2}{E_q}\right)^k(E_q - x)\\
= &\ 1 + \frac{x}{(1-x)^2}\frac{E_q - x}{1 - \frac{x}{(1-x)^2}\frac{\Delta_q^2}{E_q}} = 1 + \frac{x(E_q - x)}{(1-x)^2 - x\frac{\Delta_q^2}{E_q}}\\
= &\ 1 + \frac{E_q x(E_q - x)}{(1-x)^2 E_q - x(E_q - 1)^2} = 1 + \frac{E_q x}{1 - xE_q}\\
= &\ (1 - xE_q)^{-1} = LHS\ of\ (2.8).
\end{aligned}
$$

This complete the proofs of (2.6)-(2.8).

The proof of (4.20) is straightforward:

$$
\begin{aligned}
(1+E_q)^x &= (2+\Delta_q)^x = 2^x\left(1 + \frac{\Delta}{2}\right)^x\\
&= 2^x\sum_{k\geq 0}\frac{(x)_k}{k!2^k}\Delta_q^k.
\end{aligned}
$$

To prove (4.21), we use (5.30) as follows:

$$
\begin{aligned}
(1+E_q)^x &= 1 + \sum_{j\geq 1}\binom{x}{j}E_q^j\\
&= 1 + \sum_{j\geq 1}\binom{x}{j}\sum_{k\geq 0}\left[\binom{k+j}{2k}\frac{\Delta_q^{2k}}{E_q^k} + \binom{k+j}{2k+1}\frac{\Delta_q^{2k+1}}{E_q^{k+1}}\right]\\
&= \sum_{k\geq 0}\left[\frac{\Delta_q^{2k}}{E_q^k}\sum_{j\geq k}\binom{x}{j}\binom{k+j}{2k} + \frac{\Delta_q^{2k+1}}{E_q^{k+1}}\sum_{j\geq k+1}\binom{x}{j}\binom{k+j}{2k+1}\right]\\
&= \sum_{k\geq 0}\left[\binom{x}{k}\ {}_2F_1(k-x, 2k+1; k+1; -1)\frac{\Delta_q^{2k}}{E_q^k}\right.\\
&\quad \left. + \binom{x}{k+1}\ {}_2F_1(k+1-x, 2k+2; k+2; -1)\frac{\Delta_q^{2k}}{E_q^{k+1}}\right],
\end{aligned}
$$

which implies (4.21). (4.22) can be proved similarly using Everett's symbolic expression (5.32).

For (4.23), we first have

$$
\begin{aligned}
(1+E_q)^x &= \left(1+e^{L_q}\right)^x = \sum_{j\geq 0}\binom{x}{j}e^{jL_q} && (5.33)\\
&= \sum_{j\geq 0}\binom{x}{j}\sum_{k\geq 0}\frac{(jL_q)^k}{k!} = \sum_{k\geq 0}\left(\sum_{j\geq 0}\frac{(k)_j j^k}{j!}\right)\frac{L_q^k}{k!}. && (5.34)
\end{aligned}
$$

Using (5.24), we may write the part in the parenthesis of the rightmost term as $2^x P_k^{(x)}(1/2, 1/2, \ldots)$ to finish.

References

[1] L. Comtet, Advanced Combinatorics, Dordrecht: Reidel, 1974 (Chapters 1, 3).

[2] Q. Fang, M. Xu, and T. Wang, A symbolic operator approach to Newton series, manuscript, 2009.

[3] T.-X. He, L. C. Hsu, P. J.-S. Shiue, and D. C. Torney, A symbolic operator approach to several summation formulas for power series, J. Comput. Appl. Math., 177(2005), 17-33.

[4] T.-X. He, L. C. Hsu, and P. J.-S. Shiue, Symbolization of generating functions; an application of the Mullin-Rota theory of binomial enumeration, Computers and Mathematics with Applications. 54 (2007), 664-678.

[5] Ch. Jordan, Calculus of Finite Differences, Chelsea, 1965.

[6] X.H. Wang, L.C. Hsu, A summation formula for power series using Eulerian fractions, Fibonacci Quarterly 41 (2003), 23-30.