On Multivariate Abel-Gontscharoff Interpolation

Tian-Xiao He

Dedicated to the memory of Dieter Gaier and Will Light.

Abstract. By using Gould's annihilation coefficients, we obtain an explicit fundamental polynomials of Multivariate Abel-Gontscharoff Interpolation and its remainder expression.

§1. Introduction

In [1,2,11,12], Kergin, Micchelli and Milman, and Cavaretta, Micchelli and Sharma studied a method for extending univariate interpolation procedures to higher dimensions. Their idea is based on the requirement that the multivariate extension is related to its univariate analog on the class of ridge functions. In particular, the implicit multivariate Abel-Gontscharoff without the remainder was studied.

Recently, as motivated by some special identities of Abel-type, Gould [7] has investigated a kind of general algebraic identity of the form

$$\sum_{k=0}^{n}\binom{n}{k}c(t|k;\beta)(t-\beta_k)^{n-k}=t^n, \tag{1.1}$$

where $\beta_k \in \mathbb{C}$, $\beta_0 \neq 0$, $c(t|0;\beta)=1$, and the uniquely determined coefficients $\binom{n}{k}c(t|k;\beta) \equiv \binom{n}{k}c(t|k;\beta_0,\cdots,\beta_{k-1})$ $(0 \le k \le n)$ are called (by Gould) "annihilation coefficients" and satisfy the recurrence relations

$$c(t|n;\beta)=\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n}{j}c(t|j;\beta)\beta_j^{n-j}, \quad n \ge 1, \tag{1.2}$$

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Evidently (1.2) is implied by (1.1) with $t = 0$. A few $c(k) \equiv c(t|k;\beta)$-polyno-mials can be readily found by using (1.2), *i.e.*, $c(1) = \beta_0$, $c(2) = 2\beta_0\beta_1 - \beta_0^2$, $c(3) = 6\beta_0\beta_1\beta_2 - 3\beta_0^2\beta_2 - 3\beta_0\beta_1^2 + \beta_0^3$. Actually it is easily observed that $c(t|k;\beta)$ $(k \geq 1)$ is a certain kind of homogeneous polynomial of degree k in $\beta_0, \cdots, \beta_{k-1}$. Also, it is obvious that (1.1) implies $c(t|k;\alpha,\cdots,\alpha) = \alpha^k$ by setting $\beta_k = \alpha$.

Recently, Hsu, Shiue and the author (see Proposition 2.1 in [9]) found that Gould's polynomials $c(k) \equiv c(t|k;t-\alpha_0,\ldots,t-\alpha_{k-1})$ is identical with the fundamental polynomials of Abel-Gontscharoff interpolation. In other words, the operator

$$\Lambda_n g(t) = \sum_{k=0}^{n} \frac{g^{(k)}(\alpha_k)}{k!} c(t|k;t-\alpha_0,\ldots,t-\alpha_{k-1}) \tag{1.3}$$

is the Abel-Gontscharoff interpolation operator; *i.e.*, the operator $\Lambda_n : C^n([a,b]) \to \pi_n(\mathbb{R})$, where $[a,b]$ contains all nodes $\alpha_0, \alpha_1, \cdots, \alpha_n$, satisfies

$$(\Lambda_n g)^{(k)}(\alpha_k) = g^{(k)}(\alpha_k) \tag{1.4}$$

for all $k = 0, 1, \cdots, n$. Here, $\alpha_0, \alpha_1, \cdots, \alpha_n$ are not necessary distinct, and $\pi_n(\mathbb{R})$ denotes the collection of all polynomials of degrees $\leq n$ defined on $\mathbb{R}$. Moreover, [9] and [10] gave an explicit expression of the form for $c(t|k;\beta)$ as follows.

$$\begin{aligned} c(t|k;\beta) &= (-1)^{k-1}\beta_0^k \\ &+ \sum_{(j)} \frac{(-1)^{k-r-1}k!}{j_1!(j_2-j_1)!\cdots(k-j_r)!} \beta_0^{j_1}\beta_{j_1}^{j_2-j_1}\beta_{j_2}^{j_3-j_2}\cdots\beta_{j_r}^{k-j_r}, \end{aligned} \tag{1.5}$$

where $(j) \equiv \{j_1, \cdots, j_r\}$ denotes an ordered subset of the ordered set $\{1, 2, \cdots, k-1\}$ with $1 \leq j_1 < j_2 < \cdots < j_r \leq k-1$, $(1 \leq r \leq k-1)$, and the summation is taken over all the ordered subsets of $\{1, 2, \cdots, k-1\}$ for $r = 1, \cdots, k-1$, so that the right-hand of (1.5) consists of 2^{k-1} terms.

In this paper, we will use Gould's polynomial to find an explicit fundamental polynomials of multivariate Abel-Gontscharoff interpolation and the remainder expression.

§2. Main Results

Denote

$$\begin{aligned} c(D|k;x-x^0,x-x^1,\cdots,x-x^{k-1}) &= (-1)^{k-1}D^k_{x-x^0} \\ + \sum^{(j)} \frac{(-1)^{k-r-1}k!}{j_1!(j_2-j_1)!\cdots(k-j_r)!} & D^{j_1}_{x-x^0} D^{j_2-j_1}_{x-x^{j_1}} D^{j_3-j_2}_{x-x^{j_2}} \cdots D^{k-j_r}_{x-x^{j_r}}, \end{aligned} \tag{2.1}$$

where (j) is as the same as that in (1.5), $x, x^0, x^1, \cdots, x^n \in \mathbb{R}^d$, and $D^i_{x-x^j}$ is the ith order directional derivative operator along the direction $x - x^j$. It is obvious that $D^i_{x-x^j} g(\lambda \cdot t) = (\lambda \cdot x - \lambda \cdot x^j)^i g^{(i)}(\lambda \cdot t)$ $(t \in \mathbb{R}^d)$ for all $\lambda, x, x^j \in \mathbb{R}^d$ and for all $i \in \mathbb{N}$. In the following, we will use K to denote the convex hull, conv$\{x^0, x^1, \cdots, x^n\}$, of points $x^0, x^1, \cdots, x^n \in \mathbb{R}^d$, and use $\pi_n(\mathbb{R}^d)$ to denote the collection of all polynomials of degrees $\leq n$ defined on $\mathbb{R}^d$.

Abel-Gontscharoff interpolation problem is a type of Hermite-Birkhoff interpolation problem. There exists a corresponding incidence matrix

$$E = [e_{i,k}]_{i=1,k=0}^{m\ \ \ell}, \quad m \geq 1, \ell \geq 0,$$

where elements $e_{i,k}$ are 0 or 1. Denote $|E| = \sum_{i,k} e_{i,k} = n+1$, $n \in \mathbb{N}$. In addition, we do not allow empty rows in E, *i.e.*, an i for which $e_{i,k} = 0$, $k = 0, 1, \ldots, \ell$. A set of nodes $T = \{t_1, \ldots, t_m\}$ consists of m distinct points of the set A that is either an interval or a circle. The elements E, T, and the data $c_{i,k}$ (defined for $e_{i,k} = 1$) determine a Hermite-Birkhoff interpolation problem which consists in finding a polynomial $p = p(t) \in \pi_n(\mathbb{C})$ that satisfies

$$p^{(k)}(t_i) = c_{i,k}, \quad e_{i,k} = 1. \tag{2.2}$$

It is easily seen that the above problem (2.2) has a unique solution if and only if the determinant

$$D(E;T) := det \left[\frac{t_i^{\alpha-k}}{(\alpha-k)!} \right]_{(i,k)\in \mathbf{e}}^{0\leq\alpha\leq n-1} \tag{2.3}$$

is nonzero, where $\mathbf{e} := \{(i,k) : e_{i,k} = 1\}$ (cf. [12] and [13]). Problem (2.2) is said to be complex poised if it has a unique solution. For Abel-Gontscharoff interpolation problem, the corresponding determinant defined as in (2.3)

$$D(E;T) = det \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ 0 & 1 & \frac{2!}{1!}t_1 & \cdots & \frac{n!}{(n-1)!}t_1^{n-1} \\ 0 & 0 & \frac{2!}{0!} & \cdots & \frac{n!}{(n-2)!}t_1^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n!}{0!} \end{pmatrix}$$

is $1!2!\cdots n!$. Therefore, Abel-Gontscharoff interpolation problem is complex poised. From [1,2], it follows that there exist a multivariate extension of Abel-Gontscharoff interpolatory operator (1.3). The multivariate extension of the operator shown in (1.3) is also denoted as Λ_n.

Theorem 1. *The multivariate Abel-Gontscharoff interpolation map can be described as*

$$\Lambda_n f(x) = \sum_{k=0}^{n} \frac{1}{k!} c(D|k; x - x^0, x - x^1, \cdots, x - x^{k-1}) f(x^k), \tag{2.4}$$

where $f : \mathbb{R}^d \to \mathbb{R}$, $x, x^0, x^1, \cdots, x^n \in \mathbb{R}^d$ are not necessary distinct, and $c(D|k; x - x^0, x - x^1, \cdots, x - x^{k-1})$ is defined by (2.1). In addition, the operator $\Lambda_n : C^n(K) \to \pi_n(\mathbb{R}^d)$ satisfies

$$q_\ell(\Lambda_n f)(x^\ell) = q_\ell(f)(x^\ell) \tag{2.5}$$

for every ℓth $(0 \le \ell \le n)$ order differential operator with constant coefficients. Moreover, these interpolation conditions completely determine the polynomial operator Λ_n.

Proof: For $f(x) = g(\lambda \cdot x)$, $g : \mathbb{R} \to \mathbb{R}$, from Eq. (2.1) we find that

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{1}{k!} c(D|k; x - x^0, x - x^1, \cdots, x - x^{k-1}) f(x^k) \\
&= \sum_{k=0}^{n} \frac{1}{k!} c(D|k; x - x^0, x - x^1, \cdots, x - x^{k-1}) g(\lambda \cdot x^k) \\
&= \sum_{k=0}^{n} \frac{1}{k!} c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{k-1}) g^{(k)}(\lambda \cdot x^k) \\
&= \Lambda_n g(\lambda \cdot x) = \Lambda_n f(x).
\end{aligned} \tag{2.6}$$

Here, the step before the last step is due to Eq. (1.3). By using (1.4) and noting that the ridge functions $g(\lambda \cdot x)$, $\lambda \in \mathbb{R}^k$, $g \in C^n(\mathbb{R})$, are dense in $C^n(K)$, we obtain (2.6) for all $f \in C^n(K)$, and this proves (2.5). As for the uniqueness of the interpolation operator, the proof is similar to the proof of Proposition 2 in [2], which is omitted here. □

Theorem 2. *For $f : C^{n+1}(K) \to \mathbb{R}$, the remainder of its (multivariate) Abel-Gontscharoff interpolation defined in Theorem 1 is*

$$\begin{aligned}
\rho_n(f)(x) &= \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} c(D|k; x - x^0, \cdots, x - x^{k-1}) \\
&\times \int_0^1 \sigma^{n-k} D_{x-x^k}^{n+1-k} f^{(k)}(\sigma x + (1-\sigma) x^k) d\sigma.
\end{aligned}$$

Proof: For $f(x) = g(\lambda \cdot x)$, we have

$$\begin{aligned}
&\rho_n(f)(x) = \rho_n(g)(\lambda \cdot x) \\
&= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{k-1}) \\
&\quad \times \Big[\sigma^{n-k}(\lambda \cdot x - \lambda \cdot x^k)^{n-k} g^{(n)}(\sigma\lambda \cdot x + (1-\sigma)\lambda \cdot x^k)\ |_0^1 \\
&\quad -(n-k)\int_0^1 \sigma^{n-1-k} D_{x-x^k}^{n-k} g^{(k)}(\sigma\lambda \cdot x + (1-\sigma)\lambda \cdot x^k)d\sigma\Big] = \frac{g^{(n)}(\lambda \cdot x)}{n!} \\
&\quad \times \sum_{k=0}^{n}\binom{n}{k}c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{k-1})(\lambda \cdot x^k - \lambda \cdot x)^{n-k} \\
&\quad - \frac{1}{n!}c(\lambda \cdot x|n; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{n-1})g^{(n)}(\lambda \cdot x^n) \\
&\quad + \frac{1}{(n-1)!}\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}c(D|k; x - x^0, \cdots, x - x^{k-1}) \\
&\quad \times \int_0^1 \sigma^{n-1-k} D_{x-x^k}^{n-k} f(\sigma x + (1-\sigma)x^k)d\sigma = \rho_{n-1}(g)(\lambda \cdot x) \\
&\quad - \frac{g^{(n)}(\lambda \cdot x^n)}{n!}c(\lambda \cdot x|n; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{n-1}). \qquad (2.7)
\end{aligned}$$

The last step of (2.7) is due to

$$\sum_{k=0}^{n}\binom{n}{k}c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{k-1})(\lambda \cdot x^k - \lambda \cdot x)^{n-k} = 0,$$

which is a special case of (1.1) with $t = 0$ and $\beta_k = \lambda \cdot x - \lambda \cdot x^k$. Repeating the procedure shown in (2.7) yields

$$\begin{aligned}
&\rho_n(g)(\lambda \cdot x) = \rho_0(g)(\lambda \cdot x) \\
&\quad - \sum_{k=1}^{n}\frac{g^{(k)}(\lambda \cdot x^k)}{k!}c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{k-1}) \\
&= \int_0^1 D_{x-x^0} g(\sigma\lambda \cdot x + (1-\sigma)\lambda \cdot x^k)d\sigma \\
&\quad - \sum_{k=1}^{n}\frac{g^{(k)}(\lambda \cdot x^k)}{k!}c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \lambda \cdot x - \lambda \cdot x^{k-1}) \\
&= g(\lambda \cdot x) - \sum_{k=0}^{n}\frac{g^{(k)}(\lambda \cdot x^k)}{k!}c(\lambda \cdot x|k; \lambda \cdot x - \lambda \cdot x^0, \cdots, \\
&\quad \lambda \cdot x - \lambda \cdot x^{k-1}) = f(x) - \Lambda_n f(x).
\end{aligned}$$

This completes the proof. □

It is obvious that if $x^0 = x^1 = \cdots = x^n$, then Theorems 1 and 2 give multivariate Taylor theorem; *i.e.*, let $f : \mathbb{R}^d \to \mathbb{R}$ be defined and have continuous partial derivatives of order $n+1$ in an open rectangle in $\mathbb{R}^d$ containing the point x^0, then

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} D^k_{x-x^0} f(x^0)$$
$$+ \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \int_0^1 \sigma^{n-k} D^{n+1}_{x-x^0} f(\sigma x + (1-\sigma)x^0) d\sigma.$$

Denote by $\Gamma \equiv (\Gamma; +, \cdot)$ the commutative ring of formal power series over $\mathbb{R}^d$, in which formal differentiation and integration of power series are defined as usual (cf. Comtet [1]).

Theorem 3. *Let $\alpha_k \in \mathbb{R}^d$, $\beta_k \in \mathbb{R}^d$ $(k = 0, 1, 2 \cdots)$, and let $c(D|0; \beta) = 1$ and $c(D|k; \beta)$ be defined as in (2.1). Then for any $f \in \Gamma$ we have formally*

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} c(D|k; x-\alpha_0, \cdots, x-\alpha_{k-1}) f(\alpha_k), \tag{2.8}$$

and

$$f(x) = \sum_{k=0}^{\infty} \frac{c(D|k; \beta)}{k!} \sum_{m=0}^{\infty} \frac{1}{m!} D^m_{x-\beta_k} f(0), \tag{2.9}$$

where $D_b^{(k)}(a)$ denotes the kth formal directional derivative of $f(x)$ at $x = a$ along b $(x, a, b \in \mathbb{R}^d)$.

Proof: Both (2.8) and (2.9) could be formally verified by using the identity (1.1) and the substitutions $\beta_j = z - \alpha_j$ $(j \geq 0)$ as follows. For $f(x) = g(\lambda \cdot x)$ $(\lambda, x \in \mathbb{R}^d)$, from the right-hand side of Eq. (2.8), we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} c(D|k; x-\alpha_0, \cdots, x-\alpha_{k-1}) f(\alpha_k)$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} c(\lambda \cdot x|k; x-\alpha_0, \cdots, x-\alpha_{k-1}) g^{(k)}(\lambda \cdot \alpha_k)$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} c(\lambda \cdot x|k; x-\alpha_0, \cdots, x-\alpha_{k-1}) \sum_{j=0}^{\infty} \frac{g^{(k+j)}(0)}{j!} (\lambda \cdot \alpha_k)^j$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} c(\lambda \cdot x | k; x - \alpha_0, \cdots, x - \alpha_{k-1}) \sum_{j=k}^{\infty} \frac{g^j(0)}{(j-k)!} (\lambda \cdot x - \lambda \cdot \beta_k)^{j-k}$$

$$= \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!} \sum_{k=0}^{j} \binom{j}{k} c(\lambda \cdot x | k; x - \alpha_0, \cdots, x - \alpha_{k-1}) (\lambda \cdot x - \lambda \cdot \beta_k)^{j-k}$$

$$= \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!} (\lambda \cdot x)^j = g(\lambda \cdot x) = f(x). \ \Box$$

At the end of this paper, we will discuss briefly the construction of a type of multivariate identities by using Gould's polynomials. Recalling the notation for multinomial coefficient

$$\binom{k}{j_1, \cdots, j_r} = \frac{k!}{j_1! \cdots j_r!}, \quad (k = j_1 + \cdots + j_r),$$

we can rewrite expression (1.5) for $c(a|k; \beta) = c(a|k; \beta_0, \cdots, \beta_{k-1})$ in the form

$$c(a|k; \beta) = \sum_{(k; r \geq 1)} (-1)^{k+r} \binom{k}{j_1, \cdots, j_r} \beta_0^{j_1} \beta_{j_1}^{j_2} \beta_{j_1+j_2}^{j_3} \cdots \beta_{j_1+\cdots+j_{r-1}}^{j_r}, \tag{2.10}$$

where the summation is taken over the set, denoted by $(k, r \geq 1)$, of all positive integer compositions $(j_1, \cdots, j_r)$ of k into r parts with $r \geq 1$, or in other words, over all the positive integer solutions of the equation $j_1 + \cdots + j_r = k$ for $r = 1, \cdots, k$.

We can find multivariate identities by using expression (2.10). For instance, for any number $\lambda, z, t \in \mathbb{R}^d$, we have the identity

$$\sum_{(k; r \geq 1)} (-1)^r \binom{k}{j_1, \cdots, j_r} (\lambda \cdot z)^{j_1} (\lambda \cdot z + j_1 \lambda \cdot t)^{j_2} \cdots$$
$$(\lambda \cdot z + (j_1 + \cdots + j_{r-1}) \lambda \cdot t)^{j_r} = (-1)^k (\lambda \cdot z)(\lambda \cdot z + k \lambda \cdot t)^{k-1} \tag{2.11}$$

Actually this is a consequence of (2.10) for the case $\beta_k = \lambda \cdot z + k \lambda \cdot t$ which leads to $c(\lambda \cdot z | k; \beta) = (\lambda \cdot z)(\lambda \cdot z + k \lambda \cdot t)^{k-1}$, which yields the following multivariate Abel's identity from (1.1), which is a multivariate extension of the wellknown Abel's identity in univariate setting (cf. [8] and [15]).

$$\sum_{k=0}^{n} \binom{n}{k} (\lambda \cdot z)(\lambda \cdot z + k \lambda \cdot t)^{k-1} (\lambda \cdot x - \lambda \cdot z - k \lambda \cdot t)^{n-k} = (\lambda \cdot x)^n.$$

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Tian-Xiao He
Department of Mathematics and Computer Science
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
the@iwu.edu