## Chapter 6

# MRA Frame Wavelets with Certain Regularities Associated with the Refinable Generators of Shift Invariant Spaces 

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Abstract
In this paper, we will start the discussion with the refinable generators of the shift invariant (SI) spaces in $L^{2}(R)$ that possess the largest possible regularities and required vanishing moments. For the pseudo-scaling generators, the corresponding MRA frame wavelets with certain regularities are constructed. In addition, the stability of the refinable SI spaces and the corresponding complementary spaces, biorthogonality of the SI spaces, and the approximation property of the spaces are also discussed.
Keywords: Shift invariant spaces, frames, tight frame wavelets, MRA frame wavelets, pseudo-scaling generators (functions), stability, biorthogonality, stable basis.

### 6.1 Introduction

[10] pointed out that to achieve higher regularity by increasing vanishing moments of scaling functions and wavelets is not efficient, because $80 \%$ of zero moments are wasted. In this paper, we give a method for constructing pseudo-scaling functions and the corresponding MRA frame wavelets with the largest possible regularities and required vanishing moments.

We start by setting some notations. Define low-pass filter as

$$
\begin{equation*}
m_{0}(\xi)=2^{-1} \sum_{n} h_{n} e^{-i n \xi} . \tag{1.1}
\end{equation*}
$$

Here, we assume that only finitely many $h_{n}$ are non-zero. However, some of our results can be extended to infinite sequences that have sufficient decay for $|n| \rightarrow \infty$. Next, we define $\phi$ by

$$
\begin{equation*}
\hat{\phi}(\xi)=\Pi_{j=1}^{\infty} m_{0}\left(2^{-j} \xi\right) \tag{1.2}
\end{equation*}
$$

This infinite product converge only if $m_{0}(0)=1$ i.e., if $\sum_{n} h_{n}=2$. In this case, the infinite products in (1.2) converge uniformly and absolutely on compact sets, so that $\hat{\phi}$ is well-defined $C^{\infty}$ functions. Obviously,

$$
\begin{equation*}
\hat{\phi}(\xi)=m_{0}(\xi / 2) \hat{\phi}(\xi / 2) \tag{1.3}
\end{equation*}
$$

or, equivalently, $\phi(t)=\sum_{n} h_{n} \phi(2 t-n)$ at least in the sense of distributions. From Lemma 3.1 in [3], $\phi$ has a compact support.

A shift invariant (SI) space is a closed subspace of $L_{2}(\mathbf{R})$ that is invariant under the operator $S_{k}(f):=f(\cdot-k)(k \in \mathbf{Z})$. For $\phi \in L_{2}(\mathbf{R})$, we say that $V=S(\phi):=$ $\overline{\operatorname{span}}\{\phi(\cdot-k) \mid k \in \mathbf{Z}\}$ is generated by $\phi$. In addition, if $\phi$ is refinable, then $\phi$ is said to be a refinable generator of $S(\phi)$, and $S(\phi)$ is called a refinable SI space. Each element $\phi \in \Phi$ is a refinable generator of the corresponding SI space $S(\phi)$. A refinable generator is said to be a pseudo-scaling (refinable) generator (see [19]) if it satisfies Equation (1.3) and

$$
\begin{equation*}
\left|m_{0}(\xi)\right|^{2}+\left|m_{0}(\xi+\pi)\right|^{2}=1 \tag{1.4}
\end{equation*}
$$

We now consider the simplest possible masks $m_{0}(\xi)$ of refinable generators with the following form.
Definition 1. Denote by $\Phi$ the set of all functions $\phi(t)$ that have Fourier transform $\hat{\phi}(\xi)=m_{0}(\xi / 2) \hat{\phi}(\xi / 2)$. The filter $m_{0}(\xi)=2^{-1} \sum_{n} h_{n} e^{-i n \xi}$ is in set $M$ that contains all filters with the form

$$
\begin{equation*}
m_{0}^{N}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{N} F(\xi) \tag{1.5}
\end{equation*}
$$

where

$$
\begin{equation*}
F(\xi)=e^{-i k^{\prime} \xi} \sum_{j=0}^{k} a_{j} e^{-i j \xi} \tag{1.6}
\end{equation*}
$$

Here, all coefficients of $F(\xi)$ are real, $F(0)=1 ; N$ and $k$ are positive integers; and $k^{\prime} \in \mathbf{Z}$. Hence, the corresponding $\phi$ can be written as follows.

$$
\begin{equation*}
\hat{\phi}(\xi)=\left(\frac{1+e^{-i \xi / 2}}{2}\right)^{N} F(\xi / 2) \hat{\phi}(\xi / 2) \tag{1.7}
\end{equation*}
$$

Clearly, $\phi$ is a B-spline of order $N$ if $F(\xi)=1$. The vanishing moments of $\phi$ are completely controlled by the exponents of its "spline factor," $\left(\frac{1+e^{-i \xi}}{2}\right)^{N}$. In
addition, the regularity of $\phi$ is justified by the factors $F(\xi)$, and are independent of its vanishing moments.

A frame in a Hilbert space $H$ is a family $\left\{f_{n}, n \in I\right\}$ of elements in $H$ for which there exist two positive constants, $0<A \leq B<\infty$, such that

$$
\begin{equation*}
A\|f\|^{2} \leq \sum_{n \in I}\left|\left\langle f, f_{n}\right\rangle\right|^{2} \leq B\|f\|^{2}, \tag{1.8}
\end{equation*}
$$

for all $f \in H$. If $A=B,\left\{f_{n}\right\}$ is called a tight frame. For tight frames, the frames constants $A=B$ can be assumed to be 1 , simply by dividing each frame generator $f_{n}$ by $\sqrt{A}$. In other words, the tight frames can be defined by

$$
\begin{equation*}
\sum_{n \in I}\left|\left\langle f, f_{n}\right\rangle\right|^{2}=\|f\|^{2} . \tag{1.9}
\end{equation*}
$$

The index set $I$ for the family $\left\{f_{n}\right\}$ can be quite general. We assume it to be countable and, in particular, we are often dealing with the case where $\left\{f_{n}\right\}$ is the sequence of translates $\{\phi(\cdot-n)\}$ of a function $\phi \in H \subset L_{2}(\mathbf{R}), n \in \mathbf{Z}$, or the sequence of $\left\{2^{j / 2} \phi\left(2^{j} \cdot-k\right)\right\}$ with the index set consists of the pairs $(j, k) \in \mathbf{Z} \times \mathbf{Z}$. Definition 2. [14, 17] A function $\psi \in L_{2}(\mathbf{R})$ is a tight frame wavelet (TFW) if the system $\left\{\psi_{j k}\right\}_{j, k \in \mathbf{Z}}, \psi_{j k}(x)=2^{j / 2} \psi\left(2^{j} x-k\right)$, is a tight frame for $L_{2}(\mathbf{R})$; that is, for all $f \in L_{2}(\mathbf{R})$,

$$
\begin{equation*}
\sum_{j, k \in \mathbf{z}}\left|\left\langle f, \psi_{j k}\right\rangle\right|^{2}=\|f\|_{2}^{2} . \tag{1.10}
\end{equation*}
$$

The applications of TFW are based on the following expression that is equivalent to the condition (1.10) (see [15], p.334).

$$
\begin{equation*}
f(t)=\sum_{j, k \in \mathbf{z}}\left\langle f, \psi_{j k}\right\rangle \psi_{j k}(t), \tag{1.11}
\end{equation*}
$$

where the equation holds unconditionally for all $f \in L_{2}(\mathbf{R})$.
Definition 3. A TFW $\psi$ is an MRA (Multiresolution Analysis) TFW if there exists a pseudo-scaling generator defined by Equation (1.3) such that

$$
\begin{equation*}
\hat{\psi}(\xi)=e^{i \xi / 2} \overline{m_{0}(\xi / 2+\pi)} \hat{\phi}(\xi / 2) ; \tag{1.12}
\end{equation*}
$$

i.e., the symbol of $\psi$ is $m_{1}(\xi)=e^{i \xi / 2} \overline{m_{0}(\xi / 2+\pi)}$.

Since vanishing moment conditions $\int x^{\ell} \psi(x) d x=0, \ell=0,1, \cdots, L$, are equivalent to $\left.\frac{d^{\ell}}{d \xi^{\ell}} \hat{\psi}\right|_{\xi=0}=0, \ell=0,1, \cdots, L$, we immediately know that the maximum order of vanishing moment for $\psi$ is $N-1$ if $\psi$ is given by Equation (1.12) and $\phi$ is defined by Equation (1.7). Therefore, the vanishing moments of $\phi$ are completely controlled by the exponents of its respective "spline factors," $\left(\frac{1+e^{-i \xi}}{2}\right)^{N}$. In addition, the regularities of $\psi$ is justified by the factors $F(\xi)$, and are independent of their vanishing moments.

From [14, 17], we have the following result regarding the characterization of TFWs.
Theorem 4. [14, 17] A function $\psi \in L_{2}(\mathbf{R})$ is a TFW if and only if $\psi$ satisfies

$$
\begin{equation*}
\sum_{j \in \mathbf{Z}}\left|\hat{\psi}\left(2^{j} \xi\right)\right|^{2}=1 \quad \text { a.e. } \tag{1.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j \geq 0} \hat{\psi}\left(2^{j} \xi\right) \overline{\hat{\psi}\left(2^{j}(\xi+2 q \pi)\right)}=0 \quad \text { a.e. } \tag{1.14}
\end{equation*}
$$

for all $q \in 2 \mathbf{Z}+1$; i.e., for all odd integers, $q$.
In [2], [3], and [11], the following concepts were introduced that are important in our discussion.
Definition 5. Denote $\mathbf{T}=[0,2 \pi)$. The bracket operator $[]:, L_{2}(\mathbf{R}) \times L_{2}(\mathbf{R}) \rightarrow$ $L_{1}(\mathbf{T})$ is defined by

$$
\begin{equation*}
[f, g]=\sum_{k \in \mathbf{Z}} f(\xi+2 \pi k) \overline{g(\xi+2 \pi k)} \tag{1.15}
\end{equation*}
$$

For $f \in L_{2}(\mathbf{R})$, the function $[f, f] \in L_{1}(\mathbf{T})$ is called the auto-correlation of $f$.
If $f, g$ are compactly supported, then $[\hat{f}, \hat{g}]$ is a trigonometric polynomial and has the Fourier expansion

$$
\begin{equation*}
[\hat{f}, \hat{g}](\xi)=\sum_{k \in \mathbf{Z}}\langle f(\cdot), g(\cdot+k)\rangle e^{i k \xi} \tag{1.16}
\end{equation*}
$$

Definition 6. Let $S(\phi)$ be a shift invariant space that is generated by $\phi .\{\phi(\cdot-$ $k)\}_{k \in \mathbf{Z}}$ is called a stable basis of $S(\phi)$ if there exist constants $0<A \leq B<\infty$ such that for every $\mathbf{c}=\left\{c_{k}\right\}_{k \in \mathbf{Z}} \in \ell_{2}(\mathbf{Z})$

$$
\begin{equation*}
A\|\mathbf{c}\|_{\ell_{2}(\mathbf{Z})}^{2} \leq\left\|\sum_{k \in \mathbf{Z}} c_{k} \phi(\cdot-k)\right\|_{L_{2}(\mathbf{R})}^{2} \leq B\|\mathbf{c}\|_{\ell_{2}(\mathbf{Z})}^{2} \tag{1.17}
\end{equation*}
$$

Obviously, a stable basis of $S(\phi)$ is a basis of $S(\phi)$.
Theorem 7. [17] Let $\phi \in L_{2}(\mathbf{R})$ and let $0<A \leq B<\infty$. Inequality (1.17) is equivalent to

$$
A \leq[\hat{\phi}, \hat{\phi}] \leq B, \text { a.e. }
$$

In Section 2, we will give the conditions for the coefficients $\left\{a_{j}\right\}$ such that the corresponding refinable pseudo-scaling generator $\phi$ is in $L_{2}(\mathbf{R})$ and the corresponding $\psi$ defined by Equation (1.12) is an MRA TFW. Section 3 discusses the stability of the SI space, $S(\phi)$, generated by $\phi$ and the stability of the corresponding complementary spaces. We then construct stable scaling function $\tilde{\phi} \in \Phi$ such that $\{\phi(\cdot-k)\}_{k \in Z}$ and $\{\tilde{\phi}(\cdot-k)\}_{k \in Z}$ are biorthogonal. Thus, $\{\phi(\cdot-k)\}_{k \in Z}$ and $\{\tilde{\phi}(\cdot-k)\}_{k \in Z}$ are
stable bases in the subspace that they generate. In addition, biorthogonal wavelets that possess the largest possible regularities and required vanishing moments can be obtained accordingly. We will also discuss the $L_{2}$ approximation from $S(\phi)$ in the section.

### 6.2 MRA TFWs with certain regularities

We construct pseudo-scaling generators and MRA TFWs using the following theorem.
Theorem 8. Let $\phi \in \Phi$ be defined by Definition 1; i.e., $\phi=\Pi_{j=1}^{\infty} m_{0}^{N}\left(2^{-j} \xi\right)$, where $m_{0}^{N}(\xi) \in M$ is defined by Equations (1.5) and (1.6): $m_{0}^{N}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{N} F(\xi)$ and $F(\xi)=e^{-i k^{\prime} \xi} \sum_{j=0}^{k} a_{j} e^{-i j \xi}, N, k \in \mathbf{Z}_{+}$and $k^{\prime} \in \mathbf{Z}$, where $F(0)=1$. If $F$ also satisfies $F(\pi) \neq-1$ and its coefficients satisfy

$$
\begin{equation*}
(k+1) \sum_{j=0}^{k} a_{j}^{2}<2^{2 N-1}, \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=k^{\prime}}^{N+k+k^{\prime}} \sum_{\ell=0}^{k} \sum_{\tilde{\ell}=0}^{k}\binom{N}{j-\tilde{\ell}-k^{\prime}}\binom{N}{j+2 n-\ell-k^{\prime}} a_{\tilde{\ell}} a_{\ell}=2^{2 N-1} \delta_{n 0}, \tag{2.2}
\end{equation*}
$$

where $\delta_{n 0}$ is the Kronecker symbol and $n=0, \pm 1, \pm 2, \cdots$, then $\phi \in L^{2}(\mathbf{R})$ and is a pseudo-scaling generator. In addition, The corresponding $\psi$ defined by (1.12) is an MRA TFW in $C^{\alpha}$. Here $\alpha$ is more than

$$
N-\frac{1}{2} \log _{2}\left((k+1) \sum_{j=0}^{k} a_{j}^{2}\right) .
$$

Condition (2.1) can be replaced by the following weaker condition.

$$
\begin{equation*}
C\left(\left\{a_{j}\right\}, k\right)<2^{2 N-1}, \tag{2.1}
\end{equation*}
$$

where $C\left(\left\{a_{j}\right\}, k\right)$ equals $k \sum_{j=0}^{k} a_{j}^{2}$ if $k \geq 1$ and equals $a_{0}^{2}$ if $k=0$. Hence, the corresponding regularities of $\psi$ is determined by $\psi \in C^{\alpha^{\prime}}$, where $\alpha^{\prime}$ is more than $N-\frac{1}{2} \log _{2}\left(2 C\left(\left\{a_{j}\right\}, k\right)\right)$.

We shall break up the proof of Theorem 8 into several lemmas, in which Lemma 9 is given in [13]. Here, we provide a simpler alternative proof. Lemmas 10 and 11 are also shown in [13]. However, we list them as follows for readers' convenience and the completeness of the paper.
Lemma 9. Assume that $\phi \in \Phi$ that is defined by Equation (1.7) in Definition 1. If $F(\pi) \neq-1$ and

$$
C\left(\left\{a_{j}\right\}, k\right)<2^{2 N-1}
$$

then $\phi=\Pi_{j=1}^{\infty} m_{0}^{N}\left(2^{-j} \xi\right)$ is in $L^{2}(\mathbf{R})$.
Proof. It is sufficient to prove the boundedness of the following integral

$$
\begin{align*}
& \int_{|\xi| \geq \pi}|\hat{\phi}(\xi)|^{2} d \xi= \\
& =\sum_{\ell=1}^{\infty} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \Pi_{j=1}^{\infty}\left|\left(\frac{1+e^{-i 2^{-j} \xi}}{2}\right)^{N} F\left(2^{-j} \xi\right)\right|^{2} d \xi \\
& =\sum_{\ell=1}^{\infty} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi}\left|\frac{1-e^{-i \xi}}{i \xi}\right|^{2 N} \Pi_{j=1}^{\infty}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi  \tag{2.3}\\
& \leq C \sum_{\ell=1}^{\infty} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \frac{1}{|\xi|^{2 N}} \Pi_{j=1}^{\infty}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi \\
& \leq C \sum_{\ell=1}^{\infty} \frac{1}{2^{2 \ell N}} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \Pi_{j=1}^{\ell}\left|F\left(2^{-j} \xi\right)\right|^{2} \Pi_{j=1}^{\infty}\left|F\left(2^{-\ell-j} \xi\right)\right|^{2} d \xi \\
& \leq C \sum_{\ell=1}^{\infty} 4^{-\ell N} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \Pi_{j=1}^{\ell}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi
\end{align*}
$$

We now prove the boundedness of the last integral in inequality (2.3). Denote

$$
\begin{equation*}
T f(\xi)=\left|F\left(\frac{\xi}{2}\right)\right|^{2} f\left(\frac{\xi}{2}\right)+\left|F\left(\frac{\xi}{2}+\pi\right)\right|^{2} f\left(\frac{\xi}{2}+\pi\right) \tag{2.4}
\end{equation*}
$$

Hence, for any $2 \pi$-periodic continuous function $f$, we have

$$
\begin{gather*}
\int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} f\left(2^{-\ell} \xi\right) \Pi_{j=1}^{\ell}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi \\
=\int_{-\pi}^{\pi} T^{\ell} f(\xi) d \xi \leq \sqrt{2 \pi}\left\|T^{\ell} f\right\|_{L^{2}} \leq \sqrt{2 \pi}| | f\left\|_{L^{2}}\right\| T^{\ell} \| \tag{2.5}
\end{gather*}
$$

Let $\rho(T)$ be the spectral radius of the operator $T$. Since $F(0)=1$ and $F(\pi) \neq-1$, we have $\rho(T)>0$ (also see [8]). In fact, considering the Fourier expansion

$$
|F(\xi)|^{2}=\sum_{\ell=-k}^{k} b_{\ell} e^{i \ell \xi}
$$

where $k$ is a positive integral and

$$
b_{t}=\sum_{j=0}^{k-|t|} a_{k-|t|-j} a_{k-j}
$$

$(t=-k, \cdots, k)$, we find that the matrix of $T$ restricted to

$$
E_{k}=\left\{\sum_{\ell=-k}^{k} c_{\ell} e^{i \ell \xi}:\left(c_{-k}, \ldots, c_{k}\right) \in C^{2 k+1}\right\}
$$

is

$$
M_{T}=\left(2 b_{i-2 j}\right)_{i, j=-k, \ldots, k}=2\left[\begin{array}{ccccc}
b_{k} & 0 & 0 & \cdots & 0  \tag{2.6}\\
b_{k-2} & b_{k-1} & b_{k} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{-k} & b_{-k+1} & b_{-k+2} & \cdots & b_{k} \\
0 & 0 & b_{-k} & \cdots & b_{k-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & b_{-k}
\end{array}\right] .
$$

Noting that $|F(0)|^{2}=\sum_{\ell=-k}^{k} b_{\ell}=1$ and $|F(\pi)|^{2}=\sum_{\ell=-k}^{k}(-1)^{\ell} b_{\ell}=\alpha \neq-1$, it follows that

$$
\sum_{\ell} b_{2 \ell}=\sum_{\ell} b_{2 \ell+1}=(\alpha+1) / 2
$$

and for the vector $u=(1, \ldots, 1) \in C^{2 k+1}$,

$$
T u=u M=(\alpha+1) u
$$

Thus, $T$ has at least one eigenvalue $\alpha+1 \neq 0$.
For every $\epsilon>0$, there is an integer $\ell(\epsilon)$ such that

$$
\left\|T^{\ell}\right\| \leq(\rho(T)+\epsilon)^{\ell}, \quad \ell>\ell(\epsilon)
$$

It follows from (2.3) that

$$
\int_{|\xi| \geq \pi}|\hat{\phi}(\xi)|^{2} d \xi \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N \ell}\left\|T^{\ell}\right\|+C \sum_{\ell=\ell(\epsilon)+1}^{\infty} 4^{-N \ell}(\rho(T)+\epsilon)^{\ell}
$$

so $\rho(T)$ must be estimated if we are to choose an $\epsilon>0$ small enough for the series to converge. Regardless of how small an $\epsilon>0$ is chosen, the contribution

$$
C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N \ell}\left\|T^{\ell}\right\| \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N \ell}\|T\|^{\ell}
$$

is finite, although possibly very large.
To evaluate $\rho(T)$, we consider the matrix of $T, M_{T}$, which was shown in Equation (2.6). It is clear that $\rho(T)=\rho\left(M_{T}\right)$. We write $M_{T}=2 H$, where

$$
H=\left(b_{i-2 j}\right)_{i, j=-k, \cdots, k}
$$

Obviously, $b_{\beta}$ can be written as

$$
b_{\beta}=\sum_{j=0}^{k-|\beta|} a_{k-|\beta|-j} a_{k-j}, \quad \beta=-k, \cdots, k
$$

Hence, $b_{\beta}=b_{-\beta}$, for all $\beta=-k, \cdots, k$. It is also clear that $b_{k}$ is an eigenvalue of $H$ with multiplicity 2 . To estimate bounds of the eigenvalues of $H$, we establish

$$
\left|b_{\beta}\right| \leq b_{0}, \quad \beta=-k, \cdots, k
$$

In fact,

$$
\begin{aligned}
\left|b_{\beta}\right| \leq & \sum_{j=0}^{k-|\beta|}\left|a_{k-|\beta|-j} a_{k-j}\right| \leq \sum_{j=0}^{k-|\beta|}\left[\frac{1}{2} a_{k-|\beta|-j}^{2}+\frac{1}{2} a_{k-j}^{2}\right]= \\
& =\frac{1}{2} \sum_{j=\beta}^{k} a_{k-j}^{2}+\frac{1}{2} \sum_{j=0}^{k-\beta} a_{k-j}^{2} \leq \sum_{j=0}^{k} a_{k-j}^{2}=b_{0}
\end{aligned}
$$

It is obvious that the spectral radius of $H$ is $b_{0}$ if $k=0$. For $k \geq 1$, the characteristic polynomial of H is $\left(b_{k}-\lambda\right)\left(b_{-k}-\lambda\right)$ multiplied by the characteristic polynomial of the core matrix, $H_{c}$, which consists of all rows and columns of H except its first and last rows and columns. Hence, the spectral radius of $H$ is

$$
\begin{gathered}
\rho(H)=\max \left\{b_{k}, \rho\left(H_{c}\right)\right\} \leq \max \left\{b_{k},\left\|H_{c}\right\|_{1}\right\} \\
=\max \left\{b_{k}, \sum_{i=-k+1}^{k-1}\left|b_{i-2 j}\right|: j=-k+1, \cdots, k-1\right\} \leq k b_{0}
\end{gathered}
$$

Therefore, $\rho(T)=2 \rho(H) \leq 2 C\left(\left\{a_{j}\right\}, k\right)$. Here, $C\left(\left\{a_{j}\right\}, k\right)$ was defined in Theorem 7. If $C\left(\left\{a_{j}\right\}, k\right)<2^{2 N-1}$, then $\rho(T)<2^{2 N}$, so we choose

$$
\epsilon=\frac{1}{2}\left(2^{2 N}-\rho(T)\right)
$$

Thus

$$
\rho(T)+\epsilon<2^{2 N}
$$

and we obtain the estimation

$$
\int_{|\xi| \geq \pi}|\hat{\phi}(\xi)|^{2} d \xi \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N \ell}\|T\|^{\ell}+\sum_{\ell=\ell(\epsilon)+1}^{\infty}\left(\frac{\rho(T)+\epsilon}{4^{N}}\right)^{\ell}
$$

The tail of the series is a convergent geometric series, thus completing the proof of $\phi \in L_{2}(\mathbf{R})$ if condition $(2.1)^{\prime}, C\left(\left\{a_{j}\right\}, k\right)<2^{2 N-1}$, holds.

Obviously, $C\left(\left\{a_{j}\right\}, k\right) \leq(k+1) \sum_{j=0}^{k} a_{j}^{2}$. Hence, we have proved the Lemma 9 and the first part of Theorem 8; i.e., $\phi \in L^{2}(\mathbf{R})$ under condition (2.1) or condition (2.1)'.

It is well-known (see $[1,4]$ ) that $\psi \in C^{r}$ and $r=N-\frac{1}{2} \log _{2}(\rho(T))$, where $\rho(T)$ is the spectral radius of the matrix representing operator $T$, which is defined by equation (2.4). Hence, we obtain the following result.
Lemma 10. $\psi \in C^{\alpha^{\prime}}$, where $\alpha^{\prime}$ is more than

$$
N-\frac{1}{2} \log _{2}\left(2 C\left(\left\{a_{j}\right\}, k\right)\right)
$$

Obviously, if $\langle\phi(t), \phi(t-n)\rangle=0$, then

$$
\begin{equation*}
\frac{1}{2} \sum_{j} h_{j} h_{j+2 n}=\delta_{n 0} \tag{2.7}
\end{equation*}
$$

where $n$ is an integer. From equations (2.7), noting that $h_{j}=\left(\sum_{\ell=0}^{k}\binom{N}{j-\ell-k^{\prime}}\right.$ $\left.a_{\ell}\right) / 2^{N-1}, j=k^{\prime}, \cdots, N+k+k^{\prime}$, we immediately obtain the following lemma.
Lemma 11. If $\phi$ defined by (1.7) satisfies $\langle\phi(t), \phi(t-n)\rangle=\delta_{n 0}$, then

$$
\begin{equation*}
\sum_{j=k^{\prime}}^{N+k+k^{\prime}} \sum_{\ell=0}^{k} \sum_{\tilde{\ell}=0}^{k}\binom{N}{j-\tilde{\ell}-k^{\prime}}\binom{N}{j+2 n-\ell-k^{\prime}} a_{\tilde{\ell}} a_{\ell}=2^{2 N-1} \delta_{n 0} \tag{2.8}
\end{equation*}
$$

where $\delta_{n 0}$ is the Kronecker symbol and $n=0, \pm 1, \pm 2, \cdots$.
For sufficient conditions of $\langle\phi(t), \phi(t-n)\rangle=\delta_{n 0}$, we have the following result.
Lemma 12. Let $\phi$ be the function defined by (1.7). Assume that $\phi$ satisfies (2.8) and either
(i) $\quad \sum_{j=0}^{k}\left|a_{j}\right|<2^{N-1 / 2}$,
or

$$
\text { (ii) } \quad(k+1) \sum_{j=0}^{k} a_{j}^{2}<2^{2 N-1}
$$

Then $\langle\phi(t), \phi(t-n)\rangle=\delta_{n 0}$, and $|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-1 / 2-\alpha}$, where $\alpha=N-1 / 2-$ $\log _{2} B_{2^{n}}, B_{2^{n}}=\max _{\xi}\left|\Pi_{j=0}^{2^{n}-1} F\left(2^{j} \xi\right)\right|^{1 / 2^{n}}$.
Proof. If

$$
\begin{equation*}
B_{2^{n}}=\max _{\xi}\left|\Pi_{j=0}^{2^{n}-1} F\left(2^{j} \xi\right)\right|^{1 / 2^{n}}<2^{N-1 / 2} \tag{2.9}
\end{equation*}
$$

for some integer $n>0$, then by using Propositions 4.8 and 4.9 in [3] we have $\langle\phi(t), \phi(t-n)\rangle=\delta_{n 0}$ and $|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-1 / 2-\alpha}$.

On the other hand, $B_{2^{n}} \leq \max _{\xi}|F(\xi)|=\sum_{j=0}^{k}\left|a_{j}\right|$. Hence, by using the CauchySchwartz inequality, we obtain

$$
B_{2^{n}} \leq\left[(k+1) \sum_{j=0}^{k} a_{j}^{2}\right]^{1 / 2}
$$

where the equal sign does not hold if $\left\{a_{j}\right\}$ satisfies condition (2.8), because all $\left\{a_{j}\right\}$ can not be the same. Hence, if $(k+1) \sum_{j=0}^{k} a_{j}^{2}<2^{2 \tilde{N}-1}$, then $B_{2^{n}}<2^{\tilde{N}-1 / 2}$. This completes the proof of Lemma 12.
Proof of Theorem 8. Let $\phi \in \Phi$ be the function defined by (1.7) (in Definition 1) that satisfies $F(\pi) \neq-1$ and the Condition (2.1). Then, from Lemma $9, \phi$ is in $L_{2}(\mathbf{R})$. Noting Lemmas 11 and 12, we have that Conditions (2.1) and (2.2) imply $\langle\phi(t), \tilde{\phi}(t-n)\rangle=\delta_{n 0}$, which is equivalent to

$$
\sum_{k \in \mathbf{Z}}|\hat{\phi}(\xi+2 k \pi)|^{2}=1 \quad \text { for a.e. } \xi \in \mathbf{R} .
$$

By using (1.3), the above equation can be written as an equivalent form (1.4). Hence, $\phi$ is a pseudo-scaling generator. To prove that $\psi$ defined in (1.12) is an MRA TFW, it is sufficient to prove that $\psi$ is a TFW; i.e., it satisfies Equations (1.13) and (1.14). From (1.12) and (1.4),

$$
\begin{gathered}
\sum_{j \in \mathbf{Z}}\left|\hat{\psi}\left(2^{j} \xi\right)\right|^{2}= \\
=\sum_{j \in \mathbf{Z}}\left|m_{0}\left(2^{j-1} \xi+\pi\right)\right|^{2}\left|\hat{\phi}\left(2^{j-1} \xi\right)\right|^{2} \\
=\lim _{n \rightarrow \infty} \sum_{j=-n}^{n}\left|m_{0}\left(2^{j-1} \xi+\pi\right)\right|^{2}\left|\hat{\phi}\left(2^{j-1} \xi\right)\right|^{2}= \\
=\lim _{n \rightarrow \infty} \sum_{j=-n}^{n}\left[1-\left|m_{0}\left(2^{j-1} \xi\right)\right|^{2}\right]\left|\hat{\phi}\left(2^{j-1} \xi\right)\right|^{2}= \\
=\lim _{n \rightarrow \infty}\left[\left|\hat{\phi}\left(2^{-n-1} \xi\right)\right|^{2}-\left|\hat{\phi}\left(2^{n} \xi\right)\right|^{2}\right] .
\end{gathered}
$$

Since $\phi \in L_{2}(\mathbf{R}), \lim _{n \rightarrow \infty}\left|\hat{\phi}\left(2^{n} \xi\right)\right|^{2}=0$ for a.e. $\xi$. From condition $F(0)=1$, we have $\lim _{n \rightarrow \infty}\left|m_{0}\left(2^{-n} \xi\right)\right|=1$. Hence, taking limit $n \rightarrow \infty$ on the both sides of Equation (1.2) and noting that

$$
\lim _{n \rightarrow \infty}|\hat{\phi}(\xi)|=\lim _{n \rightarrow \infty} \Pi_{j=1}^{\infty}\left|m_{0}\left(2^{-j} \xi\right)\right| \neq 0
$$

we obtain (see [19]) $\lim _{n \rightarrow \infty}\left|\hat{\phi}\left(2^{-n} \xi\right)\right|=1$, which derives $\sum_{j \in \mathbf{Z}}\left|\hat{\psi}\left(2^{j} \xi\right)\right|^{2}=1$. Therefore, Equation (1.13) holds for our $\psi$.

For any odd integer $q$,

$$
\begin{gathered}
\sum_{j \geq 0} \hat{\psi}\left(2^{j} \xi\right) \overline{\hat{\psi}\left(2^{j}(\xi+2 q \pi)\right)}= \\
=\sum_{j>0} e^{i 2^{j-1} \xi} \overline{m_{0}\left(2^{j-1} \xi+\pi\right)} \hat{\phi}\left(2^{j-1} \xi\right) \overline{e^{i 2^{j-1} \xi \overline{m_{0}\left(2^{j-1} \xi+\pi\right)} \hat{\phi}\left(2^{j-1} \xi+2^{j} q \pi\right)}+} \\
+e^{i 2^{-1} \xi} \overline{m_{0}\left(2^{-1} \xi+\pi\right)} \hat{\phi}\left(2^{-1} \xi\right) \overline{e^{i 2^{-1} \xi+\pi} \overline{m_{0}\left(2^{-1} \xi\right)} \hat{\phi}\left(2^{-1} \xi+q \pi\right)}= \\
=\sum_{j>0}\left|m_{0}\left(2^{j-1} \xi+\pi\right)\right|^{2} \hat{\phi}\left(2^{j-1} \xi\right) \overline{\hat{\phi}\left(2^{j-1} \xi+2^{j} \pi\right)}- \\
-\overline{m_{0}\left(2^{-1} \xi+\pi\right)} \hat{\phi}\left(2^{-1} \xi\right) m_{0}\left(2^{-1} \xi\right) \overline{\hat{\phi}\left(2^{-1} \xi+\pi\right)}= \\
=\sum_{j>0}\left[1-\left|m_{0}\left(2^{j-1} \xi\right)\right|^{2}\right] \hat{\phi}\left(2^{j-1} \xi\right) \overline{\hat{\phi}\left(2^{j-1} \xi+2^{j} \pi\right)}-\hat{\phi}(\xi) \overline{\hat{\phi}(\xi+2 \pi)}= \\
=\sum_{j>0}\left[\hat{\phi}\left(2^{j-1} \xi\right) \overline{\hat{\phi}\left(2^{j-1} \xi+2^{j} \pi\right)}-\hat{\phi}\left(2^{j} \xi\right) \overline{\hat{\phi}\left(2^{j} \xi+2^{j+1} \pi\right)}\right]-\hat{\phi}(\xi) \overline{\hat{\phi}(\xi+2 \pi)}= \\
=-\lim _{n \rightarrow \infty} \hat{\phi}\left(2^{n} \xi\right) \overline{\hat{\phi}\left(2^{n} \xi+2^{n+1} \pi\right)}=0,
\end{gathered}
$$

where $\hat{\phi}\left(2^{j-1} \xi+2^{j} q \pi\right)=\hat{\phi}\left(2^{j-1} \xi+2^{j} \pi\right), q \in 2 \mathbf{Z}+1$, because $m_{0}(\xi)$ is $2 \pi-$ periodic. Therefore, (1.14) also holds, and the proof of Theorem 8 is complete.

We now give a general algorithm to construct the pseudo-scaling generator $\phi$ such that the corresponding MRA TFW $\psi$ is of the largest possible regularity and the required vanishing moments. In fact, this method can be described as an optimization problem of finding suitable $F(\xi)$, or, equivalently, suitable coefficient set, $\mathbf{a}=\left\{a_{0}, \cdots, a_{k}\right\}$, of $F(\xi)$, such that $\sum_{j=0}^{k} a_{j}^{2}$ is the minimum under conditions (2.1) and (2.2) of Theorem 8. Thus, the above optimization problem can be written as
follows.

$$
\begin{align*}
\min _{\mathbf{a}} & \sum_{j=0}^{k} a_{j}^{2},  \tag{2.10}\\
\text { subject to } & \left(\sum_{j=0}^{k}(-1)^{j} a_{j}+1\right)^{2}>0,  \tag{2.11}\\
& \sum_{j=0}^{k} a_{j}^{2}<2^{2 N-1} /(k+1),  \tag{2.12}\\
& \sum_{j=k^{\prime}}^{N+k+k^{\prime}}\left(\sum_{\ell=0}^{k}\binom{N}{j+2 n-\ell-k^{\prime}} a_{\ell}\right)\left(\sum_{\tilde{\ell}=0}^{k}\binom{N}{j-\tilde{\ell}-k^{\prime}} a_{\tilde{\ell}}\right) \\
& =2^{2 N-1} \delta_{n 0}, \quad n=0, \pm 1, \pm 2, \cdots, \tag{2.13}
\end{align*}
$$

where object (2.10) will give the largest possible regularity, condition (2.11) is from the definition of $F(\pi) \neq-1$, and conditions (2.12) and (2.13) come from conditions (2.1) and (2.2) of Theorem 8.

Problem (2.10)-(2.13) can be written in a form without the inequality conditions by defining parameters $s, t \neq 0$ as $s^{2}=2^{2 N-1} /(k+1)-\sum_{j=0}^{k} a_{j}^{2}$ and $t^{2}=\left(\sum_{j=0}^{k}(-1)^{j} a_{j}+1\right)^{2}$, respectively. Hence, the optimization problem becomes

$$
\min _{\mathbf{a}} \quad \sum_{j=0}^{k} a_{j}^{2}+\frac{1}{s^{2}}+\frac{1}{t^{2}}
$$

$$
\text { subject to } t^{2}=\left(\sum_{j=0}^{k}(-1)^{j} a_{j}+1\right)^{2}
$$

$$
s^{2}+\sum_{j=0}^{k} a_{j}^{2}=2^{2 N-1} /(k+1)
$$

$$
\sum_{j=k^{\prime}}^{N+k+k^{\prime}}\left(\sum_{\ell=0}^{k}\binom{N}{j+2 n-\ell-k^{\prime}} a_{\ell}\right)\left(\sum_{\tilde{\ell}=0}^{k}\binom{N}{j-\tilde{\ell}-k^{\prime}} a_{\tilde{\ell}}\right)=
$$

$$
=2^{2 N-1} \delta_{n 0}, \quad n=0, \pm 1, \pm 2, \cdots
$$

As examples, we choose $N=1$ and $k=k^{\prime}=0$, then the solution of problem (2.10)-(2.13) is $a_{0}=1$ and we obtain the Haar function. If we choose $N=$ $2, k=1$, and $k^{\prime}=0$, then the solutions of the problem are $a_{0}=\frac{1 \pm \sqrt{3}}{2}$ and $a_{1}=\frac{1 \mp \sqrt{3}}{2}$. Hence, the corresponding $\phi$ is defined by $\hat{\phi}(\xi)=\Pi_{j=1}^{\infty} \tilde{m}_{0}\left(2^{-j} \xi\right)$, where $m_{0}(\xi)=\frac{1 \pm \sqrt{3}}{2}\left(1-e^{-i \xi}\right)\left(\frac{1+e^{-i \xi}}{2}\right)^{2}$. In addition, the regularities of $\psi$ is more than $2-\log _{2}(4) / 2=1$; i.e., $\psi \in C^{1}$.

### 6.3 Stable generators of an SI space and its complementary space

We now discuss the stability of $\phi$. From [16], we obtain a necessary and sufficient condition for a refinable function defined by Definition 1 to be stable. Here, a function $\phi$ is stable if it is a stable generator of space $S(\phi)$, or, equivalently, the integer translates of $\phi$ form a stable basis of the space.
Lemma 13. The function $\phi$ defined by Definition 1 is stable if and only if $F(\xi)$ satisfies the following two conditions.
(i) $F(\xi)$ does not have any symmetric zeroes on $\mathbf{T}=[0,2 \pi)$;
(ii) For any odd integer $m>1$ and a primitive mth root $\omega=e^{-i 2 n \pi / m}$ of unity (i.e., $n$ is an integer relatively prime to $m$ ), there exists an integer $d, 0 \leq d<\operatorname{ord}_{m} 2$, such that $F\left(-2^{d+1} n \pi / m\right) \neq 0$, where $p=\operatorname{ord}_{m} 2$ is the smallest positive integer with $2^{p} \equiv 1(\bmod m)$.

In addition, if $\phi$ is stable, then $|\hat{\phi}(\xi)|^{2}$ and

$$
\begin{align*}
& \bar{F}(\xi)=\Pi_{j=1}^{\infty}\left|F\left(2^{-j}(\xi+2 \pi u)\right)\right|^{2} / \Pi_{j=1}^{\infty}\left|F\left(2^{-j} \xi\right)\right|^{2} \\
& =\frac{\Pi_{j=1}^{\infty}\left|\sum_{\ell=0}^{k} a_{\ell} e^{-i \ell 2^{-j}(\xi+2 \pi u)}\right|^{2}}{\Pi_{j=1}^{\infty}\left|\sum_{\ell=0}^{k} a_{\ell} e^{-i \ell 2^{-j} \xi}\right|^{2}} \tag{3.1}
\end{align*}
$$

have no roots in $\mathbf{T}=[0,2 \pi)$.
Proof. By using Lemma 6.6 of [1] and Theorem 1 of [16], we obtain that $\phi$ is stable. In addition, since

$$
\begin{equation*}
|\hat{\phi}(\xi)|^{2}=\left|\frac{1-e^{-i \xi}}{i \xi}\right|^{2 N} \Pi_{j=1}^{\infty}\left|F\left(2^{-j} \xi\right)\right|^{2}, \tag{3.2}
\end{equation*}
$$

we have

$$
|\hat{\phi}(\xi+2 \pi u)|^{2}=\bar{F}(\xi)|\hat{\phi}(\xi)|^{2} \frac{\xi^{2 N}}{(\xi+2 \pi u)^{2 N}},
$$

where $\bar{F}(\xi)$ is defined as (3.1). Therefore,

$$
\begin{aligned}
& {[\hat{\phi}, \hat{\phi}]=\sum_{u \in \mathbf{Z}}|\hat{\phi}(\xi+2 \pi u)|^{2}=} \\
& =|\hat{\phi}(\xi)|^{2} \bar{F}(\xi)(\xi)^{2 N} \sum_{u \in \mathbf{Z}} \frac{1}{(\xi+2 \pi u)^{2 N}}
\end{aligned}
$$

By using formula (4.2.7) in [6], we can rewrite the last expression as

$$
\begin{align*}
& {[\hat{\phi}, \hat{\phi}]=} \\
& =|\hat{\phi}(\xi)|^{\frac{\bar{F}}{}(\xi)(\xi / 2)^{2 N}} \sin ^{2 N}(\xi / 2)  \tag{3.3}\\
& \sum_{u=-\infty}^{\infty}\left|\hat{B}_{N}(\xi+2 \pi u)\right|^{2} l
\end{align*}
$$

where $\hat{B}_{N}(\xi)$ is the Fourier transform of the B-spline of order $N$. In addition, the sum on the right-hand side of Equation (3.3) can be evaluated by using formula (4.2.10) in [6]:

$$
\sum_{u=-\infty}^{\infty}\left|\hat{B}_{N}(2 \xi+2 \pi u)\right|^{2}=\frac{-\sin ^{2 N} \xi}{(2 N-1)!} \frac{d^{2 N-1}}{d \xi^{2 N-1}} \cot \xi
$$

Thus, noting that $\phi$ is stable and applying Theorem 7 to the above $[\hat{\phi}, \hat{\phi}]$, we immediately know that $|\hat{\phi}(\xi)|^{2}, \bar{F}(\xi) \neq 0$ for all $\xi \in \mathbf{T}$. This completes the proof of Lemma 13.

From Theorem 8 and Lemma 13, we have the following result.
Theorem 14 Let $\phi \in \Phi$ be defined as in Definition 1. If $F(\xi)$ satisfies $F(\pi) \neq-1$, conditions (i) and (ii) in Lemma 13, and (2.1) or (2.1)', then the corresponding $\phi$ is in $L_{2}(\mathbf{R})$ and is stable.

Let $V_{j}:=\overline{\operatorname{span}}\left\{\phi\left(2^{j} t-k\right): k \in \mathbf{Z}\right\}$. Following [11], for any $\phi \in L_{2}(\mathbf{R})$, we define the (natural) dual $\bar{\phi}$ by its Fourier transform,

$$
\widehat{\hat{\phi}}:=\frac{\hat{\phi}}{[\hat{\phi}, \hat{\phi}]},
$$

where we interpret $0 / 0=0$. Thus, from Equation (3.3), the dual function's Fourier transform of $\phi$ is

$$
\begin{gathered}
\widehat{\bar{\phi}}=\frac{\hat{\phi}}{[\hat{\phi}, \hat{\phi}]} \\
=\sin ^{2 N}(\xi / 2) \\
(\xi / 2)^{2 N} \overline{\hat{\phi}} \bar{F}(\xi) \sum_{u=-\infty}^{\infty}\left|\hat{B}_{N}(\xi+2 \pi u)\right|^{2}
\end{gathered}
$$

The properties of the dual function $\bar{\phi}$ can be found in $[6,11]$.
It is clear that $S(\phi) \subset V_{1}=\operatorname{span}\{\phi(2 \cdot-k): k \in \mathbf{Z}\}$. We now consider the complementary space of $S(\phi)$ in $V_{1}$, which is generated by a function $\psi \in V_{1}$. We say a function $f \in L_{2}(\mathbf{T})$ is in $\mathbf{W}$, the Wiener Algebra, if its Fourier series $\sum_{k \in \mathbf{Z}} f_{k} e^{-i k \xi}$ satisfies $\left\{f_{k}\right\} \in \ell_{1}(\mathbf{Z})$. From [11], we can establish the following lemma.
Theorem 15. Let $\phi \in \Phi$ satisfy all conditions of Theorem 14, where $\Phi$ is defined in Definition 1, and let $\psi \in V_{1}:=\overline{\operatorname{span}}\{\phi(2 \cdot-k): k \in \mathbf{Z}\}$ have the symbol $m_{1}(\xi) \in \mathbf{W}$, the Wiener Algebra, such that

$$
\begin{equation*}
\left|m_{1}(\xi)\right|^{2}+\left|m_{1}(\xi+\pi)\right|^{2}>0, \quad \xi \in \mathbf{T} . \tag{3.4}
\end{equation*}
$$

Then $\psi$ is stable (i.e., a stable generator for $S(\psi)$ ).
Proof. Since $\psi \in V_{1}$, using the two-scale relation of $\psi$ yields

$$
\begin{aligned}
& {[\hat{\psi}, \hat{\psi}](\xi)=} \\
= & \sum_{i \in \mathbf{Z}}\left|m_{1}\left(\frac{\xi}{2}+\pi i\right)\right|^{2}\left|\hat{\phi}\left(\frac{\xi}{2}+\pi i\right)\right|^{2}= \\
= & \sum_{i \in \mathbf{Z}}\left|m_{1}\left(\frac{\xi}{2}+2 \pi i\right)\right|^{2}\left|\hat{\phi}\left(\frac{\xi}{2}+2 \pi i\right)\right|^{2}+ \\
+ & \sum_{i \in \mathbf{Z}}\left|m_{1}\left(\frac{\xi}{2}+\pi+2 \pi i\right)\right|^{2}\left|\hat{\phi}\left(\frac{\xi}{2}+\pi+2 \pi i\right)\right|^{2}= \\
= & \left|m_{1}\left(\frac{\xi}{2}\right)\right|^{2}\left[\hat{\phi}\left(\frac{\xi}{2}\right), \hat{\phi}\left(\frac{\xi}{2}\right)\right]+ \\
+ & \left|m_{1}\left(\frac{\xi}{2}+\pi\right)\right|^{2}\left[\hat{\phi}\left(\frac{\xi}{2}+\pi\right), \hat{\phi}\left(\frac{\xi}{2}+\pi\right)\right] .
\end{aligned}
$$

From Theorem 8 and Lemma $13, \phi$ is in $L_{2}(\mathbf{R})$ and is stable. Thus, Theorem 7 shows there exist $0<A \leq B<\infty$ such that

$$
A \leq[\hat{\phi}, \hat{\phi}] \leq B \quad \text { a.e. }
$$

Thus we can bound the auto-correlation of $\psi$ by

$$
A M_{1}(\xi) \leq[\hat{\psi}, \hat{\psi}](\xi) \leq B M_{1}(\xi)
$$

where

$$
M_{1}(\xi):=\left|m_{1}\left(\frac{\xi}{2}\right)\right|^{2}+\left|m_{1}\left(\frac{\xi}{2}+\pi\right)\right|^{2} .
$$

Since $m_{1}(\xi) \in C(\mathbf{T})$, from the condition (3.4) we have

$$
\bar{A}:=\min _{\xi \in \mathbf{T}} M_{1}(\xi)>0
$$

On the other hand, $\bar{B}:=\left\|m_{1}(\xi)\right\|_{C(\mathbf{T})}<\infty$. Therefore,

$$
0<\bar{A} A \leq[\hat{\psi}, \hat{\psi}](\xi) \leq \bar{B} B<\infty, \quad \text { a.e. }
$$

By using Theorem 7, we have proved that $\psi$ is a stable generator for $S(\psi)$.
We can also construct a biorthogonal system from $\phi \in \Phi$ as follows. Here, the set $\Phi$ was defined in Definition 1. Assume that $\tilde{\phi} \in \Phi$; i.e., $\tilde{\phi}(t)=\sum_{n} \tilde{h}_{n} \tilde{\phi}(2 t-n)$ or equivalently, $\hat{\tilde{\phi}}(\xi)=\tilde{m}_{0}\left(\frac{\xi}{2}\right) \hat{\tilde{\phi}}\left(\frac{\xi}{2}\right)$ with $\tilde{m}_{0}(\xi)=2^{-1} \sum_{n} \tilde{h}_{n} e^{-i n \xi} \in M$, which is defined in Definition 1. Therefore, we can write

$$
\begin{equation*}
\tilde{m}_{0}(\xi)=\tilde{m}_{0}^{\tilde{N}}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{\tilde{N}} \tilde{F}(\xi) \tag{3.5}
\end{equation*}
$$

Here $\tilde{F}(\xi)$ is defined by

$$
\tilde{F}(\xi)=e^{-i \tilde{k}^{\prime} \xi} \sum_{j=0}^{\tilde{k}} \tilde{a}_{j} e^{-i j \xi}
$$

where, $\tilde{F}(0)=1$; all coefficients of $\tilde{F}(\xi)$ are real; $\tilde{N}$ and $\tilde{k}$ are positive integers; and $\tilde{k}^{\prime} \in \mathbf{Z}$. Hence, the corresponding $\phi$ and $\tilde{\phi}$ can be written as follows.

$$
\begin{align*}
& \hat{\phi}(\xi)=\left(\frac{1+e^{-i \xi / 2}}{2}\right)^{N} F(\xi / 2) \hat{\phi}(\xi / 2), \\
& \hat{\tilde{\phi}}(\xi)=\left(\frac{1+e^{-i \xi / 2}}{2}\right)^{\tilde{N}} \tilde{F}(\xi / 2) \hat{\tilde{\phi}}(\xi / 2) . \tag{3.6}
\end{align*}
$$

We also define the corresponding $\psi$ and $\tilde{\psi}$ by

$$
\begin{align*}
& \hat{\psi}(\xi)=e^{i \xi / 2} \overline{\tilde{m}_{0}(\xi / 2+\pi)} \hat{\phi}(\xi / 2) \\
& \hat{\tilde{\psi}}(\xi)=e^{i \xi / 2} \overline{m_{0}(\xi / 2+\pi)} \overline{\hat{\phi}}(\xi / 2), \tag{3.7}
\end{align*}
$$

or, equivalently,

$$
\begin{align*}
\psi(x) & =\sum_{n}(-1)^{n-1} \tilde{h}_{-n-1} \phi(2 x-n) \\
\tilde{\psi}(x) & =\sum_{n}(-1)^{n-1} h_{-n-1} \phi(2 x-n) \tag{3.8}
\end{align*}
$$

Similar to $\phi, \tilde{\phi}$ is also a B-spline of order $\tilde{N}$ if $\tilde{F}(\xi)=1$. Since vanishing moment conditions $\int x^{\ell} \psi(x) d x=0, \ell=0,1, \cdots, L$, are equivalent to $\left.\frac{d^{\ell}}{d \xi^{\ell}} \hat{\psi}\right|_{\xi=0}=0$, $\ell=0,1, \cdots, L$, we immediately know that the maximum orders of vanishing moment for $\psi$ and $\tilde{\psi}$ are $N-1$ and $\tilde{N}-1$, respectively. Therefore, following the similar argument in Section 2, the vanishing moments of $\phi$ and $\tilde{\phi}$ are completely controlled by the exponents of their respective "spline factors," $\left(\frac{1+e^{-i \xi}}{2}\right)^{N}$ and $\left(\frac{1+e^{-q i \xi}}{2}\right)^{\tilde{N}}$. In addition, the regularities of $\phi$ and $\tilde{\phi}$ are justified by the factors $F(\xi)$ and $\tilde{F}(\xi)$, respectively, and are independent of their vanishing moments.

Similar to $\phi$, if $\tilde{\phi}$ defined in (3.6) satisfies $\tilde{F}(\pi) \neq-1$ and

$$
\begin{equation*}
(\tilde{k}+1) \sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}<2^{2 \tilde{N}-1} \tag{3.9}
\end{equation*}
$$

then $\tilde{\phi} \in L_{2}(\mathbf{R})$.
From [8], the stability of $\phi, \tilde{\phi}, \psi$, and $\tilde{\psi}$ are implied by their biorthogonality. In fact, $[8]$ gave the following results.

Lemma 16. If $\phi, \tilde{\phi} \in L_{2}(\mathbf{R})$ satisfy $\langle\phi(t), \tilde{\phi}(t-n)\rangle=\delta_{n 0}$, then $\{\phi(t-k)\}_{k \in \mathbf{Z}}$ and $\{\tilde{\phi}(t-k)\}_{k \in \mathbf{Z}}$ are stable; i.e., they are stable bases (Riesz bases) in the subspace that they generate. In addition, the corresponding biorthogonal wavelet functions $\psi$ and $\tilde{\psi}$ are also stable; i.e., they are stable bases (Riesz bases) in the subspace that they generate.

Hence, we have the following results.
Theorem 17. Let $\phi, \tilde{\phi} \in \Phi$ be defined in Definition 1; that is, $\phi=\Pi_{j=1}^{\infty} m_{0}^{N}\left(2^{-j} \xi\right)$ and $\tilde{\phi}=\Pi_{j=1}^{\infty} \tilde{m}_{0}^{\tilde{N}}\left(2^{-j} \xi\right)$, where $m_{0}^{N}(\xi)$ and $\tilde{m}_{0}^{\tilde{N}}(\xi)$ are defined by Equations (1.5) and (3.5). Assume $F(\pi), \tilde{F}(\pi) \neq-1$, and conditions (2.1) and (3.9) are satisfied by $m_{0}^{N}(\xi)$ and $\tilde{m}_{0}^{\tilde{N}}(\xi)$ and

$$
\begin{equation*}
\sum_{j=\mu}^{\nu} \sum_{\ell=0}^{k} \sum_{\tilde{\ell}=0}^{\tilde{k}}\binom{\tilde{N}}{j-\tilde{\ell}-\tilde{k}^{\prime}}\binom{N}{j+2 n-\ell-k^{\prime}} \tilde{a}_{\tilde{\ell}} a_{\ell}=2^{N+\tilde{N}-1} \delta_{n 0} \tag{3.10}
\end{equation*}
$$

holds for all $n \in \mathbf{Z}$, where $\mu=\min \left\{k^{\prime}, \tilde{k}^{\prime}\right\} ; \nu=\max \left\{N+k+k_{\tilde{\varphi}}^{\prime}, \tilde{N}+\tilde{k}+\tilde{k}^{\prime}\right\}$ and $\delta_{n 0}$ is the Kronecker symbol, then $\phi, \tilde{\phi} \in L^{2}(\mathbf{R})$ satisfy $\langle\phi(t), \tilde{\phi}(t-i)\rangle=\delta_{i, 0}$ for all $i \in \mathbf{Z}$, and thus they are stable. In addition, The corresponding biorthogonal wavelets $\psi$ and $\tilde{\psi}$ are in $C^{\alpha}$ and $C^{\tilde{\alpha}}$, respectively. Here $\alpha$ and $\tilde{\alpha}$ are more than

$$
N-\frac{1}{2} \log _{2}\left((k+1) \sum_{j=0}^{k} a_{j}^{2}\right)
$$

and

$$
\begin{equation*}
\tilde{N}-\frac{1}{2} \log _{2}\left((\tilde{k}+1) \sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}\right) \tag{3.11}
\end{equation*}
$$

respectively.
Proof. By using Lemma 9 , we have $\phi, \tilde{\phi} \in L_{2}(\mathbf{R})$. Similar to the proofs of Lemmas 11 and 12, from Propositions 4.8 and 4.9 in [9], $\phi$ and $\tilde{\phi}$ are biorthogonal if (2.1), (3.9), and the following conditions hold.

$$
\begin{equation*}
\frac{1}{2} \sum_{j} \tilde{h}_{j} h_{j+2 n}=\delta_{n 0}, \quad n \in \mathbf{Z} \tag{3.12}
\end{equation*}
$$

where $h_{j}$ and $\tilde{h}_{j}$ are respectively the coefficients of two-scale relations of $\phi$ and $\tilde{\phi}$. From equations (3.12), noting that

$$
h_{j}=\left(\sum_{\ell=0}^{k}\binom{N}{j-\ell-k^{\prime}} a_{\ell}\right) / 2^{N-1}
$$

$\left(j=k^{\prime}, \cdots, N+k+k^{\prime}\right)$, and

$$
\tilde{h}_{j}=\left(\sum_{\ell=0}^{\tilde{k}}\binom{\tilde{N}}{j-\ell-\tilde{k}^{\prime}} \tilde{a}_{\ell}\right) / 2^{\tilde{N}-1}
$$

$\left(j=\tilde{k}^{\prime}, \cdots\right), \tilde{N}+\tilde{k}+\tilde{k}^{\prime}$, we immediately obtain (3.10).
It is well-known (see [7, 9]) that $\psi \in C^{r}$ and $r=N-\frac{1}{2} \log _{2}(\rho(T))$, where $\rho(T)$ is the spectral radius of the matrix representing operator $T$, and $T$ is defined by equation (2.4). Hence, we obtain that the corresponding biorthogonal wavelets $\psi$ and $\tilde{\psi}$ are in $C^{\alpha}$ and $C^{\tilde{\alpha}}$, respectively, and $\alpha$ and $\tilde{\alpha}$ are more than the quantities shown in (3.11).
Remark 1. Conditions (2.1) and (3.9) can be replaced by the following weaker conditions.

$$
\begin{equation*}
C\left(\left\{a_{j}\right\}, k\right)<2^{2 N-1}, \tilde{C}\left(\left\{\tilde{a}_{j}\right\}, \tilde{k}\right)<2^{2 \tilde{N}-1}, \tag{3.13}
\end{equation*}
$$

where $C\left(\left\{a_{j}\right\}, k\right)$ equals $k \sum_{j=0}^{k} a_{j}^{2}$ if $k \geq 1$ and equals $a_{0}^{2}$ if $k=0$ while $\tilde{C}\left(\left\{\tilde{a}_{j}\right\}, \tilde{k}\right)$ equals $\tilde{k} \sum_{j=0}^{k} \tilde{a}_{j}^{2}$ if $\tilde{k} \geq 1$ and equals $\tilde{a}_{0}^{2}$ if $\tilde{k}=0$. Hence, the corresponding regularities of $\phi$ and $\tilde{\phi}$ are determined by $\phi \in C^{\alpha^{\prime}}$ and $\tilde{\phi} \in C^{\tilde{\alpha}^{\prime}}$, where $\alpha^{\prime}$ and $\tilde{\alpha}^{\prime}$ are more than

$$
\begin{equation*}
N-\frac{1}{2} \log _{2}\left(2 C\left(\left\{a_{j}\right\}, k\right)\right) \tag{3.14}
\end{equation*}
$$

and

$$
\tilde{N}-\frac{1}{2} \log _{2}\left(2 \tilde{C}\left(\left\{\tilde{a}_{j}\right\}, \tilde{k}\right),\right.
$$

respectively.
Remark 2. If $F(\xi)=1$ (i.e., $k, k^{\prime}=0$ ), then condition (3.10) can be written as

$$
\begin{equation*}
\sum_{\ell=0}^{\tilde{k}} \sum_{j=\tilde{k}^{\prime}}^{\max \left\{N, \tilde{N}+\tilde{k}+\tilde{k}^{\prime}\right\}}\binom{\tilde{N}}{j-\ell-\tilde{k}^{\prime}}\binom{N}{j+2 n} \tilde{a}_{\ell}=2^{N+\tilde{N}-1} \delta_{n 0}, \tag{3.15}
\end{equation*}
$$

where $n=0, \pm 1, \cdots$. Condition (3.15) can be reduced again as follows for $\tilde{N}=N-\tilde{k}$ and $\tilde{k}^{\prime}=0$ :

$$
\begin{equation*}
\sum_{\ell=0}^{\tilde{k}}\left(\sum_{j=0}^{N}\binom{N-\tilde{k}}{j-\ell}\binom{N}{j+2 n}\right) \tilde{a}_{\ell}=2^{2 N-\tilde{k}-1} \delta_{n 0}, \tag{3.16}
\end{equation*}
$$

where $n=0, \pm 1, \pm 2, \cdots$.
We now give a general algorithm to construct biorthogonal scaling functions $\phi$ and $\tilde{\phi}$ with the largest possible regularity and the required vanishing moments. Similar the algorithm shown in Section 2, this method can be described as an optimization problem of finding suitable $F(\xi)$ and $\tilde{F}(\xi)$, or, equivalently, suitable coefficient sets, $\mathbf{a}=\left\{a_{0}, \cdots, a_{k}\right\}$ and $\tilde{\mathbf{a}}=\left\{\tilde{a}_{0}, \cdots, \tilde{a}_{k}\right\}$, of $F(\xi)$ and $\tilde{F}(\xi)$, respectively, such that $\sum_{j=0}^{k} a_{j}^{2}$ and $\sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}$ are the minimum under conditions (2.1) and (3.9). From the wavelet analysis of spline approximation, we assume $F(\xi)=1$; i.e., the corresponding $\phi$ is the B -spline of order $N$. Then, the above optimization problem
can be written as follows.
$\min _{a}$

$$
\begin{align*}
& \sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2},  \tag{3.17}\\
& \left(\sum_{j=0}^{\tilde{k}}(-1)^{j} \tilde{a}_{j}+1\right)^{2}>0,  \tag{3.18}\\
& \sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}<2^{2 \tilde{N}-1} /(\tilde{k}+1),  \tag{3.19}\\
& \sum_{\ell=0}^{\tilde{k}} \sum_{\substack{\max \left\{N, \tilde{N}+\tilde{k}+\tilde{k}^{\prime}\right\}}} \sum_{j=\tilde{k^{\prime}}}\binom{\tilde{N}}{j-\ell-\tilde{k}^{\prime}}\binom{N}{j+2 n} \tilde{a}_{\ell}  \tag{3.20}\\
& =2^{N+\tilde{N}-1} \delta_{n 0},
\end{align*}
$$

where $n=0, \pm 1, \cdots$, object (3.17) will give the largest possible regularity, condition (3.18) is from the definition of $\tilde{F}$, and conditions (3.19) and (3.20) come from conditions (3.9) and (3.10).

Problem (3.17)-(3.20) can be written in a form without the inequality conditions by defining parameters $s, t \neq 0$ by $s^{2}=2^{2 \tilde{N}-1} /(\tilde{k}+1)-\sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}$ and $t^{2}=\left(\sum_{j=0}^{\tilde{k}}(-1)^{j} \tilde{a}_{j}+1\right)^{2}$, respectively. Hence, the optimization problem becomes

$$
\begin{array}{ll}
\min _{\mathbf{a}} \quad & \sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}+\frac{1}{s^{2}}+\frac{1}{t^{2}}, \\
\text { subject to } \quad & t^{2}=\left(\sum_{j=0}^{\tilde{k}}(-1)^{j} \tilde{a}_{j}+1\right)^{2} \\
& \sum_{j=0}^{\tilde{k}} \tilde{a}_{j}^{2}+s^{2}=2^{2 \tilde{N}-1} /(\tilde{k}+1), \\
& \sum_{\ell=0}^{\tilde{k} \max \left\{N, \tilde{N}+\tilde{k}+\tilde{k}^{\prime}\right\}} \sum_{j=\tilde{k}^{\prime}}\binom{\tilde{N}}{j-\ell-\tilde{k}^{\prime}}\binom{N}{j+2 n} \tilde{a}_{\ell} \\
& =2^{N+\tilde{N}-1} \delta_{n 0}
\end{array}
$$

where $n=0, \pm 1, \cdots$.
As examples, we consider $m_{0}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)$; i.e., $\phi$ is B -spline of order 1 . If we choose $\tilde{N}=1$ and $\tilde{k}=\tilde{k}^{\prime}=0$, then the solution of problem (3.17)-(3.19) is $\tilde{a}_{0}=1$ and we obtain the Haar function. If we choose $\tilde{N}=2, \tilde{k}=1$, and $\tilde{k}^{\prime}=0$, then the solutions of the problem are $\tilde{a}_{0}=3 / 2$ and $\tilde{a}_{1}=-1 / 2$. Hence, the corresponding $\tilde{\phi}$ is defined by $\hat{\tilde{\phi}}(\xi)=\Pi_{j=1}^{\infty} \tilde{m}_{0}\left(2^{-j} \xi\right)$, where $\tilde{m}_{0}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{2}\left(\frac{3-e^{-i \xi}}{2}\right)$. Clearly,

Both $\phi$ and $\tilde{\phi}$ satisfy all the conditions of Theorem 17. Hence, they are in $L_{2}(\mathbf{R})$ and are stable. In addition, the regularities of $\phi$ and $\tilde{\phi}$ are respectively 1 and $2-\log _{2}(5) / 2=0.839036$.

We now discuss the error of $L_{2}$ approximation from $S(\phi)$. Denote

$$
E(f, S(\phi))_{L_{2}}:=\inf _{g \in S(\phi)}\|f-g\|_{L_{2}}
$$

We have the following result.
Theorem 18. Let $\phi \in \Phi$ defined by Definition 1; i.e., $\phi=\Pi_{j=1}^{\infty} m_{0}^{N}\left(2^{-j} \xi\right)$, where $m_{0}^{N}(\xi) \in M$ is defined by Equations (1.5) and (1.6): $m_{0}^{N}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{N} F(\xi)$ and $F(\xi)=e^{-i k^{\prime} \xi} \sum_{j=0}^{k} a_{j} e^{-i j \xi}, N, k \in \mathbf{Z}_{+}$and $k^{\prime} \in \mathbf{Z}$, where $F(\xi)$ satisfies Conditions (i) and (ii) of Lemma $13, F(0)=1$, and $F(\pi) \neq-1$. If inequality (2.1) or (2.1)' holds, then for any function $f \in W_{2}^{N+1}(\mathbf{R})$, the Sobolev space,

$$
E\left(f, S(\phi)^{h}\right)_{2}=C_{\phi}^{N}\|f\|_{W_{2}^{N}(\mathbf{R})}+O\left(h^{N+1}\right)
$$

where

$$
S(\phi)^{h}:=\{f(\cdot / h) \mid f \in S(\phi)\}
$$

and

$$
C_{\phi}^{N}=\frac{1}{N!} \sqrt{\sum_{u \neq 0}\left|\hat{\phi}^{(N)}(2 \pi u)\right|^{2}}
$$

Proof. Following [4], we define the error kernel

$$
\Gamma_{\phi}:=\left(1-\frac{|\hat{\phi}|^{2}}{[\hat{\phi}, \hat{\phi}]}\right)^{\frac{1}{2}}
$$

where $0 / 0$ is interpreted as 0 . From Equation (3.3), we have

$$
\frac{|\hat{\phi}|^{2}}{[\hat{\phi}, \hat{\phi}]}=\frac{\sin ^{2 N}(\xi / 2)}{(\xi / 2)^{2 N} \bar{F}(\xi) \sum_{u=-\infty}^{\infty}\left|\hat{B}_{N}(\xi+2 \pi u)\right|^{2}}
$$

Noting Equation (3.1) and the inequality (see [6])

$$
\sum_{u=-\infty}^{\infty}\left|\hat{B}_{N}(\xi+2 \pi u)\right|^{2} \leq 1
$$

we obtain $|\xi|^{-N} \Gamma_{\phi}(\xi) \in L_{\infty}(\mathbf{T})$. Thus, from Theorems 1.6 and 2.20 of [4], $S(\phi)$ provides approximation order $N$; i.e.,

$$
E\left(f, S(\phi)^{h}\right)_{2} \leq C_{\phi}^{N} h^{N}\|f\|_{W_{2}^{N}(\mathbf{R})}
$$

where $C_{\phi}^{N}$ can be found as follows. By observing that $\phi \in \mathbf{E}_{N}:=\{f| | f(x) \mid \leq$ $\left.C(1+|x|)^{-(N+1+\epsilon)}, \epsilon>0\right\}$ is stable and using the following lemma shown in [5], we immediately obtain the Theorem 18.
Lemma 19. [5] Assume $\phi \in \mathbf{E}_{m}$ is stable while $\hat{\phi}(0)=1$ and provides $L_{2}$ approximation order $m$. Then for any function $f \in W_{2}^{m+1}(\mathbf{R})$,

$$
E\left(f, S(\phi)^{h}\right)_{2}=C_{\phi}^{m} h^{m}|f|_{W_{2}^{m}(\mathbf{R})}+O\left(h^{m+1}\right)
$$

where $C_{\phi}^{m}=\frac{1}{m!} \sqrt{\sum_{u \neq 0}\left|\hat{\phi}^{(m)}(2 \pi u)\right|^{2}}$.

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