Some dense subset of real numbers and an application

Tian-Xiao He[*], Zachariah Sinkala and Xiaoya Zha[†]

Abstract

In this paper we first give a collection of subsets which are dense in the set of real numbers. Then, as an application, we show that: for a continuous function f on $\mathbb{R}\backslash\{0\}$, the integrals $F_{p,f}(x) = \int_x^{px} f(t)dt$ and $F_{q,f}(x) = \int_x^{qx} f(t)dt$ (where $\frac{\ln p}{\ln q} \notin \mathbb{Q}$) are constant functions of x if and only if $f = \dfrac{c}{x}$, $c = f(1) \in \mathbb{R}$.

1 Introduction

Throughout this paper we use $\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-, \mathbb{Q}, \mathbb{Z}, \mathbb{Z}^+$ to denote real numbers, positive real numbers, negative real numbers, rational numbers, integers, positive integers, respectively. This work is motivated by the calculus problem of finding the derivative of $F(x) = \displaystyle\int_x^{2x} \frac{1}{t}\, dt,\ x \neq 0$ (this problem is designed for applying the Fundamental Theorem of Calculus and the chain rule). It is easy to see that $F'(x) = 0$, which implies a nice geometric fact: for any given $x, x \neq 0$, the area between the curves of $y = 1/t$ and $y = 0$ from x to $2x$ is a constant. Clearly, the function $y = c/t,\ c \in \mathbb{R}$ also has this interesting property. It is natural to ask whether the converse is true or not; i.e., letting f be a continuous function on $\mathbb{R}\backslash\{0\}$ and $F_{2,f}(x) = \displaystyle\int_x^{2x} f(t)\, dt$ a constant function of x ($x \neq 0$), is $f(t) = c/t$, for some constant

[*]Department of Mathematics and Computer Science, Illinois Wesleyan University, Bloomington, Illinois 61702

[†]Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, Tennessee 37132

$c \in \mathbb{R}$? Examples in Section 3 show that in fact $F_{2,f}(x)$ being a constant function of x is not sufficient to guarantee that $f(x) = c/x$ for some $c \in \mathbb{R}$. They also demonstrate that there exists a function g such that $F_{3,g}(x) = \int_x^{3x} g(t)\, dt$ being a constant function of x is not sufficient to guarantee that $g(x) = c/x,\ c \in \mathbb{R}$. However, one may ask whether there is a function h with both $F_{2,h}(x)$ and $F_{3,h}(x)$ are constant functions of x, does it force that $h(t) = c/t$? or a possible function k can be constructed by "combining" f and g so that both $F_{2,k}(x)$ and $F_{3,k}(x)$ are constant functions of x but $k(t) \neq c/t,\ c \in \mathbb{R}$? We study this problem in Section 3. We provide a necessary and sufficient condition for $f(t)$ being a constant multiple of $1/t$. The study of this problem relies on dense subsets of $\mathbb{R}$. In Section 2 we give a collection of "small" subsets which are dense in $\mathbb{R}$.

2 Dense subsets of real numbers

In this section, we construct a collection of infinitely many dense subsets (see p. 32 of [1] for the definition of dense subsets) of $\mathbb{R}$ that are very "small". These dense subsets may not seem dense in $\mathbb{R}$ at the first glance. For example, we show that both $S_{2,3} = \{\pm 2^n 3^m : n, m \in \mathbb{Z}\}$ and $S_{\pi,e} = \{\pm \pi^n e^m : n, m \in \mathbb{Z}\}$ are dense in $\mathbb{R}$. In general, we prove the following main theorem.

Theorem 1 *Suppose p and q are in $\mathbb{R}^+ \backslash \{1\}$. Let*

$$S_{p,q} = \{\pm p^n q^m : n, m \in \mathbb{Z}\}.$$

Then $S_{p,q}$ is dense in $\mathbb{R}$ if and only if $\dfrac{\ln p}{\ln q} \notin \mathbb{Q}$.

We first prove a technical lemma.

Lemma 2 *Let p, q and $S_{p,q}$ be defined as in Theorem 2.1. Then*

(i) There exists a sequence $\{a_n, n \geq 1, a_n \in S_{p,q}\}$ such that $1 < a_1 < a_2 < \ldots < a_n \ldots$ and $\lim_{n\to\infty} a_n = 1^+$;

(ii) There exists a sequence $\{b_n, n \geq 1, b_n \in S_{p,q}\}$ such that $b_1 < b_2 < \ldots < b_n < \ldots < 1$ and $\lim_{n\to\infty} b_n = 1^-$.

Proof. Since $S_{p,q}$ is closed under reciprocals, we may assume that $1 < p < q$. Let $s_1 = \ln p$ and $t_1 = \ln q$. Then $0 < s_1 < t_1$. Since $\ln p$ is not a rational multiple of $\ln q$, there exists an integer $n_1 \in \mathbb{Z}^+$ such that $n_1 s_1 < t_1 < (n_1+1)s_1$ and $t_1 \neq \dfrac{n_1 s_1 + (n_1+1)s_1}{2}$. Let

$$s_2 = \min\{t_1 - n_1 s_1, (n_1+1)s_1 - t_1\}$$

and

$$t_2 = \max\{t_1 - n_1 s_1, (n_1+1)s_1 - t_1\}.$$

We have

(2.1) $0 < s_2 < \frac{1}{2}s_1 < t_2 < s_1 < t_1$
(since $t_1 \neq \dfrac{n_1 s_1 + (n_1+1)s_1}{2}$ and $n_1 s_1 < t_1 < (n_1+1)s_1$);

(2.2) t_2 is not a rational multiple of s_2
(otherwise $(n_1+1)s_1 - t_1 = k(t_1 - n_1 s_1), k \in \mathbb{Q}$ which implies that t_1 is a rational multiple of s_1, i.e $\dfrac{\ln p}{\ln q} \in \mathbb{Q}$, a contradiction);

(2.3) s_2 and t_2 are linear combination of $\ln p$ and $\ln q$ with integer coefficients, respectively.

For $i = 3, 4, ...$, construct $s_3, t_3, s_4, t_4, ...$ inductively. Suppose s_i and t_i are constructed, satisfying

(2.4) $0 < s_i < \frac{1}{2}s_{i-1} < t_i < s_{i-1} < t_{i-1}$;

(2.5) t_i is not a rational multiple of s_i;

(2.6) s_i and t_i are linear combination of $\ln p$ and $\ln q$ with integer coefficients, respectively.

Let

$$s_{i+1} = \min\{t_i - n_i s_i, (n_i+1)s_i - t_i\}$$

and

$$t_{i+1} = \max\{t_i - n_i s_i, (n_i+1)s_i - t_i\},$$

where n_i is the unique integer satisfying $n_i s_i < t_i < (s_i+1)s_i$. It is easy to show inductively that (2.4) - (2.6) are true for s_{i+1} and t_{i+1}. Therefore, two sequences $\{s_n, n \geq 1\}$ and $\{t_n, n \geq 1\}$ are constructed, satisfying (2.4) - (2.6). By (2.4), $s_1 > s_2 > ... > s_n > ... > 0$ and $\lim_{n\to\infty} s_n = 0^+$. For $n = 1, 2, ...$, let $a_n = e^{s_n}$. Then

$$a_n \in S_{p,q}, a_1 > a_2 > ... > a_n > ... > 1 \text{ and } \lim_{n\to\infty} a_n = 1^+.$$

Thus Lemma 2.2 (i) is true. Let $b_n = 1/a_n, n \geq 1$. Then $b_1 < b_2 < ... < b_n < ... < 1$ and $\lim_{n\to\infty} b_n = 1^-$. Therefore Lemma 2.2 (ii) is also true. □

Proof of Theorem 2.1. The proof for the necessary condition is straightforward. Suppose $\frac{\ln p}{\ln q} \in \mathbb{Q}$, say $\ln p = \frac{a}{b} \ln q,\ a, b \in \mathbb{Z}^+$. Then $p = q^{\frac{a}{b}}$ and

$$S_{p,q} = \{q^{\frac{a}{b}s+t},\ s, t \in \mathbb{Z}\}.$$

Clearly for any given $a, b \in \mathbb{Z}^+$, the set $A = \{\frac{a}{b}s + t,\ s, t \in \mathbb{Z}\}$ is not dense in $\mathbb{R}$. Hence $S_{p,q}$ is not dense in $\mathbb{R}^+$, which contradicts the assumption.

We now show that the condition is sufficient; i.e., if $\frac{\ln p}{\ln q} \notin \mathbb{Q}$, then $S_{p,q}$ is dense in $\mathbb{R}$. Let $c \in \mathbb{R}^+$ be any positive real number and assume that $c \notin S_{p,q}$. Again since $S_{p,q}$ is closed under reciprocals, we may assume that $1 < p < q$. We need to show that there exits a sequence of numbers $x_1, x_2, ..., x_n, ... \in S_{p,q}$ such that $\lim_{n\to\infty} x_n = c$. Suppose the claim is not true. Let $S^- = \{x \in S_{p,q}, 0 < x < c\}$ and $S^+ = \{x \in S_{p,q}, x > c\}$. Since $\lim_{k\to\infty} 1/p^k = 0$ and $\lim_{k\to\infty} p^k = \infty$, both S^- and S^+ are not empty. Let $\alpha = \sup S^-$ and $\beta = \inf S^+$. Then $0 < \alpha \leq c \leq \beta$. We now show that $\alpha = c = \beta$. By symmetry we only need to show that $\alpha = c$.

Assume $\alpha < c$. Let us consider the following two cases. Case 1: $\alpha \in S_{p,q}$. By Lemma 2.2, there exists a sequence $\{a_n, n \geq 1\}$ and $\lim_{n\to\infty} a_n = 1^+$. Choose a_i such that $1 < a_i < 1 + \dfrac{c-\alpha}{\alpha}$. Then $\alpha a_i \in S_{p,q}$, but $\alpha a_i > \alpha$ and $\alpha a_i < \alpha(1 + \dfrac{c-\alpha}{\alpha}) = c$ which contradicts the fact that $\alpha = \sup S^-$.

Case 2: $\alpha \notin S_{p,q}$. Then there exists a sequence $x_n, n \geq 1, x_n \in S_{p,q}, x_n < \alpha$, and $\lim_{n\to\infty} x_n = \alpha$. Again, as in the proof of Case 1, we choose a_i such that $1 < a_i < 1 + \dfrac{c-\alpha}{2\alpha}$. Then we choose $x_{n_0} \in \{x_n\}$

such that $x_{n_0} > \dfrac{\alpha}{a_i}$. Therefore $x_{n_0} a_i \in S_{p,q}$, but $x_{n_0} a_i > \frac{\alpha}{a_i} a_i = \alpha$ and $x_{n_0} a_i < \alpha(1 + \dfrac{c-\alpha}{2\alpha}) = \dfrac{c+\alpha}{2} < c$. This again contradicts the fact that $\alpha = \sup S^-$.

A similar argument shows that $\beta = c$. Therefore $S_{p,q}$ is dense in $\mathbb{R}^+$ and also in $\mathbb{R}$ since all negative elements in $S_{p,q}$ will be dense in $\mathbb{R}^-$. $\square$

3 An Application

In this section we apply Theorem 2.1 to solve a problem originated from the calculus, as stated in the introduction.

Let $f(t)$ be a continuous function on $\mathbb{R}\backslash\{0\}$, and

$$F_{\lambda,f}(x) = \int_x^{\lambda x} f(t)dt, \quad \lambda \in \mathbb{R}^+\backslash\{1\}.$$

If $f(t) = \dfrac{c}{t}, c \in \mathbb{R}$, then for $x \neq 0$, $F_{\lambda,f}(x) = c \ln \lambda$ is a constant function of x. However, the following example shows that the inverse is not true; i.e., $F_{\lambda,f}(x)$ being a constant function of x, $(x \neq 0)$ does not imply that $f(t) = \dfrac{c}{t}, c \in \mathbb{R}$.

Example 3.1. Inductively define $f_2(t)$ as follows:

(i) for $t \in [1, 2)$, let

$$f_2(t) = \begin{cases} t-1, & t \in [1, \frac{3}{2}) \\ -t+2, & t \in [\frac{3}{2}, 2) \end{cases};$$

(ii) for $t \in [2^{n-1}, 2^n), n = 2, 3, ...,$ let $f(t) = \dfrac{1}{2} f(\dfrac{t}{2})$;

(iii) for $t \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}), n = 1, 2, ...,$ let $f(t) = 2f(2t)$;

(iv) for $t \in (-\infty, 0)$, let $f(t) = -f(-t)$.

Then $f(t)$ is a continuous function on $\mathbb{R}\backslash\{0\}$ and

$$F_{2,f}(x) = \int_x^{2x} f(t)dt = \frac{1}{4}.$$

In fact, Example 3.1 can be generalized to functions with any parameter $\lambda \in \mathbb{R}^+ \backslash \{1\}$.
Example 3.2 For $\lambda \in \mathbb{R}^+ \backslash \{1\}$ and $\lambda > 1$, define $f_\lambda(t)$ on $\mathbb{R} \backslash \{0\}$ as follows:

(i) for $t \in [\lambda^{n-1}, \lambda^n),\ n \in \mathbb{Z},$, let

$$f_\lambda(t) = \begin{cases} \frac{1}{\lambda^{2n-2}}(t - \lambda^{n-1}), & t \in [\lambda^{n-1}, \lambda^{n-1}(\frac{\lambda+1}{2})) \\ -\frac{1}{\lambda^{2n-2}}(t - \lambda^n) & x \in [\lambda^{n-1}(\frac{\lambda+1}{2}), \lambda^n) \end{cases} ;$$

(ii) for $t \in (-\infty, 0)$, let $f_\lambda(t) = -f(-x)$.

For $\lambda \in \mathbb{R}^+ \backslash \{1\}$ and $\lambda < 1$, define $f_\lambda(t)$ on $\mathbb{R} \backslash \{0\}$ similar to the case $\lambda > 1$, with the only change being replacing λ by $\frac{1}{\lambda}$.

The function $f(t)$ is continuous on $\mathbb{R} \backslash \{0\}$, and

$$F_{\lambda,f}(x) = \int_x^{\lambda x} f(t)dt = \frac{1}{4}(\lambda - 1)^2$$

is a constant function of x, $(x \neq 0)$. However, $f(t) \neq c/t, c \in \mathbb{R}$.

On the other hand, by the same approach as in the proof of Theorem 2.1, it is not hard to show that if $f(t)$ is a continuous function on $\mathbb{R} \backslash \{0\}$ and $F_{n,f}(x) = \displaystyle\int_x^{nx} f(t)dt$ is a constant function of $x, (x \neq 0)$ for all positive integer n, then $f(t)$ and a function c/t agree on all rational numbers. Hence by the continuity of f, $f(t) = c/t$, $c = f(1) \in \mathbb{R}$. Certainly, the condition that, for all $n \in \mathbb{Z}^+$, the function $F_{n,f}(x)$ is a constant function of x for $x \neq 0$ is very strong. The next theorem shows that in fact we can weaken the sufficient condition quite a lot.

Theorem 3 *Suppose $p, q \in \mathbb{R}^+ \backslash \{1\}$, and $\dfrac{\ln p}{\ln q} \notin \mathbb{Q}$. Let $f(t)$ be a continuous function on $\mathbb{R} \backslash \{0\}$. Then $f(t) = \dfrac{c}{t}$, where $c = f(1)$ is an arbitrary real number if and only if both $F_{p,f}(x) = \int_x^{px} f(t)dt$ and $F_{q,f}(x) = \int_x^{qx} f(t)dt$ are constant functions of $x, (x \neq 0)$.*

Proof. The necessary condition is obvious. We only need to prove the sufficient condition. The function $f(t)$ is continuous, and hence $F_{p,f}(x)$

is differentiable over $\mathbb{R}\backslash\{0\}$. Since $F_{p,f}(x)$ is a constant function of x, we have

$$F'_{p,f}(x) = pf(px) - f(x) = 0,$$

and in turn

$$f(px) = \frac{f(x)}{p}.$$

Inductively, for $l \in \mathbb{Z}^+$ and $l > 1$, we have

$$f(p^l x) = \frac{f(p^{l-1}x)}{p} = ... = \frac{f(x)}{p^l}.$$

Furthermore,

$$f(x) = f(p\frac{x}{p}) = \frac{f(\frac{x}{p})}{p}.$$

and therefore

$$f(\frac{x}{p}) = pf(x).$$

Inductively we have

$$f(\frac{x}{p^n}) = p^n f(x).$$

Similarly, for $k \in \mathbb{Z}^+$,

$$f(q^k x) = \frac{f(x)}{q^k} \text{ and } f(\frac{x}{q^k}) = q^k f(x).$$

It follows that

$$f(p^k q^l x) = \frac{f(x)}{p^k q^l},\ k, l \in \mathbb{Z}.$$

Let $x \in S_{p,q}$. Then $x = \pm p^k q^l,\ k, l \in \mathbb{Z}$. If $x = p^k q^l$, then

$$f(x) = f(p^k q^l) = \frac{f(1)}{p^k q^l} = \frac{f(1)}{x}.$$

If $x = -p^k q^l$, then

$$f(x) = f(p^k q^l(-1)) = \frac{f(-1)}{p^k q^l} = -\frac{f(-1)}{x}.$$

If x is a positive real number, there exists a sequence of numbers $\{x_k : x_k \in S_{p,q}\}$ such that $\lim_{k\to\infty} x_k = x$. Since f is continuous on $\mathbb{R}\backslash\{0\}$,

$$f(x) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \frac{f(1)}{x_k} = \frac{f(1)}{x}.$$

If x is a negative real number, then a similar argument shows that

$$f(x) = -\frac{f(-1)}{x}.$$

Since $F_{p,f}(x)$ is a constant function of x,

$$\int_{-1}^{-p} -\frac{f(-1)}{t} dt = \int_{1}^{p} \frac{f(1)}{t} dt.$$

Evaluating the above integrals, we obtain $-f(-1)\ln p = f(1)\ln p$. Thus $-f(-1) = f(1)$, completing the proof. $\square$

The collection of the dense subsets in $\mathbb{R}$ constructed in Section 2 has a wide range applications, such as pointwise approximation, data fitting, etc (see [2]). Some of these applications will be presented in a future paper.

References

[1] W. Rudin, Principles of Mathematical Analysis, Third Edition, McGraw-Hill, Inc., New York, 1976.

[2] R. A. Devore and G. G. Lorentz, Constructive Approximation, Springer-Verlag, New York, 1991.