Dimensionality-Reducing Expansion, Boundary Type Quadrature Formulas, and the Boundary Element Method

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Dedicated to Professor L. C. Hsu on the Occasion of his 80th Birthday

Abstract

This paper discusses the connection between boundary quadrature formulas constructed by using solutions of partial differential equations and boundary element schemes.

AMS Subject Classfication: 65D30, 65D32, 65N38.

Key Words and Phrases: boundary quadrature formula, boundary element.

1 Introduction

Dimensionality-reducing expansion (DRE) is a technique for numerical integration that reduces a higher dimensional integral to lower dimensional integrals with or without a remainder. Obviously, a DRE can be used to reduce the computational load of many very high dimensional numerical integrations. For instance, the multivariate integral over the 180-dimensional unit cube in the CMO (collateralized mortgage obligation) problem (see [3]) can be reduced to integrals of any dimension by successive applications of certain DRES; it can even be directly reduced to a one-dimensional integral by a single application of the measure theory. In most cases, the computation needed for the reductions and the final integration is miniscule compared to that of the original integration. Most DREs are based on Green's Theorem in the real or complex field. In 1963, using the theorem, Hsu [15] devised a way

to construct a DRE with algebraic precision (degree of accuracy) for multivariate integrations. From 1978 to 1986, building on [15], Hsu, Wang, Zhou, and the author (see [17], [18], [20], [21], and [22]) gave a general process for constructing a DRE with algebraic precision and estimating its remainder. In 1972, with the aid of Green's Theorem and the Schwarz function, P. J. Davis [4] gave an exact DRE, or a DRE without a remainder, for a double integral over a complex field. In 1979, also utilizing Green's Theorem, Kratz [24] constructed an exact DRE for a function that satisfies a type of partial differential equations. Lastly, to complete the introduction, we mention Burrows' DRE for measurable functions, developed in the 1980's. As noted above, his DRE can reduce a multivariate integration to a one dimensional integral. Some important and common applications of DRE include the construction of boundary type quadrature formulas and boundary element schemes.

A Boundary type quadrature formulas (BTQF) is an approximate integration formula with all its of evaluation points lying on the boundary of the integration domain. Such a formula is particularly useful for cases where the values of the integrand function and its derivatives inside the domain are not given or are not easily determined. Boundary quadrature formulas are not really new. Indeed, from the viewpoint of numerical analysis, the classic Euler-Maclaurin summation formula and the Hermite two-end multiple nodes quadrature formula may be regarded as one-dimensional BTQFs since they use only the values of the integrand function and their derivatives at the limits of integration. The earliest example of a BTQF with some algebraic precision for multivariate integration is possibly the formula of algebraic precision degree 5 for a triple integral over a cube given by Sadowsky [28] in 1940. He used 42 points on the surface of a cube to construct the quadrature, which has been modified by the author to one with 32 points, the fewest possible boundary points (see [7] and [8]). Some 20 years later, Levin [26] and [27], Federenko [6], and Ionescu [23] individually investigated certain optimal BTQFs for double integration over squares using partial derivatives at some boundary points of the region. Despite these advances, however, both the general principle and the general technique of construction remained lacking for many years.

In 1978, Hsu, Wang, and Zhou ([18]) developed a general method for constructing BTQFs using the basic principles of multivariate integration; i.e., Hsu's DRE (see [15]). Since then, Hsu, Zhou, Yang, and the author have developed several different methods for constructing BTQFs ([7], [8], [10], [13], [17], [19], [20], [21], and [22]). In this paper, we will look at a recent development on this topic, using BTQFs to develop boundary element schemes.

In next section, we will recall the Kratz's result (see [24]) of replacing a double integral with a single integral and its higher dimension setting shown in [30]. By using this approach, BTQFs related to solutions of partial differential equations will be constructed. Then, the BTQFs will be applied in Section 3 to give a scheme of the boundary element method, which is a method for solving partial differential equations numerically.

2 Construction of BTQFs

In the following, we use $\Omega \equiv V_n$ to denote a bounded and closed region in $\mathbb{R}^n$. Suppose that the boundary of V_n, S_{n-1}, can be described by a system of parametric equations. In particular, points $(x_1, \cdots, x_n)$ on S_{n-1} satisfy equation

$$\Phi(x_1, \cdots, x_n) = 0, \tag{1}$$

where Φ has continuous partial derivatives. In addition, $\Phi(x_1, \cdots, x_n) \leq 0$ for all points in V_n.

We begin with the second order differential operator L defined by

$$Lu = \sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X) \frac{\partial u}{\partial x_i} + c(X)u, \tag{2}$$

where $a_{ij}(X) \in H_n^2(\Omega)$, $b_i(X)$ and $c(X) \in H_n^1(\Omega)$, and $H_n^\alpha(\Omega)(\alpha \geq 1)$ is the collection of all functions $f(X) = f(x_1, \cdots, x_n)$ that have continuous partial derivatives $D^{(i_1, \cdots, i_n)} f$, $0 \leq i_1 + \cdots + i_n \leq \alpha n$, $0 \leq i_k \leq \alpha$, $k = 1, 2, \cdots, n$.

It is well-known that the conjugate operator of L is

$$Mv = \sum_{i,j=1}^{n} \frac{\partial^2 (v a_{ij}(X))}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (v b_i(X))}{\partial x_i} + c(X)v. \tag{3}$$

If we denote by $r_i(X)$ the following expression

$$r_i(X) = -v \sum_{j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} + u \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} + uv \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} - b_i uv, \tag{4}$$

then we have

$$uMv - vLu = \sum_{i=1}^{n} \frac{\partial r_i}{\partial x_i}. \tag{5}$$

Similarly, if we denote

$$p_i = p_i(X) = \sum_{j=1}^{n} a_{ij}(X) \frac{\partial v}{\partial x_j} + \sum_{j=1}^{n} v \frac{\partial a_{ij}(X)}{\partial x_j} - b_i(X)v, \tag{6}$$

then from Equation (5) we obtain

$$uMv - vLu = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(up_i - v \sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right).$$

From this relation, Equation (5), and Green's formula, we have the following result, which is an alternative form of Theorem 1 shown in [30].

Theorem 1 *Let $\Omega \in \mathbb{R}^n$ be an n-dimensional bounded closed domain with the boundary $\partial\Omega$ being a piecewise smooth surface or a simple closed curve with finite length when $n = 2$. Let $u = u(X)$ and $v = v(X)$ be functions in $C^2(\Omega)$, and let L and M be differential operators defined by (2) and (3), respectively. Then we have the identities*

$$\int_\Omega (uMv - vLu)\ dV = \int_{\partial\Omega} \left(\sum_{i=1}^{n} r_i \frac{\partial x_i}{\partial \nu} \right)\ dS \tag{7}$$

and

$$\begin{aligned} & \int_\Omega (uMv - vLu)\ dV \\ = & \int_{\partial\Omega} \left[u \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial \nu} \right]\ dS - \int_{\partial\Omega} \left[v \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right) \frac{\partial x_i}{\partial \nu} \right]\ dS, \end{aligned} \tag{8}$$

where $r_i(X)$ and $p_i(X)$ are defined respectively by (4) and (6); dV and dS are volume element and surface element respectively; and $\frac{\partial x_i}{\partial \nu}$ is the outer normal derivative of x_i on the surface $\partial\Omega$.

Furthermore, if $v = v(X)$ satisfies $Mv = 1$ on Ω, then for any solution, $u = u(X)$ $(X \in \Omega)$ of $Lu = g$, we have identity

$$\begin{aligned} \int_\Omega u(X)\ dV = & \int_\Omega v(X) g(X)\ dV + \int_{\partial\Omega} \left[u(X) \sum_{i=1}^{n} p_i(X) \frac{\partial x_i}{\partial \nu} \right]\ dS \\ & - \int_{\partial\Omega} \left[v(X) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right) \frac{\partial x_i}{\partial \nu} \right]\ dS. \end{aligned} \tag{9}$$

Proof. In Green's formula

$$\int_\Omega \frac{\partial f(X)}{\partial x_i}\ dX = \int_{\partial\Omega} f(X) \frac{\partial x_i}{\partial \nu}\ dS$$

with

$$\frac{\partial x_i}{\partial \nu} = \frac{\partial \Phi}{\partial x_i}\left[\left(\frac{\partial \Phi}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial \Phi}{\partial x_n}\right)^2\right]^{-1/2},$$

we replace $f(X)$ with $r_i(X)$ and immediately obtain (7) from (5). Similarly, we obtain identity (8). In particular, if $v = v(X)$ is a solution to the differential equation $Mv = 1$, then identity (8) can be reduced to (9), which is also called a DRE with the remainder $\int_\Omega v(X)g(X)\,dX$. □

If the remainder term of the identity (9) vanishes; i.e., $g(X)$ satisfies $\int_\Omega v(X)g(X)dV = 0$ (for instance, $g(X) = 0$ and u is a solution of $Lu = 0$), we obtain the following exact DRE for integral $\int_\Omega u\,dX$.

$$\begin{aligned}\int_\Omega u\,dX &= \int_{\partial\Omega}\left[u\sum_{i=1}^n p_i\frac{\partial x_i}{\partial \nu}\right]dS \\ &\quad - \int_{\partial\Omega}\left[v\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial \nu}\right]dS. \qquad (10)\end{aligned}$$

In addition, if v satisfies the boundary condition $v(X) = 0$, $X \in \partial\Omega$ (i.e., v is the solution of the boundary problem $Mv(X) = 1$ $(X \in \Omega)$ and $v(X) = 0$ $(X \in \partial\Omega)$, then expansion (10) can be reduced to

$$\int_\Omega u\,dX = \int_{\partial\Omega}\left[u\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}(X)\frac{\partial v}{\partial x_j}\right)\frac{\partial x_i}{\partial \nu}\right]dS. \qquad (11)$$

The above exact DRE is convenient for computing $\int_\Omega u(X)dV$.

We should point out that from identity (8), we can construct a boundary element scheme. If v is the fundamental solution of equation $Mv = 0$ at point $X^0 \in \partial\Omega$, then identity (8) gives a boundary integral equation, which can be used to solve the function values and derivative values of u on the bounder $\partial\Omega$. Then, we take X^0 as any interior point of Ω and substitute the corresponding fundamental solution v into (8). Thus value of $u(X^0)$ can be evaluated from equation (8). All of this topic will be discussed in next section.

We can specify DRE formula (9) by replacing operator L with classical partial differential operators as L such as elliptic, hyperbolic, and parabolic operators. As an example, let us consider the elliptic operator.

Corollary 2 *Let L shown as (2) be an elliptic operator, in which*

$$a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and v be a solution of Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} - \sum_{i=1}^{n} \frac{\partial (v b_i)}{\partial x_i} + cv = 1, & on\ V_n; \\ v = 0, & on\ S_{n-1}. \end{cases}$$

Then we obtain

$$\int_\Omega u(X)dV = \int_\Omega v(X)g(X)dV + \int_{\partial\Omega} \left[u(X) \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \frac{\partial x_i}{\partial \nu} \right) \right] dS,$$

where function $u = u(X)$ is a solution of equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = g.$$

Using DREs (10) and (11), we can construct a type of boundary quadrature formulas. For instance, let us consider the cubic domain $\Omega = V_n (0 \le x_i \le 1,\ i = 1, \cdots, n)$. From equation (11), we obtain

$$\int_{V_n} udV = \sum_{i=1}^{n} \int_{V_{n-1}} [F_i(X)]_{x_i=0}^{x_i=1} \, dV_{n-1},$$

where $F_i(X) = u(X)p_i(X)$ and

$$\begin{aligned} [F_i(X)]_{x_i=0}^{x_i=1} &= F_i(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n) \\ &\quad - F_i(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n). \end{aligned}$$

Obviously, $F_i \in H_{n-1}^1(V_n)$.

Let $r > 1$. Then for any nature number $k \le N$, it can be written as $k = k_0 + k_1 r + \cdots + k_M r^M$, $0 \le k_j < r$, $0 \le j \le M$, where $M = [\log_r N]$. Define $\phi_r(k) = k_0 r^{-1} + k_1 r^{-2} + \cdots + k_M r^{-M-1}$. Then for the first $n-1$ prime numbers $p_1, \cdots, p_{n-1}$, we call sequence $\{M_k = (\phi_{p_1}(k), \cdots, \phi_{p_{n-1}}(k)), k = 1, 2, \cdots\}$ the Halton sequence.

Denote

$$F(i, M_k; n, N) = \left[F_i \left(\phi_{p_1}(k), \cdots, \phi_{p_{i-1}}(k), x_i, \phi_{p_i}(k), \cdots, \phi_{p_{n-1}}(k) \right) \right]_{x_i=0}^{x_i=1}.$$

We have

$$\int_{V_n} udV = \frac{1}{N} \sum_{i=1}^{n} \sum_{k=1}^{N} F(i, M_k; n, N) + \rho_N,$$

where $N > p_{n-1}$ and the remainder satisfies

$$|\rho_N| \le n 4^{n-1} C \Pi_{i=1}^{n-1} \left(\frac{p_i}{\log p_i} \right) \frac{\log^{n-1} N}{N},$$

where C is a constant and this estimate can not be improved. This result was first given in [30].

3 Boundary element scheme

In this section, we will give applications of the DREs (7) and (8) in the boundary element method.

Let us consider the following boundary value problem

$$\begin{cases} Lu(X) = g(X) & X \in \Omega, \\ u = \overline{u} & X \in \partial\Omega_1, \\ g = \frac{\partial u}{\partial n} = \overline{g} & X \in \partial\Omega_2, \end{cases} \tag{12}$$

where differential operator L is defined by (2) and $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$. We now derive a method for solving the boundary value problem (12) by using the DRE shown in Section 2. Let $u \in C(\bar{\Omega})$ be the solution of $L(u) = g$ and v be a fundamental solution of $Mv = 0$; i.e., a solution of $Mv = \delta(X - X^0)$. Here M is the adjoint differential operator of L, $\delta(X)$ is the Delta function, and X^0, called the source point, is an arbitrarily fixed point in $\bar{\Omega}$. In general, the fundamental solution exists but is not unique. We now evaluate $\int_\Omega uMvdX$ for the fundamental solution v. If $X^0 \in \Omega \backslash \partial\Omega$, then

$$\int_\Omega u(X)Mv(X)dX = \int_\Omega u(X)\delta(X - X^0)dX = u(X^0).$$

DRE (8) yields the following equation of $u = u(X)$.

$$\begin{aligned} u(X^0) &= \int_\Omega vg \; dX + \int_{\partial\Omega} \left[u \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial n} \right] \; dS \\ &- \int_{\partial\Omega} \left[v \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right) \frac{\partial x_i}{\partial n} \right] \; dS. \end{aligned} \tag{13}$$

Therefore, we have the following result regarding the solution of the boundary value problem (12).

Theorem 3 *Let $\Omega \in \mathbb{R}^n$ be an n-dimensional bounded closed domain with the boundary $\partial\Omega$ being a piecewise smooth surface or a simple closed curve with finite length when $n = 2$. Let $u = u(X)$ be a function in $C^2(\Omega)$ that satisfies $Lu = g$, and let $v = v(X)$ be the fundamental solution of $Mv = \delta(X - X^0)$, where X^0 is an arbitrarily fixed point in Ω and adjoint differential operators L and M are defined respectively by (2) and (3).*

If $X^0 \in \Omega \backslash \partial\Omega$, then the solution of the boundary value problem (12) at

the point X^0 *is*

$$u(X^0) = \int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial n}\right]dS - \int_{\partial\Omega}\left[v\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial n}\right]\ dS. \tag{14}$$

For the points $X^0 \in \partial\Omega$, *if* $\epsilon^{n-1}\sum_{i=1}^{n} p_i(\epsilon)\frac{\partial x_i}{\partial n} \to C \neq 0$ *and* $v(\epsilon)\epsilon^{n-1} \to 0$ *as* $\epsilon \to 0$, *where* $\frac{\partial x_i}{\partial n}$ *is the outer normal derivative of* x_i *on the surface* $\{X : |X| = \epsilon\}$, *then*

$$C\beta u(X^0) = \int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial n}\right]\ dS - \int_{\partial\Omega}\left[v\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial n}\right]\ dS, \tag{15}$$

where $\beta = \lim_{\epsilon\to 0}\int_{K_\epsilon} d\theta$ *and* $d\theta$ *is the solid angle of* K_ϵ *with respect to* X^0. *Here* $K_\epsilon = \Omega \cap \partial B_\epsilon$; $B_\epsilon = \{X : |X - X^0| < \epsilon\}$ *is a small ball centered at* X^0.

Remark 1. If the values of $u(X)$ and its normal derivatives on the boundary $\partial\Omega$ are obtained, then expression (14) can be applied to evaluate all values of $u(X^0)$ in the interior of Ω. However, in boundary value problem (12) we are only given some of the values of $u(X)$ and some of its normal derivative values on $\partial\Omega$. Hence, we need to calculate those unknown values on the boundary by using Equation (15). In addition, If $X^0 \in \mathbb{R}^n\backslash\Omega$, then $Mv(X) = 0$ for all $X \in \bar{\Omega}$. Therefore,

$$\int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial n}\right]dS - \int_{\partial\Omega}\left[v\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial n}\right]\ dS = 0. \tag{16}$$

Proof. Noting that $g(X) = 0$, we immediately obtain Equation (14) from (13). To prove (15), we consider a small ball $B_\epsilon = \{X : |X - X^0| < \epsilon\}$ and

apply DRE (8) on the domain $\Omega_\epsilon = \Omega \setminus \overline{B_\epsilon}$. It follows that

$$\begin{aligned} & \int_{\Omega_\epsilon} (uMv - vLu)\, dX \\ = & \int_{\partial\Omega_\epsilon} \left[u \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial n} - v \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right) \frac{\partial x_i}{\partial n} \right] dS. \end{aligned} \tag{17}$$

Substituting the solution of $Lu = g$ ($u \in C(\bar{\Omega})$) into (17) and noting $Mv = 0$ for all $X \in \Omega_\epsilon$, we obtain

$$\begin{aligned} & \int_{\Omega} vg\, dX \\ = & \int_{\partial\Omega_\epsilon} \left[u \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial n} - v \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right) \frac{\partial x_i}{\partial n} \right] dS \\ = & I_\epsilon^{(1)} + I_\epsilon^{(2)}, \end{aligned} \tag{18}$$

where

$$I_\epsilon^{(1)} = \int_{\partial\Omega_\epsilon \setminus K_\epsilon} \left[u \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial n} - v \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u}{\partial x_j} \right) \frac{\partial x_i}{\partial n} \right] dS, \tag{19}$$

$K_\epsilon = \Omega \cap \partial B_\epsilon$, and

$$\begin{aligned} I_\epsilon^{(2)} = & \int_{K_\epsilon} \left[u(X) \sum_{i=1}^{n} p_i (X - X^0) \frac{\partial x_i}{\partial n} \right. \\ & \left. - v(X - X^0) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u(X)}{\partial x_j} \right) \frac{\partial x_i}{\partial n} \right] dS. \end{aligned} \tag{20}$$

Denote by $d\theta$ the solid angle of K_ϵ with respect to X^0. Since the volume element on K_ϵ is $dS = \epsilon^{n-1} d\theta$, Equation (20) becomes

$$\begin{aligned} I_\epsilon^{(2)} = & \int_{K_\epsilon} \left[u(X) \sum_{i=1}^{n} p_i(\epsilon) \frac{\partial x_i}{\partial n} \right. \\ & \left. - v(\epsilon) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial u(X)}{\partial x_j} \right) \frac{\partial x_i}{\partial n} \right] \epsilon^{n-1} d\theta. \end{aligned} \tag{21}$$

If the solid angle of K_ϵ with respect to X^0 tends to β as $\epsilon \to 0$ (i.e., $\lim\limits_{\epsilon\to 0}\int_{K_\epsilon} d\theta = \beta$), then from the mean value theorem for the integral,

$$\lim_{\epsilon\to 0} I_\epsilon^{(2)} = C\beta u(X^0).$$

Therefore, Equation (15) is followed. This completes the proof of the theorem.

□

Next, we consider the exterior boundary value problem (12); i.e., the problem over Ω and the complement of Ω, $\Omega' = \mathbb{R}^n\backslash\Omega$, is bounded. In order to obtain the unique solution to the problem, we also need for $u(x)$ to satisfy the radiation conditions

$$u(X)\|\nabla\ v(X)\| = O\left(|X|^{-n}\right),\ v(X)\|\nabla\ u(X)\| = O\left(|X|^{-n}\right) \tag{22}$$

as $|X| \to \infty$, where $v(X)$ is the fundamental solution of $Mv(X) = 0$ and ∇f is the gradient of f.

Similar to Theorem 3, we have the following result on the exterior boundary value problem.

Theorem 4 *Let $\Omega \in \mathbb{R}^n$ be an n-dimensional unbounded closed domain with the bounded complement $\Omega' = \mathbb{R}^n\backslash\Omega$, and let the boundary $\partial\Omega$ be a piecewise smooth surface or a simple closed curve with finite length when $n = 2$. Suppose that $u = u(X)$ is a function in $C^2(\Omega)$ that satisfies $Lu = g$, and $v = v(X)$ is the fundamental solution of $Mv = \delta(X - X^0)$, where X^0 is an arbitrarily fixed point in Ω and adjoint differential operators L and M are defined respectively by (2) and (3). In addition, we assume $u(X)$ satisfies radiation condition (22).*

If $X^0 \in \Omega\backslash\partial\Omega$, then the solution of the exterior boundary value problem (12) at the point X^0 is

$$\begin{aligned} u(X^0) &= \int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^n p_i\frac{\partial x_i}{\partial n}\right]dS \\ &\quad - \int_{\partial\Omega}\left[v\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial n}\right]\ dS. \end{aligned} \tag{23}$$

For the points $X^0 \in \partial\Omega$, if $\epsilon^{n-1}\sum_{i=1}^n p_i(\epsilon)\frac{\partial x_i}{\partial n} \to C \neq 0$ and $v(\epsilon)\epsilon^{n-1} \to 0$ as $\epsilon \to 0$, where $\frac{\partial x_i}{\partial n}$ is the outer normal derivative of x_i on the surface $\{X : |X| = \epsilon\}$, then

$$C\beta u(X^0) = \int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial n}\right]\ dS - \int_{\partial\Omega}\left[v\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial n}\right]\ dS, \tag{24}$$

where $\beta = \lim_{\epsilon\to 0}\int_{K_\epsilon} d\theta$ *and* $d\theta$ *is the solid angle of* K_ϵ *with respect to* X^0. *Here* $K_\epsilon = \Omega\cap\partial B_\epsilon$; $B_\epsilon = \{X : |X - X^0| < \epsilon\}$ *is a small ball centered at* X^0.

Remark 2. Similar to the Remark 1, if the values of $u(X)$ and its normal derivatives on the boundary $\partial\Omega$ are obtained, then expression (23) can be applied to evaluate all values of $u(X^0)$ in the interior of Ω. However, in boundary value problem (12) we are only given some of the values of $u(X)$ and some of its normal derivative values on $\partial\Omega$. Hence, we need to calculate those unknown values on the boundary by using Equation (24). In addition, If $X^0 \in \mathbb{R}^n\backslash\Omega$, then $Mv(X) = 0$ for all $X \in \bar{\Omega}$. Therefore,

$$\int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial n}\right] - \int_{\partial\Omega}\left[v\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial n}\right]\ dS = 0. \tag{25}$$

Combining Equations (14) and (15) or Equations (23) and (24), we have

$$\alpha u(X^0) = \int_\Omega vg\ dX + \int_{\partial\Omega}\left[u\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial \nu}\right]\ dS - \int_{\partial\Omega}\left[v\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right)\frac{\partial x_i}{\partial \nu}\right]\ dS, \tag{26}$$

where $\alpha = 1$, if $X^0 \in \Omega\backslash\partial\Omega$, and α is a positive real number less than 1, if $X^0 \in \partial\Omega$ (in particular, $\alpha = \frac{1}{2}$ if X^0 is on the smooth boundary). Thus, we obtain an equation about $u(X^0)$ and boundary type weighted integrals of u and $\partial u/\partial n$. Here $\partial u/\partial n = \sum_{i=1}^{n}(\partial u/\partial x_i)(\partial x_i/\partial\nu)$ is the outward normal derivative of u on $\partial\Omega$. Applying a quadrature formula to these boundary integrals, we obtain an algebraic equation, called the basic algebraic equation, about $u(X^0)$ and values of u and $\frac{\partial u}{\partial n}$ at the nodes on $\partial\Omega$. Replacing the

source point X^0 with each node of the BTQF in the basic algebraic equation, we have a system of linear equations about the values of u and $\partial u/\partial n$ at nodes on $\partial\Omega$. Substituting in the given boundary conditions (i.e., the given values of u and $\partial u/\partial n$ at some nodes on $\partial\Omega$), we can solve for the remaining unknown values of u and $\partial u/\partial n$ at the other nodes on $\partial\Omega$. After finding all boundary values of u and $\partial u/\partial n$ at the nodes, the value of u at any interior point X^0 in Ω can be evaluated from the basic algebraic equation with the values of u and $\partial u/\partial n$ at the nodes on $\partial\Omega$. $u(X^0)$ ($X^0 \in \Omega\backslash\partial\Omega$) can also be evaluated directly from (8) by replacing v in (8) with the fundamental solution of $Mv = 0$ and expressing the boundary integrals in (8) as numerical quadrature formulas, which are similar to the formula at the end of Section 2.

As an example, we consider the following boundary value problem of the Helmholtz's equation

$$\begin{cases} (\nabla^2 + k)u = 0 & \text{in } \Omega, \\ u = \overline{u} & \text{on } \partial\Omega_1, \\ q = \frac{\partial u}{\partial n} = \overline{q} & \text{on } \partial\Omega_2, \end{cases} \tag{27}$$

where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$. Obviously, $L = \nabla^2 + k$ is a self-conjugate operator. Therefore, $M = L = \nabla^2 + k$. The fundamental solution of $Mv = \delta(X - X^0)$ is $v = v(X, X^0) = -\frac{j}{4}H_0^{(2)}(kr)$; here $H_0^{(2)}$ is the Hankel function of the second kind of order zero, and $r = |X - X^0|$ denotes the distance from a point $X = (x, y)$ to an arbitrarily fixed source point $X^0 = (x^0, y^0) \in \Omega$. From (26) and observing $g = 0$ and $a_{ij} = \delta_{ij}$, $1 \le i, j \le 2$, we have

$$\alpha u(x^0, y^0) = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS - \int_{\partial\Omega} v\frac{\partial u}{\partial n}\, dS, \tag{28}$$

where $\alpha = 1$ if $(x^0, y^0) \in \Omega\backslash\partial\Omega$; $\alpha = \frac{1}{2}$ if $(x^0, y^0) \in \partial\Omega$ and $\partial\Omega$ is smooth. In (28), since $v = v(x, y; x^0, y^0) = -\frac{j}{4}H_0^{(2)}(kr)$, $r = |(x, y) - (x^0, y^0)|$, we have $\frac{\partial v}{\partial n} = \frac{\partial v}{\partial n}(x, y; x^0, y^0) = \frac{j}{4}kH_1^{(2)}(kr)\cos(\vec{r}, \vec{n})$, where $\vec{r} = (x, y)$, $\vec{n}$ is the outward normal vector at $(x, y) \in \partial\Omega$, and $H_1^{(2)}$ is the first order Hankel function of the second kind. Assume $\partial\Omega$ is defined by the parametric function $c(t) = (x(t), y(t))$, $0 \le t \le 1$. The solution of problem (27) for the two-dimensional scattering problem of electromagnetic wave incident on an infinitely long circular conducting body was discussed in Yashiro and Ohkawa [29], Koshiba and Suzuki [25], and Chan, Chui, and He [2] .

We denote $(x^j, y^j) = (x(t_j), y(t_j))$, $u_j = u(x^j, y^j) = u(x(t_j), y(t_j))$, $q_j = \frac{\partial u}{\partial n}(x^j, y^j) = \frac{\partial u}{\partial n}(x(t_j), y(t_j))$, and $t_j = \frac{j}{n}$, $j = 1, 2, \cdots, n$. Thus, u and

g can be expanded approximately as

$$u \doteq \sum_{j=1}^{n} u_j \phi_j^1(t) \quad \text{and} \quad q \doteq \sum_{j=1}^{n} q_j \phi_j^2(t),$$

where $\phi_j^k(t)$, $k = 1, 2$, are Lagrange interpolation basis functions. We can choose $\phi_j^k(t)$, $k = 1, 2$, either as the basis function $l_j(t) = \prod_{i=1, i\neq j}^{n} \frac{t-t_i}{t_j-t_i}$ or as some other basis functions. For instance, we can assume that $\phi_j^k(t) = \phi^k(t - j/n)$, $j = 1, \cdots, n$, and $\{\phi^k(2nt-2j)\}_{j\in Z}$, $k = 1, 2$, are the basis functions of the optimal spline Lagrange interpolation for the data at the even integers [9]. Here the optimal spline interpolation means the interpolation with a spline basis function that possesses the highest possible approximation order and the smallest possible compact support. Thus, we take

$$\begin{aligned} \phi^1(2nt) &= -\frac{1}{8}N_3(t-1) + \frac{1}{8}N_3(t) + N_3(t+1) \\ &\quad + N_3(t+2) + \frac{1}{8}N_3(t+3) - \frac{1}{8}N_3(t+4), \end{aligned}$$

where $N_3(t)$ is the B-spline of order 3. Hence, the support of $\phi^1(2nt)$ is $[-4, 4]$. It is easy to understand that $\phi^1(i/n) = \delta_{i0}$ and that the corresponding interpolation on the interval [0,1] with basis $\{\phi_j^1(t)\}$ has the optimal approximation order of $O((n)^{-3})$. As for $\phi^2(2nt)$, we assume it is $N_2(t+1)$, B-spline of order 2, which satisfies $\phi^2(i/n) = \delta_{i0}$.

Replacing (x^0, y^0) by $(x^i, y^i) = (x(\frac{i}{n}), y(\frac{i}{n}))$, $i = 1, \cdots, n$, from (28), we have

$$\frac{1}{2}u_i - \sum_{j=1}^{n} u_j \int_0^1 \phi_j^1(t) \frac{\partial v}{\partial n}(t, \frac{i}{n})\, ds(t) = -\sum_{j=1}^{n} q_j \int_0^1 \phi_j^2(t) v(t, \frac{i}{n})\, ds(t), \quad (29)$$

where $v(t, i/n) = v(x(t), y(t); x(i/n), y(i/n))$ and $\partial v/\partial n(t, i/n) = \partial v/\partial n$ $(x(t),\, y(t);\, x(i/n),\, y(i/n))$.

In order to evaluate the boundary integrals in (29), we expand v and $\partial v/\partial n$ in terms of $\psi_{mk}^*(t)$, the periodized version of $\psi_{mk}(t) = 2^{m/2}\psi(2^m - k)$. Here $\psi(t)$ is the wavelet associated with the scaling function $\phi(t) = \phi^1(nt)$ (see [9], [10], [11], [12], and [14]); i.e., $\psi(t) = \sum_k \tilde{q}_k \phi(2t - k)$, where the coefficients $\{\tilde{q}_k\}$ are determined by their two-scale symbol $\tilde{Q}(z) = \frac{1}{2}\sum_k \tilde{q}_k z^k$, $z = e^{-iw/2}$. From the papers by the author [11] and [12], we have $\tilde{Q}(z) = \frac{c(-z)}{c(z^2)}Q(z)$, where $c(z) = -\frac{1}{8}z^{-1} + \frac{1}{8}z^0 + z^1 + z^2 + \frac{1}{8}z^3 - \frac{1}{8}z^4$, $Q(z) = \frac{1}{2}\sum_k q_k z^k$, $z = e^{-iw/2}$, and

$$q_k = \begin{cases} \frac{(-1)^k}{4}\sum_{l=0}^{3} \binom{3}{l} N_6(k+1-l), & \text{if } 0 \le k \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

After finding all boundary integrals in (29), we denote $h_{ij} = \int_0^1 \phi_j^1(t)$ $\partial v/\partial n(t, i/n)\ ds(t) + \delta_{ij}/2$ and $g_{ij} = \int_0^1 \phi_j^2(t)\ v(t, i/n)\ ds(t)$. Thus, (29) can be written as $Hu = Gg$, where $H = [h_{ij}]_{1\le i,j\le n}$, $G = [g_{ij}]_{1\le i,j\le n}$, $u = (u_1, \cdots, u_n)$, and $g = (g_1, \cdots, g_n)^{\mathrm{T}}$. Substituting the boundary conditions (i.e., given values of u_j and g_j on $\partial\Omega$) into the above linear system, we can solve it for the unknown values of u_j and g_j.

If $(x^0, y^0) \in \Omega\backslash\partial\Omega$, from (28), we obtain the value of u at the point (x^0, y^0) by using u_j and g_j on the boundary $\partial\Omega$:

$$u(x^0, y^0) = \sum_{j=1}^{n} q_j \int_0^1 \phi_j^2(t) v(t, t^0)\ ds(t) - \sum_{j=1}^{n} u_j \int_0^1 \phi_j^1(t) \frac{\partial v}{\partial n}(t, t^0)\ ds(t).$$

Thus we obtain the numerical solution of u on the boundary $\partial\Omega$ and at any interior point of Ω.

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