If $F(x) = \int_x^{2x} f(t)\,dt$ Is Constant, Must $f(t) = c/t$?

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Introduction

This work is motivated by the calculus problem of finding the derivative of

$$F(x) = \int_x^{2x} \frac{1}{t}\,dt, \quad x \neq 0$$

(a problem designed for applying the Fundamental Theorem of Calculus and the Chain Rule). It is easy to see that $F'(x) = 0$, which implies a nice geometric fact: for any given $x \neq 0$, the area between the curves $y = 1/t$, $y = 0$, and the x-axis from x to $2x$ is constant. Clearly, for any constant c, the function $y = c/t$ also has this property. It is natural to ask whether the converse is true or not; that is, if f is a continuous function on $\mathbb{R}\backslash\{0\}$ and if $F_{2,f}(x) = \int_x^{2x} f(t)\,dt$ is a constant function of x ($x \neq 0$), must $f(t) = c/t$ for some constant $c \in \mathbb{R}$? The example in the next section shows that the answer is no. Similarly, there exists a function $g \neq c/x$ for which $F_{3,g}(x) = \int_x^{3x} g(t)\,dt$ is a constant function of x. However, one may ask whether both $F_{2,h}(x)$ and $F_{3,h}(x)$ being constant functions leads to $h(t) = c/t$? Equivalently, does there exist a function $k(t) \neq c/t$ that can be constructed by "combining" f and g so that both $F_{2,k}(x)$ and $F_{3,k}(x)$ are constant? We study this problem in the next section and give an answer in the negative; that is, if both $F_{2,k}(x)$ and $F_{3,k}(x)$ are constant, then $k(t)$ must be a constant multiple of $1/t$. In addition, we will show that if positive reals p and q satisfy $\ln p/\ln q \notin \mathbb{Q}$, then both $F_{p,f}(x)$ and $F_{q,f}(x)$ are constants if and only if $f(t) = f(1)/t$. The study of the problem relies on dense subsets of $\mathbb{R}$. In the next section we give a collection of "small" subsets that are dense in $\mathbb{R}$. (Only after finishing the paper did we find out that a special case of Theorem 2 in the next section was obtained by Moser and Macon [**2**]. They derived a similar result for the integer setting. Moreover, Heuer also gave an alternative proof of Moser and Macon's result in [**1**]. However, we have decided to retain the section as we generalize the result to pairs of real numbers and give a necessary and sufficient condition. In addition, our proof is different, self-contained, and elementary.) Our main results will be shown in the next section, and their proofs are given in the final section.

Main results

For simplicity, we restrict our attention to functions defined on $\mathbb{R}^+$; clearly our results and functions can be extended to hold on $\mathbb{R} - \{0\}$.

Let $f(t)$ be a continuous function on $\mathbb{R}^+$, and let

$$F_{\lambda,f}(x) = \int_x^{\lambda x} f(t)\,dt, \quad \lambda \in \mathbb{R}^+\backslash\{1\}.$$

If $f(t) = c/t$, $c \in \mathbb{R}$, then for $x > 0$, $F_{\lambda,f}(x) = c\ln\lambda$, a constant. However, the following example shows that the converse is not true; that is, $F_{\lambda,f}(x)$ being constant does not imply that $f(t) = c/t$ for some constant c.

Example. Define a function $f_2(t)$ recursively (see Figure 1):

(i) for $t \in [1, 2)$, let

$$f_2(t) = \begin{cases} t - 1, & t \in \left[1, \dfrac{3}{2}\right) \\ -t + 2, & t \in \left[\dfrac{3}{2}, 2\right); \end{cases}$$

(ii) for $t \in [2^{n-1}, 2^n)$, $n = 2, 3, \ldots$, let $f_2(t) = (1/2^n) f_2(t/2^n)$;

(iii) for $t \in [1/2^n, 1/2^{n-1})$, $n = 1, 2, \ldots$, let $f_2(t) = 2^n f_2(2^n t)$.

Then $f_2(t)$ is continuous on $\mathbb{R}^+$ and

$$F_{2,f_2}(x) = \int_x^{2x} f_2(t)\,dt = \frac{1}{4}.$$

In fact, this example can be generalized to a function $f_\lambda(t)$ with any parameter $\lambda \neq 1$ in a natural way. For $\lambda > 1$, define $f_\lambda(t)$ on $\mathbb{R}^+$ as follows: for $t \in [\lambda^{n-1}, \lambda^n)$, $n \in \mathbb{Z}$, let

$$f_\lambda(t) = \begin{cases} \dfrac{1}{\lambda^{2n-2}}(t - \lambda^{n-1}), & t \in \left[\lambda^{n-1}, \dfrac{1}{2}\lambda^{n-1}(\lambda + 1)\right) \\ -\dfrac{1}{\lambda^{2n-2}}(t - \lambda^n) & t \in \left[\dfrac{1}{2}\lambda^{n-1}(\lambda + 1), \lambda^n\right). \end{cases}$$

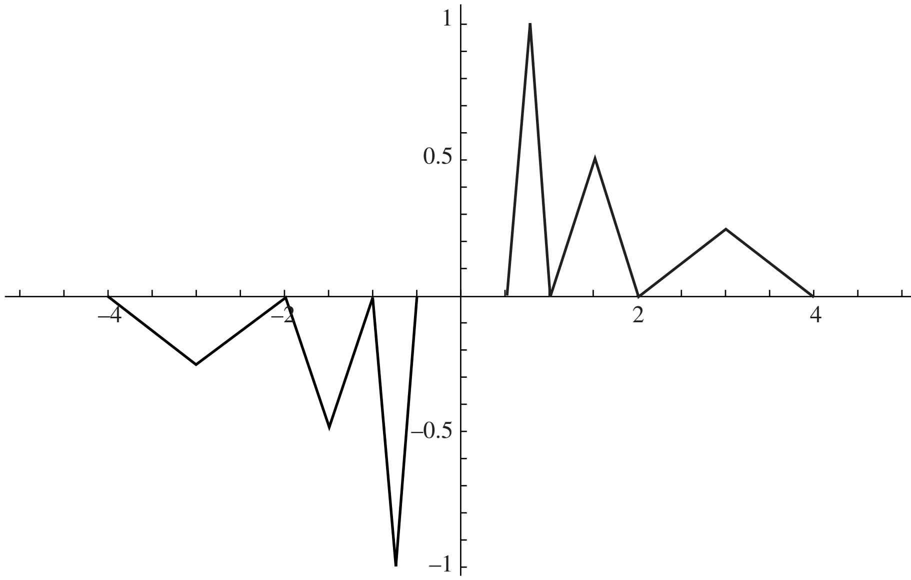

Figure 1. Figure of function $f_2(t)$ when $t \in [-4, -1/2] \cup [1/2, 4]$

For $0 < \lambda < 1$, define $f_\lambda(t)$ on $\mathbb{R}^+$ similar to the case $\lambda > 1$, with the only change being replacing λ by $1/\lambda$.

The function $f_\lambda(t)$ is continuous on $\mathbb{R}^+$, and

$$F_{\lambda, f_\lambda}(x) = \int_x^{\lambda x} f_\lambda(t)\,dt = \frac{1}{4}(\lambda - 1)^2,$$

a constant. Clearly, $f_\lambda(t) \neq c/t$ for any $c \in \mathbb{R}$.

On the other hand, it is not hard to show (by the same approach as in the proof of Theorem 1, e.g., choosing $n = 2$ and $n = 3$) that if $f(t)$ is a continuous function on $\mathbb{R}^+$ and $F_{n,f}(x) = \int_x^{nx} f(t)\,dt$ is a constant function for all positive integers n, then $f(t)$ and a function c/t agree on all rational numbers. Hence by the continuity of f, $f(t) = f(1)/t$. Certainly, the condition that, for all $n \in \mathbb{Z}^+$, the function $F_{n,f}(x)$ is constant is very strong. The following theorem shows that in fact we can weaken the sufficient condition considerably.

Theorem 1. *Let p and q be positive reals with $\ln p/\ln q$ irrational, and let function $f(t)$ be continuous on $\mathbb{R}^+$. Then $f(t) = f(1)/t$ if and only if both $F_{p,f}(x)$ and $F_{q,f}(x)$ are constant on $\mathbb{R}^+$.*

The complete proof will be given in next section, but we outline it here. Define $S_{p,q} = \{\pm p^k q^l : k, l \in \mathbb{Z}\}$. We first show that for every $t \in S_{p,q}$, $f(t) = f(1)/t$; that is, for all integers k and l, $f(p^k q^l) = f(1)/(p^k q^l)$. Thus, if $S_{p,q}$ is dense in $\mathbb{R}^+$, then $f(x) = f(1)/x$ for all $x \in \mathbb{R}^+$. It is easy to see that $S_{p,q}$ is not dense in $\mathbb{R}^+$ if $\ln p/\ln q$ is rational. However, the following theorem shows that when $\ln p/\ln q$ is irrational, $S_{p,q}$ is dense in $\mathbb{R}^+$.

Theorem 2. *Let p and q be positive reals different from 1. Then $S_{p,q}$ is dense in $\mathbb{R}^+$ if and only if $\log_q p$ is irrational.*

Proofs

To establish Theorem 2, we need a technical lemma.

Lemma. *Let p and q be positive reals different from* 1 *with* $\log_q p \notin \mathbb{Q}$. *Then*

(i) *There exists a decreasing sequence* $\{a_n\}$ *of numbers in* $S_{p,q}$ *for which*

$$\lim_{n\to\infty} a_n = 1^+;$$

(ii) *There exists an increasing sequence* $\{b_n\}$ *of numbers in* $S_{p,q}$ *for which*

$$\lim_{n\to\infty} b_n = 1^-.$$

Proof. Since $S_{p,q}$ is closed under taking reciprocals, we may assume that $1 < p < q$. Let $s_1 = \ln p$ and $t_1 = \ln q$. Then $0 < s_1 < t_1$. Since $\ln p$ is not a rational multiple of $\ln q$, there exists an integer $n_1 \in \mathbb{Z}^+$ such that

$$n_1 s_1 < t_1 < (n_1+1)s_1 \quad \text{and} \quad t_1 \neq \frac{n_1 s_1 + (n_1+1)s_1}{2}.$$

Let

$$s_2 = \min\{t_1 - n_1 s_1, (n_1+1)s_1 - t_1\}$$

and

$$t_2 = \max\{t_1 - n_1 s_1, (n_1+1)s_1 - t_1\}.$$

We have

(1) $0 < s_2 < \frac{1}{2}s_1 < t_2 < s_1 < t_1$ (since

$$t_1 \neq \frac{n_1 s_1 + (n_1+1)s_1}{2}$$

and $n_1 s_1 < t_1 < (n_1+1)s_1$);

(2) t_2 is not a rational multiple of s_2 (otherwise $(n_1+1)s_1 - t_1 = k(t_1 - n_1 s_1)$, $k \in \mathbb{Q}$, which implies that t_1 is a rational multiple of s_1, i.e.,

$$\frac{\ln p}{\ln q} (= \log_q p) \in \mathbb{Q},$$

a contradiction);

(3) s_2 and t_2 are linear combinations of $\ln p$ and $\ln q$ with integer coefficients.

Construct $s_2, t_2, s_3, t_3, \ldots$ inductively. Assume s_i and t_i are constructed, satisfying

(4) $0 < s_i < \frac{1}{2}s_{i-1} < t_i < s_{i-1} < t_{i-1}$;

(5) t_i is not a rational multiple of s_i;

(6) s_i and t_i are linear combinations of $\ln p$ and $\ln q$ with integer coefficients.

Let $s_{i+1} = \min\{t_i - n_i s_i, (n_i+1)s_i - t_i\}$ and $t_{i+1} = \max\{t_i - n_i s_i, (n_i+1)s_i - t_i\}$, where n_i is the unique integer satisfying $n_i s_i < t_i < (s_i+1)s_i$. It is easy to show inductively that (4)–(6) are true for s_{i+1} and t_{i+1}. Therefore, two sequences $\{s_n, n \geq 1\}$ and $\{t_n, n \geq 1\}$ are constructed, satisfying (4)–(6). By (4), $s_1 > s_2 > \cdots > s_n > \cdots > 0$

and $\lim_{n\to\infty} s_n = 0^+$. For $n = 1, 2, \ldots$, let $a_n = e^{s_n}$. Then

$$a_n \in S_{p,q}, \;\; a_1 > a_2 > \cdots > a_n > \cdots > 1, \quad \text{and} \quad \lim_{n\to\infty} a_n = 1^+.$$

Thus part (i) of the lemma is true. Let $b_n = 1/a_n$, $n \geq 1$. Then $b_1 < b_2 < \cdots < b_n < \cdots < 1$ and $\lim_{n\to\infty} b_n = 1^-$. Therefore part (ii) of the lemma is also true. ∎

Proof of Theorem 2. Let $c \in \mathbb{R}^+$. If $c \in S_{p,q}$, there is nothing to prove, so suppose $c \notin S_{p,q}$. We want to find a sequence $\{x_n\}$ in $S_{p,q}$ with $x_n \to c$. Let $S^- = S_{p,q} \cap (0, c)$ and $\alpha = \sup\ S^-$. Then $\alpha \leq c$, and there is a sequence $\{x_n\}$ in S^- converging to α, so it suffices to show that $\alpha = c$. We shall show that $\alpha < c$ is impossible. If $\alpha < c$ we may by the lemma choose a_i in $S_{p,q}$ such that $1 < a_i < 1 + (c - \alpha)/2\alpha$. Thus $\alpha/a_i < \alpha$, so there exists n_0 such that $\alpha/a_i < x_{n_0} < \alpha$. Then $x_{n_0} a_i \in S_{p,q}$ but

$$\alpha < x_{n_0} a_i < \alpha a_i < \alpha \left(1 + \frac{c - \alpha}{2\alpha}\right) = \frac{\alpha + c}{2} < c,$$

showing that $x_{n_0} a_i \in S^-$ and $x_{n_0} a_i > \alpha$, which is impossible because $\alpha = \sup\ S^-$. Thus $\alpha = c$. ∎

Proof of Theorem 1. The necessity of the given condition is obvious, so we need only prove the sufficiency. The function $f(t)$ is continuous, and hence $F_{p,f}(x)$ is differentiable over $\mathbb{R}\backslash\{0\}$. Since $F_{p,f}(x)$ is a constant function of x, we have $F'_{p,f}(x) = pf(px) - f(x) = 0$, and in turn $f(px) = f(x)/p$. Inductively, for $l \in \mathbb{Z}^+$ and $l > 1$, we have

$$f(p^l x) = \frac{f(p^{l-1}x)}{p} = \cdots = \frac{f(x)}{p^l}.$$

Furthermore,

$$f(x) = f\left(p\frac{x}{p}\right) = \frac{f(\frac{x}{p})}{p},$$

and therefore $f(x/p) = pf(x)$. Inductively we have $f(x/p^n) = p^n f(x)$. Similarly, for $k \in \mathbb{Z}^+$,

$$f(q^k x) = \frac{f(x)}{q^k} \quad \text{and} \quad f\left(\frac{x}{q^k}\right) = q^k f(x).$$

It follows that

$$f(p^k q^l x) = \frac{f(x)}{p^k q^l}, \;\; k, l \in \mathbb{Z}.$$

Let $x \in S_{p,q}$. Then $x = \pm p^k q^l$ for all $k, l \in \mathbb{Z}$. If $x = p^k q^l$, then

$$f(x) = f(p^k q^l) = \frac{f(1)}{p^k q^l} = \frac{f(1)}{x}.$$

If $x = -p^k q^l$, then

$$f(x) = f(p^k q^l(-1)) = \frac{f(-1)}{p^k q^l} = -\frac{f(-1)}{x}.$$

If x is a positive real number, there exists a sequence of numbers $\{x_k : x_k \in S_{p,q}\}$ such that $\lim_{k\to\infty} x_k = x$. Since f is continuous on $\mathbb{R}\backslash\{0\}$,

$$f(x) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \frac{f(1)}{x_k} = \frac{f(1)}{x}.$$

If x is negative, a similar argument shows that $f(x) = -f(-1)/x$.

Since $F_{p,f}(x)$ is a constant function of x,

$$\int_{-1}^{-p} -\frac{f(-1)}{t}\,dt = \int_{1}^{p} \frac{f(1)}{t}\,dt.$$

Evaluating the above integrals, we obtain $-f(-1)\ln p = f(1)\ln p$. Thus $-f(-1) = f(1)$, completing the proof. ■

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Numeration

Nine figures are sufficient to express any number in common practice: neverthe-less, the following table may be thought necessary.

Nonillions	*Octillions*	*Septillions*	*Sextillions*	*Quintillions*
8 5 7 3 4 2,	1 6 2 4 8 6,	3 4 5 9 8 6,	4 3 7 9 1 6,	4 2 3 1 4 7,
Quadrillions	*Trillions*	*Billions*	*Millions*	*Units*
2 4 8 0 1 6,	2 3 5 4 2 1,	2 6 1 7 3 4,	3 6 8 1 4 9,	6 2 3 1 3 7.

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page 2.