# Biorthogonal Spline Type Wavelets 

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#### Abstract

Let $\phi$ be an orthonormal scaling function with approximation degree $p-1$, and let $B_{n}$ be the B-spline of order $n$. Then, spline type scaling functions defined by $\bar{f}_{n}=f * B_{n}(n=1,2, \ldots)$ possess higher approximation order, $p+n-1$, and compact support. The corresponding biorthogonal wavelet functions are also constructed. This technique is extended to the case of biorthogonal scaling function system. As an application of the method supplied in this paper, one can easily construct a sequence of pairs of biorthogonal spline type scaling functions from one pair of biorthogonal scaling functions or an orthonormal scaling function. In particular, if both the method and the lifting scheme of Sweldens (see [1]) are applied, then all pairs of biorthogonal spline type scaling functions shown in references [2] and [3] can be constructed from the Haar scaling function. © 2004 Elsevier Science Ltd. All rights reserved.


Keywords-Biorthogonal wavelets, B-splines, Spline type scaling functions, Backward-difference, Forward-difference.

## 1. INTRODUCTION

We denote $\phi(t)$ an orthogonal scaling function that satisfies dilation equation (or refinement equation)

$$
\begin{equation*}
\phi(t)=\sum_{k} c_{k} \phi(2 t-k), \tag{1}
\end{equation*}
$$

where the constant coefficients $c_{k}$ satisfy the following four properties.
(i) $c_{k}=0$, for $k \notin\{0,1, \ldots, 2 p-1\}$;
(ii) $\sum_{k} c_{k}=2$;
(iii) $\sum_{k}(-1)^{k} k^{m} c_{k}=0$, for $0 \leq m \leq p-1$;
(iv) $\sum_{k} c_{k} c_{k-2 m}=2 \delta_{0 m}$, for $1-p \leq m \leq p-1$.

Solutions of $c_{k}$ for $1 \leq p \leq 10$ that satisfy Items (i)-(iv) can be found in [3] and [4]. For instance, for $p=1$ and $p=2$, the corresponding scaling functions are, respectively, the Haar scaling function and the Daubechies $D_{4}$ scaling function.

For each $p \geq 1$, it was shown in [3] that the dilation equation (1) has a unique solution such that $\operatorname{supp} \phi=[0,2 p-1]$. There are several ways (cf. [5] and [6]) to construct the solution $\phi$, e.g., Daubechies method (cf. [4]), upon which the scaling function is constructed by using iteration

$$
\begin{equation*}
\phi^{n}(t)=\sum_{k} c_{k} \phi^{n-1}(2 t-k), \quad n=0,1, \ldots, \tag{2}
\end{equation*}
$$

where $\phi^{0}$ is the characteristic function over $[0,1], \chi_{[0,1]}$. For $p=1, \phi^{0}$ is invariant in (2) and hence $\phi^{n}=\chi_{[0,1]}(n=1,2, \ldots)$. For $p=2, \phi \in C^{r}, r \approx 0.55$. From Property (iii) and equation (2), we obtain $\sum_{k} \phi(t-k)=1$. In general, one can show that the polynomials $1, x, \ldots, x^{p-1}$ are in the subspace spanned by $\{\phi(t-k)\}$; i.e., the approximation degree of $\phi$ is $p$. Taking integral over $\mathbb{R}$ on both sides of equation (1), we find Item (ii) is equivalent to $\int_{\mathbb{R}} \phi(t) d t=1$. Taking Fourier transform,

$$
\hat{g}(\xi)=\int_{-\infty}^{\infty} g(t) e^{-i \xi t} d t
$$

on both sides of equation (1), we have $\hat{\phi}(\xi)=m_{\phi}(\xi / 2) \hat{\phi}(\xi / 2)$, where

$$
m_{\phi}(\xi)=2^{-1} \sum_{k} c_{k} e^{-i k \xi}
$$

is called the mask of $\phi$. It is easy to find Item (iv) is equivalent to the orthonormality of $\{\phi(t-k)\}_{k \in \mathcal{Z}}$; i.e., $\langle\phi(t), \phi(t-k)\rangle=\delta_{0 k}$. The orthogonality condition of $\{\phi(t-k)\}$ becomes

$$
\sum_{k}|\hat{\phi}(\xi+2 \pi k)|^{2}=1
$$

that can be further described as $\left|m_{\phi}(\xi)\right|^{2}+\left|m_{\phi}(\xi+\pi)\right|^{2}=1$.
Given $\phi(t)$, we define the corresponding wavelet function as

$$
\begin{equation*}
\psi(t)=\sum_{k}(-1)^{k-1} c_{-k-1} \phi(2 t-k) \tag{3}
\end{equation*}
$$

Here, supp $\psi=[1-p, p]$. For $p=1$ and $p=2$, the corresponding wavelet functions are called Haar wavelet and Daubechies $D_{4}$ wavelets. For given $p$, let $c_{k}$ satisfy Properties (i)-(iv), and let $\phi$ satisfy equation (1). The corresponding wavelet defined by (3) generates an orthonormal basis, $\psi_{j k}(t)$, of $L^{2}(\mathbb{R})$, where

$$
\begin{equation*}
\psi_{j k}(t)=2^{j / 2} \psi\left(2^{j} t-k\right) \tag{4}
\end{equation*}
$$

In addition, the first $p$ moments of $\psi_{j k}$ equal to zero

$$
\int_{-\infty}^{\infty} t^{m} \psi_{j k}(t) d t=0, \quad 0 \leq m \leq p-1
$$

From (3), the mask of $\psi$ is

$$
m_{1}(\xi)=e^{i \xi} \overline{m_{\phi}(\xi+\pi)}
$$

and hence the Fourier transform of $\psi$ satisfies $\hat{\psi}=m_{1}(\xi / 2) \hat{\psi}(\xi / 2)$.
If two scaling functions, $\phi$ and $\tilde{\phi}$, satisfy $\langle\phi(t), \tilde{\phi}(t-k)\rangle=\delta_{0 k}$, we call them biorthogonal scaling functions. If the dilation equations of $\phi$ and $\tilde{\phi}$ are

$$
\begin{equation*}
\phi(t)=\sum_{k} c_{k} \phi(2 t-k), \quad \tilde{\phi}(t)=\sum_{k} \tilde{c}_{k} \phi(2 t-k) \tag{5}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\hat{\phi}(\xi)=m_{\phi}\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right), \quad \hat{\tilde{\phi}}(\xi)=\tilde{m}_{\tilde{\phi}}\left(\frac{\xi}{2}\right) \hat{\tilde{\phi}}\left(\frac{\xi}{2}\right) \tag{6}
\end{equation*}
$$

then the corresponding biorthogonal wavelets $\psi$ and $\tilde{\psi}$ can be defined by

$$
\begin{equation*}
\psi(x)=\sum_{k}(-1)^{k-1} \tilde{h}_{-k-1} \phi(2 x-k), \quad \tilde{\psi}(x)=\sum_{k}(-1)^{k-1} h_{-k-1} \tilde{\phi}(2 x-k) \tag{7}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\hat{\psi}(\xi)=e^{i \xi / 2} \overline{\tilde{m}_{\tilde{\phi}}\left(\frac{\xi}{2}+\pi\right)} \hat{\phi}\left(\frac{\xi}{2}\right), \quad \hat{\tilde{\psi}}(\xi)=e^{i \xi / 2} \overline{m_{\phi}\left(\frac{\xi}{2}+\pi\right)} \tilde{\tilde{\phi}}\left(\frac{\xi}{2}\right) \tag{8}
\end{equation*}
$$

Here, $m_{\phi}(\xi)$ and $\tilde{m}_{\tilde{\phi}}(\xi)$ are, respectively, masks of $\phi$ and $\tilde{\phi}$,

$$
m_{\phi}(\xi)=2^{-1} \sum_{k} c_{k} e^{-i k \xi}, \quad \tilde{m}_{\tilde{\phi}}(\xi)=2^{-1} \sum_{k} \tilde{c}_{k} e^{-i k \xi}
$$

and coefficients $c_{k}$ and $\tilde{c}_{k}$ satisfy Items (i)-(iii) and Item (iv)', i.e.,
(i) $c_{k}=\tilde{c}_{\tilde{k}}=0$, for $k \notin\{0,1, \ldots, 2 p-1\}$ and $\tilde{k} \notin\{0,1, \ldots, 2 \tilde{p}-1\}$;
(ii)' $\sum_{k} c_{k}=\sum_{k} \tilde{c}_{k}=2$;
(iii)' $\sum_{k}(-1)^{k} k^{m} c_{k}=\sum_{k}(-1)^{k} k^{\tilde{m}} \tilde{c}_{k}=0$, for $0 \leq m \leq p-1$ and $0 \leq \tilde{m} \leq \tilde{p}-1$;
(iv) ${ }^{\prime} \sum_{k} c_{k} \tilde{c}_{k-2 m}=2 \delta_{0 m}$, for $1-\tilde{p} \leq m \leq \tilde{p}-1$.

Let $\phi$ be an orthonormal scaling function with approximation degree $p-1$. In this paper, we construct spline type scaling functions, $\bar{f}_{n}$, by using convolution: $\bar{f}_{n}=f * B_{n}$; for $n=1,2, \ldots$, where $B_{n}$ is the B -spline of order $n$. In Section 2, we will show that $\bar{f}_{n}$ is a scaling function with approximation degree $p+n-1$, and its dilation equation will also be established. Since the orthogonality is not generally preserved by $\bar{f}_{n}$, we will construct biorthogonal scaling functions and biorthogonal wavelets from $\bar{f}_{n}$ in Section 3. In addition, we will extend this technique to biorthogonal scaling function system. An application of this type of wavelet can be found in [7], in which $\bar{f}_{1}$ was used to construct Galerkin basis functions in Galerkin-wavelet methods, which produce well-conditioned stiffness matrices and satisfy the Dirichlet boundary conditions. Another application is given in Example 3.4, in which biorthogonal scaling functions with higher regularities are constructed. Shann and Tzeng [8] studied the case $n=1$ of this technique, which is called the levering scheme there and is considered as a complementary of the lifting scheme (see [1]). As an application, using the method supplied in this paper one can construct a sequence of pairs of biorthogonal spline type scaling functions from one pair of biorthogonal spline type scaling functions. In particular, using both the method and the lifting scheme of Sweldens, one can construct all pairs of biorthogonal spline type scaling functions shown in [2] and [3] from the Haar scaling function (see Example 3.3).

## 2. SPLINE TYPE SCALING FUNCTIONS CONSTRUCTED BY USING CONVOLUTION

Denote $B_{n}(x)$ the B-spline of degree $n-1$ having notes as $0,1, \ldots, n$; i.e., $B_{n}(x)=Q_{n}(x)=$ $M(x ; 0,1, \ldots, n)$, which is defined by (1.1) on page 11 of [9]. We first give a general expression of the convolution $\bar{f}_{n}=f * B_{n}$ for bounded function $f$.

Theorem 1. Suppose that $f: \mathbb{R} \rightarrow \mathbb{C}$ is bounded. Then, its convolution with $B$-spline $B_{n}$, denoted by $\bar{f}_{n}$, exists and satisfies

$$
\begin{equation*}
\bar{f}_{n}(x)=\left(f * B_{n}\right)(x)=\int_{-\infty}^{x} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \nabla^{n} f\left(t_{1}\right) d t_{1} \cdots d t_{n} \tag{9}
\end{equation*}
$$

where $\nabla^{n}$ is the backward-difference operator defined by $\nabla f(t)=f(t)-f(t-1)$ and $\nabla^{n} f(t)=$ $\nabla\left(\nabla^{n-1} f(t)\right)$.
Proof. Since $f$ is bounded by a constant $C$ and

$$
\int_{\mathbb{R}}\left|f(x-t) B_{n}(t)\right| d t \leq C \int_{\mathbb{R}} B_{n}(t) d t=C
$$

convolution $\bar{f}_{n}=f * B_{n}$ is well defined.

We now prove formula (9) using mathematical induction. For $n=1$,

$$
\begin{aligned}
\bar{f}_{n}(x) & =\left(f * B_{1}\right)(x)=\int_{-\infty}^{\infty} f(x-t) B_{1}(t) d t=\int_{0}^{1} f(x-t) d t=\int_{x-1}^{x} f(t) d t \\
& =\int_{-\infty}^{x}(f(t)-f(t-1)) d t=\int_{-\infty}^{x} \nabla f(t) d t
\end{aligned}
$$

Assume inductively that equation (9) holds for $n=k$, we now turn to the case of $n=k+1$. Using identity (2.4) in [9] and identity (vii) in [10] and substituting $x-t_{k+1} \mapsto t_{k+1}$ yields

$$
\begin{aligned}
\bar{f}_{k+1}(x) & =\int_{\mathbb{R}} f(x-t) B_{k+1}(t) d t \\
& =\int_{[0,1]^{k+1}} f\left(x-t_{1}-t_{2}-\cdots-t_{k+1}\right) d t_{1} d t_{2} \cdots d t_{k+1} \\
& =\int_{[0,1]^{k}}\left(\int_{0}^{1} f\left(x-t_{k+1}-t_{1}-\cdots-t_{k}\right) d t_{k+1}\right) d t_{1} \cdots d t_{k} \\
& =\int_{[0,1]^{k}}\left(\int_{x-1}^{x} f\left(t_{k+1}-t_{1}-\cdots-t_{k}\right) d t_{k+1}\right) d t_{1} \cdots d t_{k} \\
& =\int_{x-1}^{x}\left(\int_{[0,1]^{k}} f\left(t_{k+1}-t_{1}-\cdots-t_{k}\right) d t_{1} \cdots d t_{k}\right) d t_{k+1} \\
& =\int_{x-1}^{x} \int_{\mathbb{R}} B_{k}(t) f\left(t_{k+1}-t\right) d t d t_{k+1}
\end{aligned}
$$

Applying the inductive assumption, the last integral can be written as

$$
\begin{aligned}
& \int_{x-1}^{x} \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k} f\left(t_{1}\right) d t_{1} \cdots d t_{k} d t_{k+1} \\
= & \int_{-\infty}^{x} \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k} f\left(t_{1}\right) d t_{1} \cdots d t_{k} d t_{k+1} \\
- & \int_{-\infty}^{x-1} \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k} f\left(t_{1}\right) d t_{1} \cdots d t_{k} d t_{k+1}
\end{aligned}
$$

Using substitution $t_{k+1} \mapsto t_{k+1}-1, t_{k} \mapsto t_{k}-1, \ldots, t_{2} \mapsto t_{2}-1$, and $t_{1} \mapsto t_{1}-1$ successively in the second integral on the right side of the above equation, we change the right side into

$$
\begin{aligned}
\int_{-\infty}^{x} & \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k} f\left(t_{1}\right) d t_{1} \cdots d t_{k} d t_{k+1} \\
& -\int_{-\infty}^{x} \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k} f\left(t_{1}-1\right) d t_{1} \cdots d t_{k} d t_{k+1} \\
= & \int_{-\infty}^{x} \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k}\left(f\left(t_{1}\right)-f\left(t_{1}-1\right)\right) d t_{1} \cdots d t_{k} d t_{k+1} \\
= & \int_{-\infty}^{x} \int_{-\infty}^{t_{k+1}} \int_{-\infty}^{t_{k}} \cdots \int_{-\infty}^{t_{2}} \nabla^{k+1} f\left(t_{1}\right) d t_{1} \cdots d t_{k} d t_{k+1}
\end{aligned}
$$

Hence, identity (9) has been derived.
REmARK 2.1. The backward-difference operator shown in equation (9) has the following property. Let $\phi: \mathbb{R} \mapsto \mathbb{R}$ be a bounded function. Then,

$$
\sum_{j=0}^{n}\binom{n}{j} \nabla^{n} f(t-j)=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} f(t-2 j)
$$

The above equation can be proved by applying the following obvious identity to the left-hand side of the equation:

$$
\left.\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{n}{k-j}=(-1)^{[k / 2]} \frac{1+(-1)^{k}}{2}\left(\begin{array}{c}
n \\
\frac{k}{2}
\end{array}\right]\right)
$$

where $k$ is any nonnegative integer.
From definition of $\bar{\phi}$ shown in (9) and Fourier transform of B-spline,

$$
\begin{equation*}
\hat{B}_{n}(\xi)=e^{-i n \xi / 2}\left(\frac{\sin (\xi / 2)}{\xi / 2}\right)^{2}=\left(\frac{1-e^{-i \xi}}{\xi}\right)^{n} \tag{10}
\end{equation*}
$$

the Fourier transform of $\bar{\phi}_{n}$ can be written as

$$
\begin{equation*}
\hat{\bar{\phi}}_{n}(\xi)=\hat{B}_{n}(\xi) \hat{\phi}(\xi)=e^{-i n \xi / 2}\left(\frac{\sin (\xi / 2)}{\xi / 2}\right)^{n} \hat{\phi}(\xi) \tag{11}
\end{equation*}
$$

We shall show that if $f$ in Theorem 1 is a scaling function, then the corresponding $\bar{f}_{n}, n=$ $1,2, \ldots$, are also scaling functions with higher approximation orders and compact supports. We now establish the refinement relation of $\bar{f}_{n}$ as follows.

It is obvious that the convolution of two refinable functions is also refinable. Since $B_{n}$ is refinable with mask (see equation (10))

$$
\begin{equation*}
m_{B_{n}}(\xi)=\frac{\hat{B}_{n}(2 \xi)}{\hat{B}_{n}(\xi)}=\left(\frac{1+e^{-i \xi}}{2}\right)^{n} \tag{12}
\end{equation*}
$$

$\bar{f}_{n}$ is refinable if $f$ is refinable. Indeed, let $f$ be $\phi$ defined as in (1) with mask

$$
m_{\phi}(\xi)=2^{-1} \sum_{k} c_{k} e^{-i k \xi}
$$

then the corresponding $\bar{\phi}_{n}$ defined as in equation (9) is refinable and possesses mask

$$
\begin{equation*}
m_{\bar{\phi}_{n}}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{n} m_{\phi}(\xi) \tag{13}
\end{equation*}
$$

In addition, we have the following proposition.
Proposition 2. Let $\phi(t)=\sum_{k=0}^{2 p-1} c_{k} \phi(2 t-k)$, and $\phi(t) \in L^{2}(\mathbb{R})$. Then, $\bar{\phi}_{n}$ defined as in (9) is also in $L^{2}(\mathbb{R})$ and satisfies dilation equation

$$
\begin{equation*}
\bar{\phi}_{n}=\sum_{k=0}^{2 p+n-1} h_{k}^{(n)} \bar{\phi}_{n}(2 t-k) \tag{14}
\end{equation*}
$$

where

$$
h_{k}^{(n)}=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} c_{k-j}
$$

Proof. From equation (13)

$$
m_{\bar{\phi}_{n}}(\xi)=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} e^{-i j \xi} \frac{1}{2} \sum_{k=0}^{2 p-1} c_{k} e^{-i k \xi}=\frac{1}{2} \sum_{k=0}^{2 p+n-1}\left(\frac{1}{2^{n}} \sum_{j=0}^{k}\binom{n}{j} c_{k-j}\right) e^{-i k \xi} .
$$

Noting $\binom{n}{j}=0$ for $j>n$, we obtain equation (14).

Equation (14) can be written as

$$
\begin{equation*}
\hat{\bar{\phi}}_{n}(\xi)=m_{\bar{\phi}_{n}}\left(\frac{\xi}{2}\right) \hat{\bar{\phi}}_{n}\left(\frac{\xi}{2}\right) \tag{15}
\end{equation*}
$$

where

$$
m_{\bar{\phi}_{n}}(\xi)=\frac{1}{2} \sum_{k=0}^{2 p+n-1} h_{k}^{(n)} e^{-i k \xi}
$$

Example 2.1. For $n=1$, from Proposition 2 we have the dilation equation for $\bar{\phi}_{1}$ as

$$
\bar{\phi}_{1}(t)=\sum_{k} \frac{c_{k}+c_{k-1}}{2} \bar{\phi}_{1}(2 t-k)
$$

For $n=2$, the dilation equation of $\phi_{2}$ is

$$
\bar{\phi}_{2}(t)=\sum_{k} \frac{c_{k}+2 c_{k-1}+c_{k-2}}{4} \bar{\phi}_{2}(2 t-k)
$$

Remark 2.2. Proposition 2 can be derived by using Theorem 1 and the first equation shown in Remark 2.1 as follows. From the equation and noting dilation equation (1) and backwarddifference expansion

$$
\nabla^{n} f(t)=\Delta^{n} f(t-n)=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} f(t-j)
$$

we have

$$
\begin{aligned}
\bar{\phi}(t) & =\int_{-\infty}^{t} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \nabla^{n} \phi\left(t_{1}\right) d t_{1} \cdots d t_{n} \\
& =\int_{-\infty}^{t} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \sum_{j=0}^{n}(-1)^{j}\binom{n}{j} \phi\left(t_{1}-j\right) d t_{1} \cdots d t_{n} \\
& =\int_{-\infty}^{t} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \sum_{j=0}^{n}(-1)^{j}\binom{n}{j} \sum_{k} c_{k} \phi\left(2 t_{1}-2 j-k\right) d t_{1} \cdots d t_{n} \\
& =\sum_{k} c_{k} \int_{-\infty}^{t} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \sum_{j=0}^{n}\binom{n}{j} \nabla^{n} \phi\left(2 t_{1}-k-j\right) d t_{1} \cdots d t_{n} \\
& =\sum_{k} \sum_{j=0}^{n} \frac{1}{2^{n}}\binom{n}{j} c_{k} \int_{-\infty}^{2 t-k-j} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \nabla^{n} \phi\left(t_{1}\right) d t_{1} \cdots d t_{n} \\
& =\sum_{k} \sum_{j=0}^{n} \frac{1}{2^{n}}\binom{n}{j} c_{k} \bar{\phi}_{n}(2 t-k-j) \\
& =\sum_{k}\left[\sum_{j=0}^{n} \frac{1}{2^{n}}\binom{n}{j} c_{k-j}\right] \bar{\phi}_{n}(2 t-k)=\sum_{k} h_{k}^{(n)} \bar{\phi}_{n}(2 t-k) .
\end{aligned}
$$

This completes an alternative proof of Proposition 2.
To find the properties of the set of dilation coefficients, $\left\{h_{k}^{(n)}\right\}_{k \in \mathbb{Z}}$, we need the following lemma. Lemma 3.

$$
\begin{equation*}
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{\ell}=0 \tag{16}
\end{equation*}
$$

for $\ell=0,1, \ldots, n-1$. In addition, equation (16) does not hold for $\ell=n$.

Proof. We first derive the following equation.

$$
\begin{equation*}
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j(j-1) \cdots(j-\ell)=0, \quad \ell=0,1, \ldots, n-2 \tag{17}
\end{equation*}
$$

In fact, for $\ell=0,1, \ldots, n-2$,

$$
\begin{aligned}
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j(j-1) \cdots(j-\ell) & =\sum_{j=\ell+1}^{n}(-1)^{j}\binom{n}{j} j(j-1) \cdots(j-\ell) \\
& =n(n-1) \cdots(n-\ell) \sum_{j=\ell+1}^{n}(-1)^{j}\binom{n-\ell-1}{j-\ell-1} \\
& =(-1)^{\ell+1} n(n-1) \cdots(n-\ell) \sum_{j^{\prime}=0}^{n-\ell-1}(-1)^{j^{\prime}}\binom{n-\ell-1}{j^{\prime}}=0
\end{aligned}
$$

Next, we prove equation (16) inductively by using (17). Equation (16) obviously holds for $\ell=0$. We assume it is true for all $\ell=0,1, \ldots, k, 0 \leq k \leq n-2$. Considering the case of $\ell=k+1$, we have

$$
\begin{aligned}
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{k+1} & =\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\left[j(j-1) \cdots(j-k)+\frac{k(k+1)}{2} j^{k}+\text { lower power terms of } j\right] \\
& =\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j(j-1) \cdots(j-k)=0
\end{aligned}
$$

Here, the induction assumption and equation (17) have been, respectively, applied in the last two steps.

To prove the last statement of the lemma, we consider the case of $\ell=n$ in (16).

$$
\begin{aligned}
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{n}= & \sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\left[j(j-1) \cdots(j-n+1)+\frac{n(n-1)}{2} j^{n-1}\right. \\
& + \text { lower power terms of } j] \\
= & \sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j(j-1) \cdots(j-n+1)=(-1)^{n} n!
\end{aligned}
$$

This completes the proof of Lemma 3.
Theorem 4. Suppose that the set of dilation coefficients, $\left\{h_{k}^{(n)}\right\}_{k \in \mathbb{Z}}$, is defined in Proposition 2. Then we have

$$
\begin{aligned}
& (\text { (i) })^{\prime \prime} h_{k}^{(n)}=0 \text {, for } k \notin\{0,1, \ldots, 2 p+n-1\} \text {; } \\
& \text { (ii) }{ }^{\prime \prime} \sum_{k} h_{k}^{(n)}=2 \text {; } \\
& (\text { (iii) })^{\prime \prime} \sum_{k}(-1)^{k} k^{m} h_{k}^{(n)}=0 \text {, for } 0 \leq m \leq p+n-1
\end{aligned}
$$

Proof. From Item (i) and the definition $\left\{h_{k}^{(n)}\right\}$ shown in Proposition 2, we immediately know Item (i)" holds. Item (ii)" also holds because

$$
\sum_{k=0}^{2 p+n-1} h_{k}^{(n)}=\sum_{k=0}^{2 p+n-1} \frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} c_{k-j}=\sum_{j=0}^{n} \frac{1}{2^{n}}\binom{n}{j} \sum_{k^{\prime}=-j}^{2 p+n-j-1} c_{k^{\prime}}=\sum_{j=0}^{n} \frac{1}{2^{n}}\binom{n}{j} \sum_{k^{\prime}=0}^{2 p-1} c_{k^{\prime}}=2 .
$$

Finally, we prove Item (iii) ${ }^{\prime \prime}$. For $m=0,1, \ldots, p+n-1$,

$$
\begin{aligned}
\sum_{k=0}^{2 p+n-1}(-1)^{k} k^{m} h_{k}^{(n)} & =\frac{1}{2^{n}} \sum_{k=0}^{2 p+n-1} \sum_{j=0}^{n}(-1)^{k} k^{m}\binom{n}{j} c_{k-j} \\
& =\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} \sum_{k=-j}^{2 p+n-j-1}(-1)^{k+j}(k+j)^{m} c_{k} \\
& =\frac{1}{2^{n}} \sum_{j=0}^{n}(-1)^{j}\binom{n}{j} \sum_{k=0}^{2 p-1}(-1)^{k} c_{k} \sum_{i=0}^{m}\binom{m}{i} k^{i} j^{m-i} \\
& =\frac{1}{2^{n}} \sum_{i=0}^{m}\binom{m}{i}\left(\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{m-i}\right)\left(\sum_{k=0}^{2 p-1}(-1)^{k} k^{i} c_{k}\right)
\end{aligned}
$$

Hence, for $0 \leq i \leq p-1$, the sum in the second parenthesis in the last line of the above equation is equal to zero (see Item (iii)). For $p \leq i \leq m$, we have $0 \leq m-i \leq m-p \leq p+n-1-p=n-1$. Therefore, from Lemma 3 the sum in the first parenthesis in the last line of the above equation becomes zero. It follows that the right-hand side of the last equation is equal to zero. This completes the proof of Theorem 4.

It is obvious that $\bar{\phi}_{n}$ does not satisfy Item (iv) that is shown at the beginning of the paper. Hence, we will construct a scaling function $\tilde{\phi}_{n}$ that is biorthogonal with $\bar{\phi}_{n}$ in the sense $\left\langle\bar{\phi}_{n}(t), \tilde{\phi}_{n}(t-k)\right\rangle=\delta_{0 k}$.

## 3. BIORTHOGONAL SPLINE TYPE SCALING FUNCTIONS AND WAVELETS

Let $\bar{\phi}_{n}$ be defined as in (9) that satisfies dilation equation (14), in which dilation coefficients possess properties shown in Items (i) ${ }^{\prime \prime}$-(iii) ${ }^{\prime \prime}$ of Theorem 4. Hence, $\bar{\phi}_{n}$ is also a scaling function with approximation degree $p+n-1$. To find another scaling function that is biorthogonal with $\bar{\phi}_{n}$ in the sense shown at the end of the last section, we first consider B-spline functions, denoted by $C_{n}(t)(n=1,2, \ldots)$, that are defined as

$$
\begin{equation*}
C_{n}(t)=M(t ;-n, \ldots,-1,0) \tag{18}
\end{equation*}
$$

where function $M$ is given in (1.1) of [9]. Therefore, the Fourier transform of $C_{n}(t)$ is

$$
\begin{equation*}
\hat{C}_{n}(\xi)=e^{i n \xi / 2}\left(\frac{\sin (\xi / 2)}{\xi / 2}\right)^{n} \tag{19}
\end{equation*}
$$

Comparing the above expression with equation (10) yields

$$
\begin{equation*}
\overline{\hat{C}}_{n}(\xi)=\hat{B}_{n}(\xi) \tag{20}
\end{equation*}
$$

Equation (20) can also be derived from the symmetry of B-splines. From the symmetry, $C_{n}(t)=$ $B_{n}(-t)$, which implies that the Fourier transform of $\widehat{C_{n}(t)}(\xi)=\widehat{B_{n}(-t)}(\xi)=\widehat{\widehat{B_{n}(t)}}(\xi)$.

Denote $\tilde{\phi}_{n}$ a function that satisfies

$$
\begin{equation*}
\left(C_{n} * \tilde{\phi}_{n}\right)(t)=\phi(t) \tag{21}
\end{equation*}
$$

where $\phi$ is an orthonormal scaling function shown at the beginning of Section 1. Taking Fourier transform on both sides of equation (21) and noting equation (19) yields

$$
\begin{equation*}
\hat{\tilde{\phi}}_{n}(\xi)=\frac{\hat{\phi}(\xi)}{\hat{C}_{n}(\xi)}=e^{-i n \xi / 2}\left(\frac{\xi / 2}{\sin (\xi / 2)}\right)^{n} \hat{\phi}(\xi) \tag{22}
\end{equation*}
$$

Theorem 5. The function $\tilde{\phi}_{n}$ is well defined by equation (21), and it satisfies the following dilation equation:

$$
\begin{equation*}
\hat{\tilde{\phi}}_{n}(\xi)=\tilde{m}_{\tilde{\phi}_{n}}\left(\frac{\xi}{2}\right) \hat{\tilde{\phi}}_{n}\left(\frac{\xi}{2}\right) \tag{23}
\end{equation*}
$$

where

$$
\tilde{m}_{\tilde{\phi}_{n}}(\xi)=\left(\frac{2}{1+e^{i \xi}}\right)^{n} m_{\phi}(\xi)
$$

and $m_{\phi}$ is the mask of $\phi$.
In addition, let $\bar{\phi}_{n}$ be defined as (9). Then $\left\{\bar{\phi}_{n}(t-k)\right\}_{k \in \mathbb{Z}}$ and $\left\{\tilde{\phi}_{n}(t-k)\right\}_{k \in \mathbb{Z}}$ are a biorthogonal set; i.e., $\bar{\phi}_{n}$ and $\tilde{\phi}_{n}$ satisfy $\left\langle\bar{\phi}_{n}(t), \tilde{\phi}_{n}(t-k)\right\rangle=\delta_{0 k}$.

Proof. To derive dilation equation (23), we start from equation (22) and apply equations (20) and (21) to give

$$
\frac{\hat{\tilde{\phi}}_{n}(\xi)}{\hat{\tilde{\phi}}_{n}(\xi / 2)}=\frac{\hat{C}_{n}(\xi / 2)}{\hat{C}_{n}(\xi)} \frac{\phi(\xi)}{\phi(\xi / 2)}=\frac{\overline{\hat{B}_{n}(\xi / 2)}}{\hat{B}_{n}(\xi)} m_{\phi}\left(\frac{\xi}{2}\right)=\left(\frac{2}{1+e^{i \xi}}\right)^{n} m_{\phi}\left(\frac{\xi}{2}\right)
$$

Hence, equation (23) is established and $\tilde{\phi}_{n}$ is well defined by (21).
To prove the biorthogonality of $\left\{\bar{\phi}_{n}(t-k)\right\}_{k \in \mathbb{Z}}$ and $\left\{\tilde{\phi}_{n}(t-k)\right\}_{k \in \mathbb{Z}}$, using the general Parseval's relation, substituting expressions shown as in (11) and (22), and noting relation (20), we have

$$
\begin{aligned}
\left\langle\bar{\phi}_{n}(t), \tilde{\phi}_{n}(t-k)\right\rangle & =\frac{1}{2 \pi}\left\langle\hat{\bar{\phi}}_{n}(\xi), \hat{\tilde{\phi}}_{n}(\xi) e^{-i k \xi}\right\rangle=\frac{1}{2 \pi} e^{i k \xi}\left\langle\hat{B}_{n}(\xi) \hat{\phi}(\xi), \frac{\hat{\phi}(\xi)}{\hat{C}_{n}(\xi)}\right\rangle \\
& =\frac{1}{2 \pi} e^{i k \xi}\langle\hat{\phi}(\xi), \hat{\phi}(\xi)\rangle=\langle\phi(t), \phi(t-k)\rangle=\delta_{0 k}
\end{aligned}
$$

where the last step is due to the orthonormality of $\{\phi(t-k)\}_{k \in \mathbb{Z}}$. This completes the proof of Theorem 5.

We now turn to the properties of the dilation coefficients of $\tilde{\phi}_{n}$.
Proposition 6. Let $\phi(t)=\sum_{k=0}^{2 p-1} c_{k} \phi(2 t-k)$. Then $\tilde{\phi}_{n}, n \leq 2 p-1$, defined as in (21) satisfies dilation equation

$$
\begin{equation*}
\tilde{\phi}_{n}=\sum_{k=n}^{2 p-1} \tilde{h}_{k}^{(n)} \tilde{\phi}_{n}(2 t-k) \tag{24}
\end{equation*}
$$

where dilation coefficients, $\left\{\tilde{h}_{k}^{(n)}\right\}_{k}$, satisfy

$$
c_{k}=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} \tilde{h}_{k+j}^{(n)}
$$

for $k=n, n+1, \ldots, 2 p-1$.
Proof. The mask of $\tilde{\phi}_{n}$ can be written as

$$
\tilde{m}_{\tilde{\phi}_{n}}(\xi)=\frac{1}{2} \sum_{k=s}^{t} \tilde{h}_{k}^{(n)} e^{-i k \xi}
$$

where $s, t$ are in $\mathbb{Z}$ or infinite, which will be determined as follows. We also assume that $\tilde{h}_{k}^{(n)}=0$
for all $k<s$ or $k>t$. Then, from the dilation equation (23) of $\tilde{\phi}_{n}$ shown in Theorem 5 , we have

$$
\begin{align*}
m_{\phi}(\xi) & =\frac{\sum_{j=0}^{n}\binom{n}{j} e^{i j \xi}}{2^{n}} \tilde{m}_{\tilde{\phi}_{n}}=\frac{1}{2^{n+1}} \sum_{j=0}^{n}\binom{n}{j} e^{i j \xi} \sum_{k=s}^{t} \tilde{h}_{k}^{(n)} e^{-i k \xi} \\
& =\frac{1}{2^{n+1}} \sum_{j=0}^{n}\binom{n}{j} \sum_{k=s-j}^{t-j} \tilde{h}_{k+j}^{(n)} e^{-i k \xi}  \tag{25}\\
& =\frac{1}{2^{n+1}} \sum_{j=0}^{n}\binom{n}{j} \sum_{k=s-n}^{t} \tilde{h}_{k+j}^{(n)} e^{-i k \xi} \\
& =\frac{1}{2} \sum_{k=s-n}^{t}\left(\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} \tilde{h}_{k+j}^{(n)}\right) e^{-i k \xi}
\end{align*}
$$

Since $m_{\phi}(\xi)=(1 / 2) \sum_{k=0}^{2 p-1} c_{k} e^{-i k \xi}$, equating coefficients of $e^{-i k \xi}$ for each $k$ on both sides of equation (25) yields $s=n, t=2 p-1$, and

$$
c_{k}=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} \tilde{h}_{k+j}^{(n)}
$$

Remark 3.1. From (24), we can evaluate $\left\{\tilde{h}_{k}^{(n)}\right\}_{k=n, \ldots, 2 p-1}$ as follows.

$$
\tilde{h}_{k}^{(n)}=2^{n} c_{k}-\left[\sum_{j=1}^{n}\binom{n}{j} \tilde{h}_{k+j}^{(n)}\right]
$$

Hence,

$$
\begin{aligned}
& \tilde{h}_{2 p-1}^{(n)}=2^{n} c_{2 p-1}, \\
& \tilde{h}_{2 p-2}^{(n)}=2^{n} c_{2 p-2}-n \tilde{h}_{2 p-1}^{(n)}, \\
& \tilde{h}_{2 p-3}^{(n)}=2^{n} c_{2 p-3}-n \tilde{h}_{2 p-2}^{(n)}-\frac{n(n-1)}{2} \tilde{h}_{2 p-1}^{(n)}, \ldots
\end{aligned}
$$

REmark 3.2. Equation (24) can be derived using a similar argument in Remark 2.2. First, similar to Theorem 1, we have

$$
\left(f * C_{n}\right)(x)=\int_{-\infty}^{x} \int_{-\infty}^{t_{n}} \cdots \int_{-\infty}^{t_{2}} \Delta^{n} f\left(t_{1}\right) d t_{1} \cdots d t_{n}
$$

where $\Delta$ is the forward-difference operator $\Delta$ that is defined as $\Delta f(t)=f(t+1)-f(t)$ and $\Delta^{n} f=\Delta\left(\Delta^{n-1} f\right)$. Secondly, from Remark 2.1, we obtain the following formula:

$$
\sum_{j=0}^{n}\binom{n}{j} \Delta^{n} f(t-j)=\sum_{j=0}^{n}\binom{n}{j} \nabla^{n} f(t+n-j)=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} f(t+n-2 j)
$$

Finally, use the above two equations to establish equation (24).
The following theorem and Theorem 4 give the summary of the properties of dilation coefficients of $\bar{\phi}_{n}$ and $\tilde{\phi}_{n}$.

Theorem 7. Suppose that the set of dilation coefficients, $\left\{\tilde{h}_{k}^{(n)}\right\}_{k \in \mathbb{Z}}$, is defined in Proposition 2, $n \leq 2 p-1$. Then we have
(i) ${ }^{\prime \prime \prime} \tilde{h}_{k}^{(n)}=0$, for $k \notin\{n, n+1, \ldots, 2 p-1\}$;
(ii) ${ }^{\prime \prime \prime} \sum_{k} \tilde{h}_{k}^{(n)}=2$;
(iii) ${ }^{\prime \prime \prime} \sum_{k}(-1)^{k} k^{m} \tilde{h}_{k}^{(n)}=0$, for $0 \leq m \leq p-n-1$.

Proof. Item (i) ${ }^{\prime \prime \prime}$ is given by Proposition 6.
Since

$$
\sum_{k=0}^{2 p-1} c_{k}=\frac{1}{2^{n}} \sum_{k=0}^{2 p-1} \sum_{j=0}^{n}\binom{n}{j} \tilde{h}_{k+j}^{(n)}=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} \sum_{k=j}^{2 p+j-1} \tilde{h}_{k}^{(n)}=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j} \sum_{k=n}^{2 p-1} \tilde{h}_{k}^{(n)}=\sum_{k=n}^{2 p-1} \tilde{h}_{k}^{(n)}
$$

and $\sum_{k=0}^{2 p-1} c_{k}=2$, we obtain Item (ii) ${ }^{\prime \prime \prime}$.
For $m=0,1, \ldots, p+n-1$, we have $\sum_{k=0}^{2 p+n-1}(-1)^{k} k^{m} c_{k}=0$. On the other hand,

$$
\begin{align*}
\sum_{k=0}^{2 p+n-1}(-1)^{k} k^{m} c_{k} & =\frac{1}{2^{n}} \sum_{k=0}^{2 p-1} \sum_{j=0}^{n}(-1)^{k} k^{m}\binom{n}{j} \tilde{h}_{k+j}^{(n)} \\
& \left.=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j}^{2 p+j-1} \sum_{k=j}^{2 p+1}\right)^{k-j}(k-j)^{m} \tilde{h}_{k}^{(n)} \\
& =\frac{1}{2^{n}} \sum_{j=0}^{n}(-1)^{j}\binom{n}{j} \sum_{k=n}^{2 p-1}(-1)^{k} \tilde{h}_{k}^{(n)} \sum_{i=0}^{m}\binom{m}{i}(-j)^{i} k^{m-i}  \tag{26}\\
& =\frac{1}{2^{n}} \sum_{i=0}^{m}(-1)^{i}\binom{m}{i}\left(\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}^{i}\right)\left(\sum_{k=n}^{2 p-1}(-1)^{k} k^{m-i} \tilde{h}_{k}^{(n)}\right)
\end{align*}
$$

Taking $m=n$ in the last line of equation (26) and noting Lemma 3 and $\sum_{k=0}^{2 p+n-1}(-1)^{k} k^{n} c_{k}=0$ yields

$$
\frac{1}{2^{n}}(-1)^{n}\left(\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{n}\right)\left(\sum_{k=n}^{2 p-1}(-1)^{k} \tilde{h}_{k}^{(n)}\right)=0
$$

Because of Lemma 3, $\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{n} \neq 0$. Hence, $\sum_{k=n}^{2 p-1}(-1)^{k} \tilde{h}_{k}^{(n)}=0$. Assume

$$
\begin{equation*}
\sum_{k=n}^{2 p-1}(-1)^{k} k^{\ell} \tilde{h}_{k}^{(n)}=0 \tag{27}
\end{equation*}
$$

for all $1 \leq \ell \leq M, M<p-n-1$. Substituting $m=\ell+n+1$ into the last line of equation (26) and noting equation (27) and $\sum_{k=0}^{2 p+n-1}(-1)^{k} k^{\ell+n+1} c_{k}=0$, we have

$$
\begin{array}{r}
\frac{1}{2^{n}} \sum_{i=n}^{\ell+n+1}(-1)^{i}\binom{\ell+n+1}{i}\left(\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{i}\right)\left(\sum_{k=n}^{2 p-1}(-1)^{k} k^{\ell+n-i+1} \tilde{h}_{k}^{(n)}\right) \\
=\frac{1}{2^{n}}(-1)^{n}\binom{\ell+n+1}{n}\left(\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} j^{n}\right)\left(\sum_{k=n}^{2 p-1}(-1)^{k} k^{\ell+1} \tilde{h}_{k}^{(n)}\right)=0 .
\end{array}
$$

Using Lemma 3 gives

$$
\sum_{k=n}^{2 p-1}(-1)^{k} k^{\ell+1} \tilde{h}_{k}^{(n)}=0
$$

Hence, we have proved Item (iii) ${ }^{\prime \prime \prime}$ holds for all $0 \leq m \leq p-n-1$.

As for Item (iv)', it is obviously true because of Theorem 5. Therefore, we have construct biorthogonal scaling functions $\bar{\phi}$ and $\tilde{\phi}$. The corresponding biorthogonal wavelets can be found by using either equations (7) or (8).
Example 3.1. For scaling functions $\bar{\phi}_{1}$ and $\bar{\phi}_{2}$ studied in Examples 2.1, the corresponding biorthogonal scaling functions are, respectively,

$$
\hat{\tilde{\phi}}_{1}(\xi)=\frac{e^{-i \xi / 2} \xi / 2}{\sin (\xi / 2)} \hat{\phi}(\xi)
$$

and

$$
\hat{\tilde{\phi}}_{2}(\xi)=e^{-i \xi}\left(\frac{\xi / 2}{\sin (\xi / 2)}\right)^{2} \hat{\phi}(\xi)
$$

Here, the masks of $\tilde{\phi}_{1}$ and $\tilde{\phi}_{2}$ are

$$
\tilde{m}_{\tilde{\phi}_{1}}(\xi)=\frac{2}{1+e^{i \xi}} m_{\phi}(\xi)
$$

and

$$
\tilde{m}_{\tilde{\phi}_{2}}(\xi)=\left(\frac{2}{1+e^{i \xi}}\right)^{2} m_{\phi}(\xi)
$$

respectively. For Daubechies' D4 scaling function with dilation coefficients

$$
c_{0}=\frac{1+\sqrt{3}}{4}, \quad c_{1}=\frac{3+\sqrt{3}}{4}, \quad c_{2}=\frac{3-\sqrt{3}}{4}, \quad \text { and } \quad c_{3}=\frac{1-\sqrt{3}}{4}
$$

using Proposition 2, we find the dilation coefficients of the corresponding $\bar{\phi}_{1}$ and $\bar{\phi}_{2}$ are, respectively,

$$
\begin{gathered}
h_{0}^{(1)}=\frac{1+\sqrt{3}}{8}, \quad h_{1}^{(1)}=\frac{2+\sqrt{3}}{4}, \quad h_{2}^{(1)}=\frac{3}{4} \\
h_{3}^{(1)}=\frac{2-\sqrt{3}}{4}, \quad h_{4}^{(1)}=\frac{1-\sqrt{3}}{8},
\end{gathered}
$$

and

$$
\begin{array}{lll}
h_{0}^{(2)}=\frac{1+\sqrt{3}}{16}, & h_{1}^{(2)}=\frac{5+3 \sqrt{3}}{16}, & h_{2}^{(2)}=\frac{5+\sqrt{3}}{8} \\
h_{3}^{(2)}=\frac{5-\sqrt{3}}{8}, & h_{4}^{(2)}=\frac{5-3 \sqrt{3}}{16}, & h_{5}^{(2)}=\frac{1-\sqrt{3}}{16} .
\end{array}
$$

From Remark 3.1, we obtain that the dilation coefficients of $\tilde{\phi}_{1}$ and $\tilde{\phi}_{2}$ are

$$
\tilde{h}_{1}^{(1)}=\frac{1+\sqrt{3}}{2}, \quad \tilde{h}_{2}^{(1)}=1, \quad h_{3}^{(1)}=\frac{1-\sqrt{3}}{2}
$$

and

$$
\tilde{h}_{2}^{(2)}=1+\sqrt{3}, \quad \tilde{h}_{3}^{(2)}=1-\sqrt{3}
$$

respectively.
Example 3.2. For the scaling function $\phi$ given in an example shown on [11, p. 320], which has mask

$$
m_{\phi}(\xi)=\frac{1-\sqrt{3}}{2}\left(1-e^{-i \xi}\right)\left(\frac{1+e^{-i \xi}}{2}\right)^{2}
$$

the corresponding $\bar{\phi}_{1}$ and $\tilde{\phi}_{1}$ are determined by their dilation coefficients

$$
\begin{gathered}
h_{0}^{(1)}=\frac{1-\sqrt{3}}{8}, \quad h_{1}^{(1)}=\frac{1-\sqrt{3}}{4}, \quad h_{2}^{(1)}=0, \\
h_{3}^{(1)}=\frac{\sqrt{3}-1}{4}, \quad h_{4}^{(1)}=\frac{\sqrt{3}-1}{8},
\end{gathered}
$$

and

$$
\tilde{h}_{1}^{(1)}=\frac{1-\sqrt{3}}{2}, \quad \tilde{h}_{2}^{(1)}=0, \quad \tilde{h}_{3}^{(1)}=\frac{\sqrt{3}-1}{2}
$$

respectively.
Although $\phi \in L^{2}(\mathbb{R})$ leads $\bar{\phi}_{n}=B_{n} * \phi \in L^{2}(\mathbb{R})$, it cannot guarantee that $\tilde{\phi}_{n}$ is also in $L^{2}(\mathbb{R})$. For instance, $\tilde{\phi}_{1}$ shown in Example 3.2 is in $L^{2}(\mathbb{R})$ (see Theorem 4 for the reason), but $\phi_{1}$ in Example 3.1 is not. Usually, $\tilde{\phi}_{n}$ is a distribution in a Sobolev space (e.g., examples $\tilde{\phi}_{1}$ and $\tilde{\phi}_{2}$ shown in Example 3.1). We shall establish a condition for $\phi$ such that $\phi$ is also in $L^{2}(\mathbb{R})$.

Denote $H^{\alpha}, \alpha \in \mathbb{R}$, the Sobolev space consists of all functions such that

$$
\int_{\mathbb{R}}|\hat{f}(\xi)|^{2}\left(\xi^{2}+1\right)^{\alpha} d \xi<\infty
$$

Hence, for $\alpha=0, H^{0}=L^{2}(\mathbb{R})$; for $\alpha=1,2, \ldots, H^{\alpha}$ is composed of ordinary $L^{2}(\mathbb{R})$ functions that possess $\alpha-1$ times derivatives and whose $\alpha^{\text {th }}$ derivative is in $L^{2}(\mathbb{R})$; for $\alpha=-1,-2, \ldots$, $H^{\alpha}$ contains all distributions with point support of order $\leq \alpha$.

If $\tilde{\phi}_{n}$ is the distribution solution to the dilation equation $(24)$, then $\hat{\tilde{\phi}}_{n}(0)=1$, and from [12], $\tilde{\phi}_{n} \in H^{-s}$ for $s>\log _{2} \sum_{k}\left|\tilde{h}_{k}^{(n)}\right|-1 / 2$. The proof follows from the fact that

$$
\left|\hat{\tilde{\phi}}_{n}(\xi)\right| \leq C(1+|\xi|)^{M}
$$

where $M=\log _{2} \sum_{k}\left|h_{k}^{(n)}\right|-1$, and hence $\tilde{\phi}_{n} \in H^{-s}$ for $s>M+1 / 2$. Therefore, scaling function $\tilde{\phi}_{n}$ generates a multiresolution analysis of $H^{-s}$.

We now give a condition for the orthogonal scaling function, $\phi$, with the following mask such that its corresponding $\tilde{\phi}_{n}$ is in $L^{2}(\mathbb{R})$.

$$
\begin{equation*}
m_{\phi}^{N}(\xi)=\left(\frac{1+e^{-i \xi}}{2}\right)^{N} F(\xi) \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
F(\xi)=e^{-i k^{\prime} \xi} \sum_{j=0}^{k} a_{j} e^{-i j \xi} \tag{29}
\end{equation*}
$$

Here, all coefficients of $F(\xi)$ are real, $F(0)=1 ; N$ and $k$ are positive integers; and $k^{\prime} \in \mathbb{Z}$.
Theorem 8. Let $\phi$ be defined as

$$
\hat{\phi}=\prod_{j=1}^{\infty} m_{0}^{N}\left(2^{-j} \xi\right)
$$

where $m_{0}^{N}(\xi) \in M$ is defined by (28) and (29). If $F(\pi) \neq-1$ and the coefficients of $F(\xi)$ satisfy

$$
\begin{equation*}
(k+1) \sum_{j=0}^{k} a_{j}^{2}<2^{2 N-1} \tag{30}
\end{equation*}
$$

then $\phi$ is in $L_{2}(\mathbb{R})$, while both $\phi$ and $\tilde{\phi}_{n}(n<N)$ are in $L^{2}(\mathbb{R})$ when

$$
\begin{equation*}
(k+1) \sum_{j=0}^{k} a_{j}^{2}<2^{2(N-n)-1} \tag{31}
\end{equation*}
$$

In addition, (30) and (31) can be replaced, respectively, by weaker conditions $C\left(\left\{a_{j}\right\}, k\right)<2^{2 N-1}$ and $C\left(\left\{a_{j}\right\}, k\right)<2^{2(N-n)-1}$, where $C\left(\left\{a_{j}\right\}, k\right)$ equals $k \sum_{j=0}^{k} a_{j}^{2}$ if $k \geq 1$ and equals $a_{0}^{2}$ if $k=0$. Proof. Condition (30) is directly from Lemma 3 of [13], and condition (31) comes from the following observation and a similar argument in the proof of Lemma 3 of [13].

$$
\begin{aligned}
\int_{|\xi| \geq \pi}|\hat{\phi}(\xi)|^{2} d \xi & =\sum_{\ell=1}^{\infty} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \prod_{j=1}^{\infty}\left|\left(\frac{2}{1+e^{i 2^{-j} \xi}}\right)^{n}\left(\frac{1+e^{-i 2^{-j} \xi}}{2}\right)^{N} F\left(2^{-j} \xi\right)\right|^{2} d \xi \\
& =\sum_{\ell=1}^{\infty} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi}\left|\frac{i \xi}{1-e^{i \xi}}\right|^{2 n}\left|\frac{1-e^{-i \xi}}{i \xi}\right|^{2 N} \prod_{j=1}^{\infty}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi \\
& \leq C \sum_{\ell=1}^{\infty} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \frac{1}{|\xi|^{2(N-n)}} \prod_{j=1}^{\infty}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi \\
& \leq C \sum_{\ell=1}^{\infty} \frac{1}{2^{2 \ell(N-n)}} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \prod_{j=1}^{\ell}\left|F\left(2^{-j} \xi\right)\right|^{2} \prod_{j=1}^{\infty}\left|F\left(2^{-\ell-j} \xi\right)\right|^{2} d \xi \\
& \leq C \sum_{\ell=1}^{\infty} 4^{-\ell(N-n)} \int_{2^{\ell-1} \pi \leq|\xi| \leq 2^{\ell} \pi} \prod_{j=1}^{\ell}\left|F\left(2^{-j} \xi\right)\right|^{2} d \xi
\end{aligned}
$$

At the end of this paper, we point out that the results we obtained for the orthogonal scaling functions can be extended to those for the biorthogonal scaling functions.
THEOREM 9. Let $f(t)=\sum_{k=0}^{2 p-1} c_{k} f(2 t-k)$ and $g(t)=\sum_{k=0}^{2 \tilde{p}-1} d_{k} g(2 t-k)$ be a biorthogonal scaling function (i.e., $\langle f(t), g(t-\ell)\rangle=\delta_{0 \ell}$ ) with approximation degree $p$ and $\tilde{p}$, respectively. Then functions $\bar{f}_{n}$ and $\tilde{g}_{n}$ defined by

$$
\begin{equation*}
\bar{f}_{n}(t)=\left(B_{n} * f\right)(t) \quad \text { and } \quad\left(C_{n} * \tilde{g}_{n}\right)(t)=g(t) \tag{32}
\end{equation*}
$$

where $B_{n}$ and $C_{n}$ are $B$-splines, are also biorthogonal scaling functions with approximation degree $p+n-1$ and $\tilde{p}-n-1$, respectively.

In addition, if $f \in L^{2}(\mathbb{R})$, then $\bar{f}_{n}$ is also in $L^{2}(\mathbb{R})$. If the Fourier transform of $g$ can be written as

$$
\hat{g}(\xi)=\prod_{j=1}^{\infty} m_{0}^{N}\left(2^{-j} \xi\right)
$$

where $m_{0}^{N}(\xi) \in M$ is defined by (28) and (29), with $F(\pi) \neq-1$ and the coefficients of $F(\xi)$ satisfy

$$
(k+1) \sum_{j=0}^{k} a_{j}^{2}<2^{2(N-n)-1}
$$

then both $g$ and $\tilde{g}_{n}$ are also in $L^{2}(\mathbb{R})$.
Example 3.3. Considering the examples given in [2] and [3], let ${ }_{N} \phi$ and ${ }_{N, \tilde{N}} \tilde{\phi}$ be a pair of biorthogonal scaling functions with masks ${ }_{N} m_{0}$ and ${ }_{N, \tilde{N}} \tilde{m}_{0}$, respectively. It is easy to check that $\overline{{ }_{N} \phi_{n}}(t)={ }_{N+n} \phi(t-k)$ and $\widetilde{{ }_{N, \tilde{N}} \tilde{\phi}_{n}}(t)={ }_{N+n, \tilde{N}-n} \tilde{\phi}(t-\ell)(n<\tilde{N})$ for some integers $k$ and $\ell$, where $\bar{f}_{n}$ and $\tilde{g}_{n}$ are defined by equations (9) and (21). Denote the masks of $\overline{{ }_{N} \phi_{n}}(t)$ and $\widetilde{N, \tilde{N} \tilde{\phi}_{n}}(t)$
by $m_{N+n}$ and $\tilde{m}_{N+n, \tilde{N}-n}$, which can be found using Proposition 2 and Remark 3.2 . Thus, we give an easy way to derive sequences of pairs of biorthogonal spline type scaling functions shown in [2] and [3]. For instance, the masks of ${ }_{1} \phi$ and ${ }_{1,3} \tilde{\phi}$ are, respectively,

$$
{ }_{1} m_{0}(\xi)=\frac{1+e^{-i \xi}}{2}
$$

and

$$
{ }_{1,3} \tilde{m}_{0}(\xi)=-\frac{e^{2 i \xi}}{16}\left(1+e^{-i \xi}\right)^{3}\left(1-4 e^{-i \xi}+e^{-2 i \xi}\right)
$$

Then masks of $\overline{{ }_{1} \phi_{n}}$ and $\widetilde{{ }_{1,3} \tilde{\phi}_{n}}$ for $n=1,2$ are, respectively,

$$
\begin{gathered}
m_{2}(\xi)=\frac{\left(1+e^{-i \xi}\right)^{2}}{4}, \quad m_{3}(\xi)=\frac{\left(1+e^{-i \xi}\right)^{3}}{8} \\
\tilde{m}_{2,2}(\xi)=-\frac{e^{i \xi}}{8}\left(1+e^{-i \xi}\right)^{2}\left(1-4 e^{-i \xi}+e^{-2 i \xi}\right) \\
\tilde{m}_{3,1}(\xi)=-\frac{1}{4}\left(1+e^{-i \xi}\right)\left(1-4 e^{-i \xi}+e^{-2 i \xi}\right)
\end{gathered}
$$

which are, respectively, ${ }_{2} m_{0},{ }_{3} m_{0},{ }_{2,2} \tilde{m}_{0}$, and ${ }_{3,1} \tilde{m}_{0}$ (a different factor $e^{i k \xi}(k \in \mathbb{Z}$ ) is allowed to be multiplied for shifting the function support) shown in papers [2] and [3]. In addition, by using the Sweldens' lifting scheme (see [1]), one can construct masks ${ }_{N} \tilde{N}+2 \tilde{m}_{0}$ from ${ }_{N, \tilde{N}} \tilde{m}_{0}$. Therefore, starting from the mask of the Haar scaling function, ${ }_{1} m_{0}={ }_{1,1} \tilde{m}_{0}$, we can obtain all ${ }_{n} m_{0}(n=2,3, \ldots)$ using Proposition 2 and all ${ }_{n, \ell} \tilde{m}_{0}$ using both the lifting scheme and the method supplied in Remark 3.2. For more details, from ${ }_{1,1} \tilde{m}_{0}$ we have all masks ${ }_{1,2 n+1} \tilde{m}_{0}$ $(n=1,2, \ldots)$ by applying the lifting scheme, then all $\ell, 2 n+2-\ell \tilde{m}_{0}$ for $\ell=2,3, \ldots, 2 n+1$ can be found by using the methods supplied in Remark 3.2.
Example 3.4. He [13] gives an algorithm for constructing biorthogonal scaling functions and wavelets with strongest possible regularities and with smallest possible support. To increase the regularities, we may use the technique shown in Theorem 5. For example, let $f$ and $g$ be biorthogonal scaling functions with masks

$$
m_{f}(\xi)=\frac{1+e^{-i \xi}}{2} \quad \text { and } \quad m_{g}(\xi)=\frac{1}{8}\left(1+e^{-i \xi}\right)^{2}\left(3-e^{-i \xi}\right)
$$

Then the corresponding $\bar{f}_{1}$ and $\tilde{g}_{1}$ have masks

$$
m_{\bar{f}_{1}}(\xi)=\frac{\left(1+e^{-i \xi}\right)^{2}}{4}
$$

and

$$
\tilde{m}_{\tilde{g}_{1}}(\xi)=\frac{e^{-i \xi}}{4}\left(1+e^{-i \xi}\right)\left(3-e^{-i \xi}\right)
$$

Hence, the regularity of $f$ is increased by $\bar{f}_{1}$, and the regularity of $g$ is reduced by $\tilde{g}_{1}$.

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