Biorthogonal Wavelets with Certain Regularities

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Abstract

In this paper, we will discuss the construction of biorthogonal wavelets that possess the largest possible regularities and required vanishing moments. For the sake of applications, we also give a general Daubechies' iteration method of constructing biorthogonal wavelets by using biorthogonal splines.

1. Introduction

Ingrid Daubechies, in her book "Ten Lectures on Wavelets," wrote: "What is more important, vanishing moments or regularity? The answer depends on the application, and is not always clear." She also pointed out that to achieve higher regularity by increasing vanishing moments is not efficient, because 80% of zero moments are wasted. In this paper, we give a method for constructing biorthogonal wavelets with the largest possible regularities and required vanishing moments. For the sake of dyadic spline approximation (see [3]), one of our scaling function must be a spline and the corresponding wavelet can be used to construct the supplementary sum for a better approximation.

Define masks as

$$m_0(\xi) = 2^{-1}\sum_n h_n e^{-in\xi}, \quad \tilde{m}_0(\xi) = 2^{-1}\sum_n \tilde{h}_n e^{-in\xi}. \tag{1}$$

Here, we assume that only finitely many h_n, $\tilde{h}_n$ are nonzero. However, some of our results can be extended to infinite sequences that have sufficient decay for $|n| \to \infty$. Then we define ϕ, $\tilde{\phi}$ by

$$\hat{\phi}(\xi) = \Pi_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde{\phi}}(\xi) = \Pi_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi). \tag{2}$$

These infinite products can only converge if

$$m_0(0) = \tilde{m}_0(0) = 1, \tag{3}$$

i.e., if

$$\sum_n h_n = \sum_n \tilde{h}_n = 2. \tag{4}$$

If (3) is satisfied, then the infinite products in (2) converge uniformly and absolutely on compact sets, so that $\hat{\phi}$ and $\hat{\tilde{\phi}}$ are well-defined C^∞ functions. Obviously,

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \quad \hat{\tilde{\phi}}(\xi) = \tilde{m}_0(\xi/2)\hat{\tilde{\phi}}(\xi/2), \tag{5}$$

or, equivalently,

$$\phi(t) = \sum_n h_n \phi(2t-n), \quad \tilde{\phi}(t) = \sum_n \tilde{h}_n \tilde{\phi}(2t-n), \tag{6}$$

at least in the sense of distributions. From Lemma 3.1 in [2], ϕ and $\tilde{\phi}$ have compact support.

We also define the corresponding ψ and $\tilde{\psi}$ by

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{\tilde{m}_0(\xi/2+\pi)}\hat{\phi}(\xi/2) \quad \hat{\tilde{\psi}}(\xi) = e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\hat{\tilde{\phi}}(\xi/2), \tag{7}$$

or, equivalently,

$$\psi(x) = \sum_n (-1)^{n-1}\tilde{h}_{-n-1}\phi(2x-n) \quad \tilde{\psi}(x) = \sum_n (-1)^{n-1} h_{-n-1}\tilde{\phi}(2x-n). \tag{8}$$

We now consider the simplest possible masks $m_0(\xi)$ and $\tilde{m}_0(\xi)$ with the following forms.

$$m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi) \quad \tilde{m}_0^{\tilde{N}}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}} \tilde{F}(\xi), \tag{9}$$

where

$$F(\xi) = e^{-ik'\xi} \sum_{j=0}^{k} a_j e^{-ij\xi} \quad \tilde{F}(\xi) = e^{-i\tilde{k}'\xi} \sum_{j=0}^{\tilde{k}} \tilde{a}_j e^{-ij\xi} \tag{10}$$

Here, all coefficients of $F(\xi)$ and $\tilde{F}(\xi)$ are real, $F(0) = \tilde{F}(0) = 1$, and $F(\pi), \tilde{F}(\pi) \neq -1$; N, $\tilde{N}$, k, and $\tilde{k}$ are positive integers; and $k', \tilde{k}' \in Z$. Hence, the corresponding ϕ and $\tilde{\phi}$ can be written as follows.

$$\hat{\phi}(\xi) = \left(\frac{1+e^{-i\xi/2}}{2}\right)^N F(\xi/2)\hat{\phi}(\xi/2), \;\; \hat{\tilde{\phi}}(\xi) = \left(\frac{1+e^{-i\xi/2}}{2}\right)^{\tilde{N}} \tilde{F}(\xi/2)\hat{\tilde{\phi}}(\xi/2). \tag{11}$$

Clearly, ϕ is a B-spline of order N if $F(\xi) = 1$ (In Section 3, we need one of the scaling functions to be a spline). Since vanishing moment conditions $\int x^\ell \psi(x)dx = 0$, $\ell = 0, 1, \cdots, L$, are equivalent to $\frac{d^\ell}{d\xi^\ell}\hat{\psi}|_{\xi=0} = 0$, $\ell = 0, 1, \cdots, L$, we immediately have the following proposition.

Proposition 1. The maximum number of vanishing moment for ψ and $\tilde{\psi}$ are $N-1$ and $\tilde{N}-1$, respectively.

Therefore, the vanishing moments of ϕ and $\tilde{\phi}$ are completely controlled by the exponents of their respective "spline factors" $\left(\frac{1+e^{-i\xi}}{2}\right)^N$ and $\left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}}$, respectively. In addition, as we pointed out at the beginning of this section, the regularities of ϕ and $\tilde{\phi}$ will be justified by factor $F(\xi)$ and $\tilde{F}(\xi)$, respectively, and are independent of their vanishing moments. In Section 2, we will give simple conditions for coefficients $\{a_j\}$ and $\{\tilde{a}_j\}$ such that the corresponding scaling functions $\phi, \tilde{\phi} \in L^2(R)$ while $\{\phi(x-i)\}$ and $\{\tilde{\phi}(x-i)\}$ are biorthogonal system, i.e., $\langle \phi(t), \tilde{\phi}(t-i)\rangle = \delta_{i0}$ for all integers i. From these conditions, we also give an algorithm for constructing biorthogonal wavelets with the largest possible regularities and the smallest possible supports. In Section 3, we will first establish the existence and uniqueness of biorthogonal spline functions. Then a method for constructing biorthogonal spline scaling functions by using Daubechies' iteration approach will be given.

2. Construction of biorthogonal scaling functions

We shall establish the following main result of this section.

Theorem 2. Let $\phi = \Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ and $\tilde{\phi} = \Pi_{j=1}^{\infty} \tilde{m}_0^{\tilde{N}}(2^{-j}\xi)$, where $m_0^N(\xi)$ and $\tilde{m}_0^{\tilde{N}}(\xi)$ are defined by Eqs. (9) and (10). If

(i) $(k+1)\sum_{j=0}^{k} a_j^2 < 2^{2N-1}$, $\quad (\tilde{k}+1)\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2 < 2^{2\tilde{N}-1}$, and

(ii) $\sum_{j=\mu}^{\nu}\sum_{\ell=0}^{k}\sum_{\tilde{\ell}=0}^{\tilde{k}} \binom{\tilde{N}}{j-\tilde{\ell}-\tilde{k}'}\binom{N}{j+2n-\ell-k'}\tilde{a}_{\tilde{\ell}} a_\ell = 2^{N+\tilde{N}-1}\delta_{n0}$,

where $\mu = min\{k', \tilde{k}'\}$; $\nu = max\{N+k+k', \tilde{N}+\tilde{k}+\tilde{k}'\}$; δ_{n0} is the Kronecker symbol; and $n = 0, \pm 1, \pm 2, \cdots$; then we have $\phi, \tilde{\phi} \in L^2(R)$ and $\langle \phi(t), \tilde{\phi}(t-i)\rangle = \delta_{i,0}$ for all $i \in Z$. The corresponding ψ and $\tilde{\psi}$ define biorthogonal wavelets (biorthogonal Riesz bases) are in C^α and $C^{\tilde{\alpha}}$, respectively. Here α and $\tilde{\alpha}$ are more than

$$N - \frac{1}{2}\log_2\left((k+1)\sum_{j=0}^{k} a_j^2\right), \text{ and } \tilde{N} - \frac{1}{2}\log_2\left((\tilde{k}+1)\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2\right),$$

respectively.

In addition, condition (i) can be replaced by the following weaker condition.

$$(i)' \; C(\{a_j\}, k) < 2^{2N-1}, \tilde{C}(\{\tilde{a}_j\}, \tilde{k}) < 2^{2\tilde{N}-1}, \text{and}$$

$$\sum_{j=0}^{k} |a_j| < 2^{N-1/2}, \sum_{j=0}^{\tilde{k}} |a_j| < 2^{\tilde{N}-1/2},$$

where $C(\{a_j\},k)$ equals $k\sum_{j=0}^{k} a_j^2$ if $k \geq 1$ and equals a_0^2 if $k=0$ and $\tilde{C}(\{\tilde{a}_j\},\tilde{k})$ equals $\tilde{k}\sum_{j=0}^{k} \tilde{a}_j^2$ if $\tilde{k} \geq 1$ and equals $\tilde{a}_0^2$ if $\tilde{k}=0$. Hence, the corresponding regularities of ϕ and $\tilde{\phi}$ are determined by $\phi \in C^{\alpha'}$ and $\tilde{\phi} \in C^{\tilde{\alpha}'}$, where α' and $\tilde{\alpha}'$ are more than $N - \frac{1}{2}\log_2\left(2C(\{a_j\},k)\right)$ and $\tilde{N} - \frac{1}{2}\log_2\left(2\tilde{C}(\{\tilde{a}_j\},\tilde{k}\right)$, respectively.

We shall break up the proof of Theorem 2 into several lemmas.

First, we give the simple conditions for $\phi, \tilde{\phi} \in L^2(R)$.

Lemma 3. If

$$C(\{a_j\},k) < 2^{2N-1},$$

then $\phi = \Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ is in $L^2(R)$. Similarly, $\tilde{\phi} = \Pi_{j=1}^{\infty} \tilde{m}_0^{\tilde{N}}(2^{-j}\xi) \in L^2(R)$ if $\tilde{C}(\{\tilde{a}_j\},\tilde{k}) < 2^{2\tilde{N}-1}$.

Proof. It is sufficient to prove the boundedness of the following integral

$$\begin{aligned}
&\int_{|\xi|\geq\pi} |\hat{\phi}(\xi)|^2 d\xi \\
&= \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^{\ell}\pi} \Pi_{j=1}^{\infty} \left| \left(\frac{1+e^{-i2^{-j}\xi}}{2}\right)^N F(2^{-j}\xi)\right|^2 d\xi \\
&= \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^{\ell}\pi} \left|\frac{1-e^{-i\xi}}{i\xi}\right|^{2N} \Pi_{j=1}^{\infty} |F(2^{-j}\xi)|^2 d\xi \\
&\leq C\sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^{\ell}\pi} \frac{1}{|\xi|^{2N}} \Pi_{j=1}^{\infty} |F(2^{-j}\xi)|^2 d\xi \\
&\leq C\sum_{\ell=1}^{\infty} \frac{1}{2^{2\ell N}} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^{\ell}\pi} \Pi_{j=1}^{\ell} |F(2^{-j}\xi)|^2 \Pi_{j=1}^{\infty} |F(2^{-\ell-j}\xi)|^2 d\xi \\
&\leq C\sum_{\ell=1}^{\infty} 4^{-\ell N} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^{\ell}\pi} \Pi_{j=1}^{\ell} |F(2^{-j}\xi)|^2 d\xi. \qquad (12)
\end{aligned}$$

We now prove the boundedness of the last integral in inequality (12). Denote

$$Tf(\xi) = \left|F\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|F\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right). \qquad (13)$$

Hence, for any 2π-periodic continuous function f, we have

$$\int_{2^{\ell-1}\pi\le|\xi|\le 2^{\ell}\pi} f(2^{-\ell}\xi)\Pi_{j=1}^{\ell}\left|F(2^{-j}\xi)\right|^2 d\xi$$
$$= \int_{-\pi}^{\pi} T^{\ell} f(\xi)d\xi \le \sqrt{2\pi}||T^{\ell}f||_{L^2} \le \sqrt{2\pi}||f||_{L^2}||T^{\ell}||. \tag{14}$$

Let $\rho(T)$ be the spectral radius of the operator T. Since $F(0)=1$ and $F(\pi)\ne -1$, it can be shown that $\rho(T)>0$ (see also [1]). For every $\epsilon>0$, there is an integer $\ell(\epsilon)$ such that

$$||T^{\ell}|| \le (\rho(T)+\epsilon)^{\ell}, \qquad \ell > \ell(\epsilon).$$

It follows from (12) that

$$\int_{|\xi|\ge\pi} |\hat{\phi}(\xi)|^2 d\xi \le C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T^{\ell}||$$
$$+\, C \sum_{\ell=\ell(\epsilon)+1}^{\infty} 4^{-N\ell}\left(\rho(T)+\epsilon\right)^{\ell},$$

so $\rho(T)$ must be estimated if we are to choose $\epsilon>0$ small enough for the series to converge. Regardless of how small $\epsilon>0$ is chosen, the contribution

$$C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T^{\ell}|| \le C\sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell}||T||^{\ell}$$

is finite, although possibly large.

To evaluate $\rho(T)$, we consider the conjugate operator, T^*, of T. It is easy to find that

$$T^* f(\xi) = 2\left|F(\xi)\right|^2 f(2\xi).$$

In fact, for any 2π-periodic continuous functions f and g,

$$\langle Tf, g\rangle = \int_{-\pi}^{\pi} |F\left(\frac{\xi}{2}\right)|^2 f\left(\frac{\xi}{2}\right)\bar{g}(\xi)d\xi + \int_{-\pi}^{\pi} |F\left(\frac{\xi}{2}+\pi\right)|^2 f\left(\frac{\xi}{2}+\pi\right)\bar{g}(\xi)d\xi$$

$$= 2\left(\int_{-\pi/2}^{\pi/2} |F(\xi)|^2 f(\xi)\,\bar{g}(2\xi)d\xi + \int_{\pi/2}^{3\pi/2} |F(\xi)|^2 f(\xi)\,\bar{g}(2\xi)d\xi\right)$$
$$= 2\int_{-\pi/2}^{3\pi/2} |F(\xi)|^2 f(\xi)\,\bar{g}(2\xi)d\xi$$
$$= 2\int_{-\pi}^{\pi} |F(\xi)|^2 f(\xi)\,\bar{g}(2\xi)d\xi = \langle f, T^* g\rangle.$$

If we consider the Fourier expansion

$$|F(\xi)|^2 = \sum_{\ell=-k}^{k} b_\ell e^{i\ell\xi},$$

then the matrix of T^* restricted to $E_k = \{\sum_{\ell=-k}^{k} c_\ell e^{i\ell\xi}, (c_{-k}, \cdots, c_k) \in C^{2k+1}\}$ is given by

$$2H = 2\left(b_{i-2j}\right)_{i,j=-k,\cdots,k}.$$

It is clear that b_t can be written as

$$b_t = \sum_{j=0}^{k-|t|} a_{k-|t|-j} a_{k-j}, \quad t = -k, \cdots, k.$$

Hence, $b_t = b_{-t}$, for all $t = -k, \cdots, k$. It is also obvious that b_k is an eigenvalue of H with multiplicity 2. To estimate the bounds of the eigenvalues of H, we give

$$|b_t| \le b_0$$

for all $t = -k, \cdots, k$. In fact,

$$|b_t| \le \sum_{j=0}^{k-|t|} |a_{k-|t|-j} a_{k-j}| \le \sum_{j=0}^{k-|t|} [\frac{1}{2} a_{k-|t|-j}^2 + \frac{1}{2} a_{k-j}^2]$$
$$= \frac{1}{2}\sum_{j=t}^{k} a_{k-j}^2 + \frac{1}{2}\sum_{j=0}^{k-t} a_{k-j}^2 \le \sum_{j=0}^{k} a_{k-j}^2 = b_0.$$

It is obvious that the spectral radius of H is b_0 if $k = 0$. For $k \geq 1$, the characteristic polynomial of H is $(b_k - \lambda)(b_{-k} - \lambda)$ multiplied by the characteristic polynomial of the core matrix, H_c, which consists of all rows and columns of H except its first and last rows and columns. Hence, the spectral radius of H is

$$\begin{aligned}\rho(H) &= max\{b_k, \rho(H_c)\} \leq max\{b_k, \|H_c\|_1\} \\ &= max\{b_k, \sum_{i=-k+1}^{k-1} |b_{i-2j}| : j = -k+1, \cdots, k-1\} \leq kb_0.\end{aligned}$$

Therefore, $\rho(T) = \rho(T^*) = 2\rho(H) \leq 2C(\{a_j\}, k)$. Here, $C(\{a_j\}, k)$ was defined in Theorem 2. If $C(\{a_j\}, k) < 2^{2N-1}$, then $\rho(T) < 2^{2N}$, so we choose

$$\epsilon = \frac{1}{2}\left(2^{2N} - \rho(T)\right).$$

Thus

$$\rho(T) + \epsilon < 2^{2N},$$

and we obtain the estimation

$$\int_{|\xi|\geq\pi} \left|\hat{\phi}(\xi)\right|^2 d\xi \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T||^\ell + \sum_{\ell=\ell(\epsilon)+1}^{\infty} \left(\frac{\rho(T)+\epsilon}{4^N}\right)^\ell.$$

The tail of the series is a convergent geometric series, thus completing the proof.

Obviously, $C(\{a_j\}, k) \leq (k+1)\sum_{j=0}^{k} a_j^2$ and $\tilde{C}(\{a_j\}, \tilde{k}) \leq (\tilde{k}+1)\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2$. Hence, we have proved the first part of Theorem 2; i.e., $\phi, \tilde{\phi} \in L^2(R)$ under condition (i) or the first half of condition (i)' in Theorem 2.

Remark Matrix H is a special case of rotation matrices $A = [a_{ij}]_{0\leq i,j\leq n}$, where each entry a_{ij} satisfies $a_{ij} = a_{n-i,n-j}$. Denote $E = \{A = [a_{ij}]_{0\leq i,j\leq n} | a_{ij} = a_{n-i,n-j}\}$ and $E' = \{A = [a_{ij}]_{0\leq i,j\leq n} | a_{ij} = a_{n-i,n-j}, A^{-1} exists\}$. Then E is a ring with respect to (regular) matrix multiplication and addition. E' is a group with respect to (regular) matrix multiplication.

It is well-known (see [1,4]) that $\psi \in C^r$ and $r = N - \frac{1}{2}\log_2(\rho(T))$, where $\rho(T)$ is the spectral radius of the matrix representing the operator T and T is defined by Eq. (13). Hence, we obtain the following result.

Lemma 4. $\psi \in C^{\alpha'}$ and $\tilde{\psi} \in C^{\tilde{\alpha}'}$, where α' and $\tilde{\alpha}'$ are more than

$$N - \frac{1}{2}\log_2\left(2C(\{a_j\}, k)\right) \quad \text{and} \quad \tilde{N} - \frac{1}{2}\log_2\left(2\tilde{C}(\{\tilde{a}_j\}, \tilde{k}\right),$$

respectively.

We now discuss the biorthogonality of ϕ and $\tilde{\phi}$. Obviously, if ϕ and $\tilde{\phi}$, defined as in Eq. (6), are biorthogonal, then we have

$$\frac{1}{2}\sum_j \tilde{h}_j h_{j+2n} = \delta_{n0}, \tag{15}$$

where n is an integer.

Lemma 5. If ϕ and $\tilde{\phi}$, defined by (5) (or, equivalently, (6)) and (9), satisfy $\langle \phi(t), \tilde{\phi}(t-\ell)\rangle = \delta_{\ell 0}$, then

$$\sum_{j=\mu}^{\nu}\sum_{\ell=0}^{k}\sum_{\tilde{\ell}=0}^{\tilde{k}} \binom{\tilde{N}}{j-\tilde{\ell}-\tilde{k}'}\binom{N}{j+2n-\ell-k'}\tilde{a}_{\tilde{\ell}}a_\ell = 2^{N+\tilde{N}-1}\delta_{n0}, \tag{16}$$

where $\mu = min\{k', \tilde{k}'\}$, $\nu = max\{N+k+k', \tilde{N}+\tilde{k}+\tilde{k}'\}$, δ_{n0} is the Kronecker symbol, and $n = 0, \pm 1, \pm 2, \cdots$. In particular, if $F(\xi) = 1$ (i.e., $k, k' = 0$), then condition (16) can be written as

$$\sum_{\ell=0}^{\tilde{k}}\sum_{j=\tilde{k}'}^{\max\{N,\tilde{N}+\tilde{k}+\tilde{k}'\}} \binom{\tilde{N}}{j-\ell-\tilde{k}'}\binom{N}{j+2n}\tilde{a}_\ell = 2^{N+\tilde{N}-1}\delta_{n0}, \tag{16$'$}$$

where $n = 0, \pm 1, \pm 2, \cdots$. Condition (16)' can be reduced again for $\tilde{N} = N - \tilde{k}$ and $\tilde{k}' = 0$ to:

$$\sum_{\ell=0}^{\tilde{k}} \left(\sum_{j=0}^{N} \binom{N-\tilde{k}}{j-\ell} \binom{N}{j+2n} \right) \tilde{a}_\ell = 2^{2N-\tilde{k}-1} \delta_{n0}, \tag{16$''$}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

Proof. From Eqs. in (15), noting that $h_j = (\sum_{\ell=0}^{k} \binom{N}{j-\ell-k'} a_\ell)/2^{N-1}$, $j = k', \cdots, N+k+k'$, and $\tilde{h}_j = (\sum_{\ell=0}^{\tilde{k}} \binom{\tilde{N}}{j-\ell-\tilde{k}'} \tilde{a}_\ell)/2^{\tilde{N}-1}$, $j = \tilde{k}', \cdots, \tilde{N} + \tilde{k} + \tilde{k}'$, we immediately obtain (16).

Remark. Condition (16)' implies the condition $\tilde{m}_0(0) = 1$ (see Eqs. in (3)). In fact, adding up Eq. (16)' for $n = 0, \pm 1, \pm 2, \cdots$, we obtain

$$\sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell \left(\sum_{j=\tilde{k}'}^{\max\{N, \tilde{N}+\tilde{k}+\tilde{k}'\}} \sum_n \binom{\tilde{N}}{j-\ell-\tilde{k}'} \binom{N}{j+2n} \right) = \sum_n 2^{N+\tilde{N}-1} \delta_{n0}.$$

Then we note that the left hand side of the above equation is equal to $2^{N+\tilde{N}-1} \sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell$ and the right hand side is $2^{N+\tilde{N}-1}$. Hence, $\tilde{m}_0(0) = \sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell = 1$.

For sufficient conditions of the biorthogonality of ϕ and $\tilde{\phi}$, we have the following result.

Lemma 6. Let ϕ and $\tilde{\phi}$ be functions defined by (5) (or (6)) and (9). Assume that Eq. (16) holds and suppose that, for some integer $n > 0$,

$$\begin{aligned} B_{2^n} &= \max_\xi \left| \Pi_{j=0}^{2^n-1} F(2^j \xi) \right|^{1/2^n} < 2^{N-1/2}, \\ \tilde{B}_{2^n} &= \max_\xi \left| \Pi_{j=0}^{2^n-1} \tilde{F}(2^j \xi) \right|^{1/2^n} < 2^{\tilde{N}-1/2}. \end{aligned} \tag{17}$$

Then $\langle \phi(t), \tilde{\phi}(t-\ell) \rangle = \delta_{\ell 0}$, $|\hat{\phi}(\xi)| \le C(1+|\xi|)^{-1/2-\alpha}$, and $|\hat{\tilde{\phi}}(\xi)| \le C(1+|\xi|)^{-1/2-\tilde{\alpha}}$, where $\alpha = N - 1/2 - \log_2 B_{2^n}$ and $\tilde{\alpha} = \tilde{N} - 1/2 - \log_2 \tilde{B}_{2^n}$. Hence, the corresponding ψ and $\tilde{\psi}$ are biorthogonal wavelets; i.e., $\{\psi(t-k)\}_k$ and $\{\tilde{\psi}(t-k)\}_k$ are biorthogonal Riesz bases of W_0.

Proof. If expression (17) is true, then by using Propositions 4.8 and 4.9 in [2] we have $\langle \phi(t), \tilde{\phi}(t-\ell)\rangle = \delta_{\ell 0}$, $|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-1/2-\alpha}$, and $|\hat{\tilde{\phi}}(\xi)| \leq C(1+|\xi|)^{-1/2-\tilde{\alpha}}$. By applying Theorem 3.2 in [1], $|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-1/2-\alpha}$ and $|\hat{\tilde{\phi}}(\xi)| \leq C(1+|\xi|)^{-1/2-\tilde{\alpha}}$ imply that the corresponding ψ and $\tilde{\psi}$ are biorthogonal wavelets in $L^2(R)$. Hence, the proof of Lemma 6 is complete.

Remark. It is easy to find that sequences $\{B_{2^n}\}_n$ and $\{\tilde{B}_{2^n}\}$ are decreasing. In fact, it comes from the following observation.

$$\begin{aligned}\left|\Pi_{j=0}^{2^n-1} F(2^j\xi)\right| &\leq \max_\xi \left|\Pi_{j=0}^{2^{n-1}-1} F(2^j\xi)\right| \max_\xi \left|\Pi_{j=2^{n-1}}^{2^n-1} F(2^{j-2^{n-1}} 2^{2^{n-1}}\xi)\right| \\ &= \max_\xi \left|\Pi_{j=0}^{2^{n-1}-1} F(2^j\xi)\right| \max_\xi \left|\Pi_{j=0}^{2^{n-1}-1} F(2^j\xi)\right| \\ &= \max_\xi \left|\Pi_{j=0}^{2^{n-1}-1} F(2^j\xi)\right|^2 .\end{aligned}$$

Hence $B_{2^n} \leq B_{2^{n-1}} \leq \cdots B_1 = \max_\xi |F(\xi)|$. Similarly, we have $\tilde{B}_{2^n} \leq \tilde{B}_{2^{n-1}} \leq \cdots \leq \tilde{B}_1 = \max_\xi |\tilde{F}(\xi)|$.

Obviously, $B_{2^n} \leq \max_\xi |F(\xi)| = \sum_{j=0}^{k} |a_j|$ and $\tilde{B}_{2^n} \leq \max_\xi |\tilde{F}(\xi)| = \sum_{j=0}^{\tilde{k}} |\tilde{a}_j|$. Hence, by using the Cauchy-Schwartz inequality, we obtain

$$B_{2^n} \leq \left[(k+1)\sum_{j=0}^{k} a_j^2\right]^{1/2},$$

where the equal sign does not hold if $\{a_j\}$ satisfies condition (16), because all $\{a_j\}$ can not be the same. Hence, if $(k+1)\sum_{j=0}^{k} a_j^2 < 2^{2\tilde{N}-1}$, then $B_{2^n} < 2^{\tilde{N}-1/2}$. We obtain the following lemma.

Lemma 7. If Eq. (16) holds and either

(i) $\sum_{j=0}^{k} |a_j| < 2^{N-1/2}, \qquad \sum_{j=0}^{\tilde{k}} |\tilde{a}_j| < 2^{\tilde{N}-1/2}$

or

(ii)$(k+1)\sum_{j=0}^{k} a_j^2 < 2^{2N-1}, \quad (\tilde{k}+1)\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2 < 2^{2\tilde{N}-1}$

holds, then $\langle \phi(t), \tilde{\phi}(t-\ell) \rangle = \delta_{\ell 0}$ and the corresponding ψ and $\tilde{\psi}$ are biorthogonal wavelets; i.e., $\{\psi(t-k)\}_k$ and $\{\tilde{\psi}(t-k)\}_k$ are biorthogonal Riesz bases of W_0.

Combining Lemmas 3, 4, 5, and 7, we obtain Theorem 2.

We now give a general algorithm to construct biorthogonal scaling functions ϕ and $\tilde{\phi}$ with the largest possible regularity and the required vanishing moments. In fact, this method can be described as the optimization problem of finding suitable $F(\xi)$ and $\tilde{F}(\xi)$, or, equivalently, suitable coefficient sets, $\mathbf{a} = \{a_0, \cdots, a_k\}$ and $\tilde{\mathbf{a}} = \{\tilde{a}_0, \cdots, \tilde{a}_k\}$, of $F(\xi)$ and $\tilde{F}(\xi)$, respectively, such that $\sum_{j=0}^{k} a_j^2$ and $\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2$ are the minimum under conditions (i) and (ii) of Theorem 2. From the wavelet analysis of spline approximation, we assume $F(\xi) = 1$; i.e., the corresponding ϕ is the B-spline of order N. Then, the above optimization problem can be written as follows.

$$\min_{\mathbf{a}} \quad \sum_{j=0}^{\tilde{k}} \tilde{a}_j^2, \tag{18}$$

$$\text{subject to} \quad \left(\sum_{j=0}^{\tilde{k}} (-1)^j \tilde{a}_j + 1 \right)^2 > 0, \tag{19}$$

$$\sum_{j=0}^{\tilde{k}} \tilde{a}_j^2 < 2^{2\tilde{N}-1}/(\tilde{k}+1), \tag{20}$$

$$\sum_{\ell=0}^{\tilde{k}} \sum_{j=\tilde{k}'}^{\max\{N, \tilde{N}+\tilde{k}+\tilde{k}'\}} \binom{\tilde{N}}{j-\ell-\tilde{k}'} \binom{N}{j+2n} \tilde{a}_\ell = 2^{N+\tilde{N}-1} \delta_{n0}, \tag{21}$$

where object (18) will give the largest possible regularity, condition (19) is from the definition of $\tilde{F}$, and conditions (20) and (21) come from conditions (i) and (ii) of Theorem 2.

Problem (18)-(21) can be written in a form without the inequality condition by defining parameters $s, t \neq 0$ by $s^2 = 2^{2\tilde{N}-1}/(\tilde{k}+1) - \sum_{j=0}^{\tilde{k}} \tilde{a}_j^2$ and $t^2 =$

$\left(\sum_{j=0}^{\tilde{k}}(-1)^j\tilde{a}_j+1\right)^2$, respectively. Hence, the optimization problem becomes

$$\min_{\mathbf{a}} \qquad \sum_{j=0}^{\tilde{k}}\tilde{a}_j^2+\frac{1}{s^2}+\frac{1}{t^2},$$

$$\text{subject to } t^2=\left(\sum_{j=0}^{\tilde{k}}(-1)^j\tilde{a}_j+1\right)^2$$

$$\sum_{j=0}^{\tilde{k}}\tilde{a}_j^2+s^2=2^{2\tilde{N}-1}/(\tilde{k}+1),$$

$$\sum_{\ell=0}^{\tilde{k}}\sum_{j=\tilde{k}'}^{\max\{N,\tilde{N}+\tilde{k}+\tilde{k}'\}}\binom{\tilde{N}}{j-\ell-\tilde{k}'}\binom{N}{j+2n}\tilde{a}_\ell=2^{N+\tilde{N}-1}\delta_{n0}.$$

As examples, we consider $m_0(\xi)=\left(\frac{1+e^{-i\xi}}{2}\right)$; i.e., ϕ is the B-spline of order 1. If we choose $\tilde{N}=1$ and $\tilde{k}=\tilde{k}'=0$, then the solution of problem (18)-(21) is $\tilde{a}_0=1$ and we obtain the Haar function. If we choose $\tilde{N}=2$, $\tilde{k}=1$, and $\tilde{k}'=0$, then the solutions of the problem are $\tilde{a}_0=3/2$ and $\tilde{a}_1=-1/2$. Hence, the corresponding $\tilde{\phi}$ is defined by $\hat{\tilde{\phi}}(\xi)=\Pi_{j=1}^{\infty}\tilde{m}_0(2^{-j}\xi)$, where $\tilde{m}_0(\xi)=\left(\frac{1+e^{-i\xi}}{2}\right)^2\left(\frac{3-e^{-i\xi}}{2}\right)$. In addition, the regularities of ϕ and $\tilde{\phi}$ are 1 and $2-\log_2(5)/2=0.839036$.

3. Biorthogonal scaling functions associated biorthogonal splines

For the sake of application, we need not the biorthogonality of $\phi=\Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ and $\tilde{\phi}=\Pi_{j=0}^{\infty}\tilde{m}_0^{\tilde{N}}(2^{-j}\xi)$, instead, we need only the biorthogonality of $\phi_n=\Pi_{j=1}^{n}m_0^N(2^{-j}\xi)$ and $\tilde{\phi}_n=\Pi_{j=0}^{n}\tilde{m}_0^{\tilde{N}}(2^{-j}\xi)$ for some $n>0$. Hence, we need to reconsider Eqs. in (6) as iteration schemes for constructing biorthogonal ϕ_n and $\tilde{\phi}_n$, where coefficients h_j and $\tilde{h}_j$ are assumed to satisfy Eqs. in (15). For instance, by substituting some biorthogonal initial functions ϕ_0 and $\tilde{\phi}_0$ on the right hand sides of (6), we may obtain biorthogonal ϕ_1 and $\tilde{\phi}_1$ from the left hand sides of (6). By using the iteration scheme (6) again, we can also obtain biorthogonal ϕ_2 and $\tilde{\phi}_2$ etc. This approach is called Daubechies' approach.

In this section, we will construct biorthogonal scaling functions, ϕ_0 and $\tilde{\phi}_0$, with compact support and certain approximation properties from biorthogonal spline functions by using Daubechies' approach.

In this section, we will construct biorthogonal scaling functions, ϕ and $\tilde{\phi}$, with compact support and certain approximation properties from biorthogonal spline functions using Daubechies' approach. The first step is to construct biorthogonal splines.

Let N_n be the cardinal B-spline of order n (i.e., degree $n-1$) and $S = span\,\{N_n(mt-k)\}_{k\in Z}$, where m is a natural number. We will construct a C^{n-2} spline function $\tilde{N}_n$ of degree $n-1$ from S such that N_n and $\tilde{N}_n$ are biorthogonal $(\langle \tilde{N}_n(t-\ell), N_n(t)\rangle = \delta_{\ell 0})$ and the shift space of $\tilde{N}_n$ contains all polynomials of degree $n-2$.

Theorem 8. Let $N_n : R \to R$ be the B-spline of order n and

$$\tilde{N}_n(t) = \sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk} N_n(m(t-j)-k) \tag{22}$$

be a spline function of order n such that $\langle \tilde{N}_n(t-\ell), N_n(t)\rangle = \delta_{\ell 0}$. Then the shift space of $\tilde{N}_n$ contains all polynomials of degree $n-1$ if and only if

$$\sum_{j=o}^{n-1} j^{\alpha} c_{jk} = \left(-\frac{k}{m}\right)^{\alpha} \tag{23}$$

for $k = 0, 1, \cdots, m-1$ and $\alpha = 0, 1, \cdots, n-1$.

Proof. *Sufficiency.* We only need to prove that condition (23) implies $\widehat{\tilde{N}_n}(0) = 1$ and $\widehat{\tilde{N}_n}^{(\beta)}(2\ell\pi) = 0$ for $\ell \in Z\backslash\{0\}$ and $\beta = 0, 1, \cdots, n-1$ (see [7]). In fact, from the biorthogonality of $\tilde{N}_n$ and N_n, we have

$$
\begin{aligned}
\widehat{\tilde{N}_n}(0) &= \langle \tilde{N}_n(t), 1\rangle \\
&= \langle \tilde{N}_n(t), \sum_{\ell\in Z} N_n(t-\ell)\rangle \\
&= \sum_{\ell\in Z} \langle \tilde{N}_n(t+\ell), N_n(t)\rangle \\
&= \sum_{\ell\in Z} \delta_{\ell 0} \\
&= 1.
\end{aligned} \tag{24}
$$

Taking a Fourier transform on both sides of expression (22), we obtain

$$
\widehat{\tilde{N}_n}(\omega) = \frac{1}{m}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk} e^{-i(j+k/m)\omega}\widehat{N_n}\left(\frac{\omega}{m}\right). \tag{25}
$$

Thus

$$
\begin{aligned}
&\widehat{\tilde{N}_n}^{(\beta)}(\omega) \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} c_{jk} \sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}(-i)^\gamma \left(j+\frac{k}{m}\right)^\gamma e^{-i(j+k/m)\omega}\widehat{N_n}^{(\beta-\gamma)}\left(\frac{\omega}{m}\right) \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}(-i)^\gamma \sum_{\alpha=0}^{\gamma}\binom{\gamma}{\alpha}\sum_{j=0}^{n-1} c_{jk} j^\alpha \left(\frac{k}{m}\right)^{\gamma-\alpha} \\
&\quad e^{-i(j+k/m)\omega}\widehat{N_n}^{(\beta-\gamma)}\left(\frac{\omega}{m}\right).
\end{aligned}
$$

Substituting $\omega = 2\ell\pi$ into the last equation and noting condition (23), we obtain

$$
\begin{aligned}
&\widehat{\tilde{N}_n}^{(\beta)}(2\ell\pi) \\
&= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}(-i)^\gamma \sum_{\alpha=0}^{\gamma}\binom{\gamma}{\alpha}(-1)^\alpha \left(\frac{k}{m}\right)^{\gamma} e^{-2i\ell k\pi/m}\widehat{N_n}^{(\beta-\gamma)}\left(\frac{2\ell\pi}{m}\right) \\
&= \frac{1}{m}\sum_{k=0}^{m-1} e^{-2i\ell k\pi/m}\widehat{N_n}^{(\beta)}\left(\frac{2\ell\pi}{m}\right)
\end{aligned}
$$

$$+\frac{1}{m}\sum_{k=0}^{m-1}\sum_{\gamma=1}^{\beta}\binom{\beta}{\gamma}(-i)^{\gamma}(1-1)^{\gamma}\left(\frac{k}{m}\right)^{\gamma}e^{-2i\ell k\pi/m}\widehat{N_n}^{(\beta-\gamma)}\left(\frac{2\ell\pi}{m}\right)$$
$$=\frac{1}{m}\left(\sum_{k=0}^{m-1}e^{-2i\ell k\pi/m}\right)\widehat{N_n}^{(\beta)}\left(\frac{2\ell\pi}{m}\right)$$
$$=0.$$

Necessity. The result in [5] by de Boor can help us to prove this part. From Eq. (25), we have

$$\widehat{\tilde{N}_n}(0)=\frac{1}{m}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}c_{jk}.$$

This equation and Eq. (24) give

$$\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}c_{jk}=m. \tag{26}$$

If $\tilde{N}_n(t-\ell)$, $\ell\in Z$, is a partition of unity, then $\widehat{\tilde{N}_n}(2\ell\pi)=0$; i.e.,

$$\sum_{k=0}^{m-1}\left(\sum_{j=0}^{n-1}c_{jk}\right)e^{-i\frac{2\ell k}{m}\pi}=0, \tag{27}$$

$\ell=1,2,\cdots,m-1$. It is easy to see that the system of equations made up by (26) and (27) has the unique solution $\sum_{j=0}^{n-1}c_{jk}=1$, $k=0,1,\cdots,m-1$. Hence, we have proved that condition (23) with $\alpha=0$ is necessary for the shift space of $\tilde{N}_n$ to contain all constants. By using mathematical induction, we can prove the necessity of condition (23) with $\alpha=1,\cdots,n-1$ for the shift space of $\tilde{N}_n$ to contain all polynomials of degree $n-1$.

Obviously, if we impose $supp(\tilde{N}_n)=[0,n]$ and the biorthogonal condition

$$\langle\tilde{N}_n(t),N_n(t-\ell)\rangle=\delta_{\ell 0}, \tag{28}$$

then the shift space of $\tilde{N}_n$ contains all polynomials of degree of at most $n-2$. Hence, we have the following result.

Theorem 9. There exists a unique spline function $\tilde{N}_n$ defined by (22) with $m = 3n-3$ and $supp(\tilde{N}_n) = [0,n]$ that satisfies Eq. (28), and the shift space of $\tilde{N}_n$ contains all polynomials of degree $n-2$.

Proof. Noting the supports of N_n and $\tilde{N}_n$, we immediately have $\langle \tilde{N}_n(t-\ell), N_n(t)\rangle = 0$ for all $\ell \geq n$ or $\ell \leq -n$. Hence, for $-n+1 \leq \ell \leq n-1$, we have

$$
\begin{aligned}
\delta_{\ell 0} &= \langle \tilde{N}_n(t), N_n(t-\ell)\rangle \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk}\langle N_n(m(t-j)-k), N_n(t-\ell)\rangle \\
&= \frac{1}{2m\pi}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk}\int_{-\infty}^{\infty}\widehat{N_n}\left(\frac{\omega}{m}\right)e^{-i(j+k/m)\omega}\overline{\widehat{N_n}(\omega)e^{-i\ell\omega}}d\omega \\
&= \frac{1}{2m\pi}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk}\int_{-\infty}^{\infty}\widehat{N_n}\left(\frac{\omega}{m}\right)e^{-i(j+k/m)\omega}\widehat{N_n}(\omega)e^{i(\ell+n)\omega}d\omega \\
&= \frac{1}{2m^{n+1}\pi}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk}\int_{-\infty}^{\infty}\widehat{N_{2n}}\left(\frac{\omega}{m}\right)\Big(1+e^{-i\omega/m}+e^{-i2\omega/m}+\cdots \\
&\quad +e^{-i(m-1)\omega/m}\Big)^n e^{i(\ell+n-j-k/m)\omega}d\omega \\
&= \frac{1}{2m^{n+1}\pi}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\sum_{|s|=n} c_{jk}\int_{-\infty}^{\infty}\widehat{N_{2n}}\left(\frac{\omega}{m}\right)\frac{n!}{s!}e^{-i\omega(s_1+2s_2+\cdots+(m-1)s_{m-1})/m} \\
&\quad e^{i(\ell+n-j-k/m)\omega}d\omega \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{m-1} a_{jk\ell}c_{jk},
\end{aligned}
$$

where $s = (s_1, s_2, \cdots, s_{m-1})$, $|s| = s_1+s_2+\cdots+s_{m-1}$, and $s! = s_1!s_2!\cdots s_{m-1}!$

and

$$a_{jk\ell} = \frac{1}{m^n}\left(\sum_{|s|=n}\frac{n!}{s!}N_{2n}(m(\ell+n-j)-k-\sum_{\alpha=1}^{m-1}\alpha s_\alpha)\right). \tag{29}$$

From the requirement of $supp(\tilde{N}_n)=[0,n]$, we need to set

$$c_{n-1,m-k}=0, \quad k=1,2,\cdots,n-1. \tag{30}$$

Therefore, the coefficients c_{jk} in expression (22) can be solved uniquely from the system consisting of Eqs. in (30); Eqs. in (23) for $k=0,1,\cdots,m-1$ and $\alpha=0,1,\cdots,n-2$; and equations $\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}a_{jk\ell}c_{jk}=\delta_{\ell 0}$ for $\ell=-n+1,-n+2,\cdots,n-1$. In fact, the biorthogonal conditions in (28) imply Eq. (26). Thus, we omit equation $\sum_{j=0}^{n-1}c_{j,m-1}=1$ from the system obtained, and the remaining system possesses an $nm\times nm$ coefficient matrix with full rank if $m=3n-3$. This completes the proof of Theorem 9.

As an example, we consider the cases of $n=2$ and 3 and $m=3$. From Theorems 8 and 9, there exists a spline function $\tilde{\phi}_n$ defined by

$$\tilde{\phi}_n(t)=\sum_{j=0}^{n-1}\sum_{k=0}^{2}c_{jk}N_n(3(t-j)-k),$$

such that

(a) $\langle\tilde{\phi}_n(t-\ell),N_n(t)\rangle=\delta_{\ell 0}$ and

(b) $\sum_{\ell\in Z}\tilde{\phi}(t-\ell)=1$,

where $\tilde{\phi}_n=N_n$ exists uniquely. $\tilde{\phi}_n$ can be found by using the following algorithm.

Step 1. Set $d_{j-1}:=\tilde{\phi}_n(j/3)$, $j=1,2,\cdots,3n-1$. Assume

$$d_j=d_{3n-1-j}. \tag{31}$$

Step 2. Use the integral formula

$$\langle P_n,Q_n\rangle_{[a,b]}=(n!)^2(b-a)\sum_{i_1+i_2=n}\sum_{j_1+j_2=n}a_{i_1,i_2}b_{j_1,j_2}\frac{\binom{i_1+j_1}{i_1}\binom{i_2+j_2}{i_2}}{(2n+1)!} \tag{32}$$

to establish a system

$$\langle \tilde{\phi}_n(t-\ell), N_n(t) \rangle = \delta_{\ell 0}, \tag{33}$$

$\ell = 0, 1, \cdots, n-1$. In the integral formula, $P_n(t) = \sum_{i_1+i_2=n} a_{i_1,i_2} B^n_{i_1,i_2}(u,v)$ and $Q_n(t) = \sum_{j_1,j_2=n} b_{j_1,j_2} B^n_{j_1,j_2}(u,v)$ are two polynomials with Bernstein-Bézier form defined on interval [a,b], where $B^n_{m,\ell}(u,v) = \frac{n!}{m!\ell!} u^m v^\ell$ and (u,v) is the barycentric coordinates of t.

Step 3. Set partition of unity condition equation

$$\sum_{k=1}^{n-1} d_{3k-1} = 1. \tag{34}$$

Step 4. Solve the system consisting of (31), (33), and (34) for all d_j, $j = 0, \cdots, 3n-2$.

By using the above algorithm for $n = 2$, we obtain

$$\tilde{N}_2(t) = -\frac{7}{6} N_2(3t) + \frac{13}{6} N_2(3t-1) + N_2(3t-2) + \frac{13}{6} N_2(3t-3) - \frac{7}{6} N_2(3t-4).$$

We can now construct biorthogonal Scaling functions from biorthogonal splines by using Daubechies' iteration approach. If $\phi(t) = \sum_{j\in J} h_j \phi(2t-j)$ and $\tilde{\phi}(t) = \sum_{j\in J} \tilde{h}_j \tilde{\phi}(2t-j)$, or equivalently, $\hat{\phi}(\omega) = m_0\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right)$ and $\hat{\tilde{\phi}}(\omega) = \tilde{m}_0\left(\frac{\omega}{2}\right) \hat{\tilde{\phi}}\left(\frac{\omega}{2}\right)$, which are defined by Eqs. (5) and (6), there exits the following relation between $m_0(\omega)$ and $\tilde{m}_0(\omega)$.

$$m_0(\omega)\overline{\tilde{m}_0(\omega)} + m_0(\omega+\pi)\overline{\tilde{m}_0(\omega+\pi)} = 1. \tag{35}$$

Similar to Daubechies' iteration scheme for constructing compactly supported orthogonal scaling functions and wavelets, we can construct our compactly supported continuous biorthogonal spline scaling functions and wavelets (In our approach moment conditions may be eliminated) as follows. Set $J = \{0, 1, 2\}$, and the initial functions are

$$\phi^{(0)}(t) = N_2(t) \qquad \tilde{\phi}^{(0)}(t) = \tilde{\phi}_2(t).$$

Eq. (35), coupled with the requirement that $\tilde{m}_0(0) = 1$ and $m_0(0) = 1$ and hence, $m_0(\pi) = \tilde{m}_0(\pi) = 0$, gives the four expressions

$$\sum_{j\in J} h_j = 2, \qquad \sum_{j\in \tilde{J}} \tilde{h}_j = 2, \tag{36}$$

$$\sum_{j\in J} (-1)^j h_j = 0, \qquad \sum_{j\in \tilde{J}} (-1)^j \tilde{h}_j = 0. \tag{37}$$

Next, we need the following biorthonormality of $\tilde{\phi}$ and ϕ.

$$\langle \tilde{\phi}(t), \phi(t-n)\rangle = \delta_{n,0}, \quad n = 0, \pm 1, \pm 2, \cdots,$$

which becomes (see also Eqs. in (15))

$$\sum_{j\in J\cap \tilde{J}} \tilde{h}_j h_{j+2n} = 2\delta_{n0} \quad n = 0, \pm 1, \pm 2, \cdots, \tag{38}$$

because $\phi^{(0)}$ and $\tilde{\phi}^{(0)}$ are biorthogonal. Finally, if we require some degree of smoothness for $\tilde{\phi}$ and ϕ, then some wavelet moments must be zero (see [4]). These convert into some additional conditions for $\{\tilde{h}_j\}$ and $\{h_j\}$. If $m_0(\xi)$ and $\tilde{m}_0(\xi)$ are taken as the same as (9), then the vanishing moment conditions for ψ and $\tilde{\psi}$ are ensured and $F(\xi)$ and $\tilde{F}(\xi)$ can be determined for the biorthogonality of ψ and $\tilde{\psi}$.

As an example, we consider the trivial case of $|J| = |\tilde{J}| = 2$ and set $J = \tilde{J} = \{0, 1\}$. Thus, Eqs. (36)-(38) have the solution $h_0 = h_1 = \tilde{h}_0 = \tilde{h}_1 = 1$. Here, $\psi = \tilde{\psi}$ is the Haar wavelet.

Another example is for $|J| = |\tilde{J}| = 3$, $J = \{0, 1, 2\}$, and $\tilde{J} = \{1, 2, 3\}$. To determine the coefficients $\{\tilde{h}_j\}$ and $\{h_j\}$, we need one of the following additional smoothness conditions.

$$\sum_{j \in J} (-1)^j j h_j = 0 \tag{39}$$

or

$$\sum_{j \in \tilde{J}} (-1)^j j \tilde{h}_j = 0. \tag{40}$$

Eqs. (39) and (40) are from the wavelet moments $\langle t, \psi(t) \rangle = 0$ and $\langle t, \tilde{\psi}(t) \rangle = 0$, respectively, where $\tilde{\psi}$ and ψ are the corresponding wavelets of $\tilde{\phi}$ and ϕ, respectively.

Solving Eqs. (36)-(40), for instance, we obtain $h_0 = h_2 = 1/2$, $h_1 = 1$, $\tilde{h}_1 = 3/2$, $\tilde{h}_2 = 1$, and $\tilde{h}_3 = -1/2$. Hence, we have the following result.

Theorem 10. There exists only one pair of functions $\phi(t)$, $t \in D$, and $\tilde{\phi}(t)$, $t \in \frac{1}{3}D$, that satisfy the following conditions, where $D = \cup_{j \in Z} D_j$ and $D_j = \{k/2^j : k \in Z\}$.

(i) ϕ and $\tilde{\phi}$ satisfy the two-scaling relations

$$\begin{aligned} \phi(t) &= h_0\phi(2t) + \phi(2t-1) + (1-h_0)\phi(2t-2), \quad t \in D, \\ \tilde{\phi}(t) &= \tilde{h}_1\tilde{\phi}(2t-1) + \tilde{\phi}(2t-2) + (1-\tilde{h}_1)\tilde{\phi}(2t-3), \quad t \in \frac{1}{3}D, \end{aligned}$$

where $h_0 = 1/2$ and $\tilde{h}_1 = 3/2$;

(ii) $\sum_{\ell \in Z} \phi(t-\ell) = 1$ and $\sum_{\ell \in Z} \tilde{\phi}(t-\ell) = 1$;

(iii) $supp(\phi) = [0, 2]$ and $supp(\tilde{\phi}) = [\frac{1}{2}, 3]$;

(iv) $\langle \tilde{\phi}(t-\ell), \phi(t) \rangle = \delta_{\ell 0}$.

The proof of Theorem 10 is similar to the argument in [6] by Pollen.

We can also easily find that ϕ is both right and left-differentiable but is not differentiable at any point in $D \cup [0, 2)$ and $\tilde{\phi}$ is both right and left-differentiable but is not differentiable at any point in $\frac{1}{3}D \cup [\frac{1}{2}, 3)$.

The third example is Daubechies' D4 biorthogonal scaling functions. The corresponding two-scale relation coefficients, $\{h_j\}$ and $\{\tilde{h}_j\}$, can be constructed from Eqs. (36)-(40) with $J = \tilde{J} = \{0, 1, 2, 3\}$. Thus, $h_0 = \tilde{h}_3 = (1 - \sqrt{3})/4$, $h_1 = \tilde{h}_2 = (3 - \sqrt{3})/4$, $h_2 = \tilde{h}_1 = (3 + \sqrt{3})/4$, and $h_3 = \tilde{h}_0 = (1 + \sqrt{3})/4$. And we have the following results.

Denote

$$D_j = \{\frac{k}{2^j} : k \in Z\} \qquad Q(\sqrt{3}) = \{\alpha + \beta\sqrt{3} : \alpha, \beta \in Q\}.$$

Theorem 11. There exist only two functions $\phi(t)$, $t \in D$, and $\tilde{\phi}(t)$, $t \in \frac{1}{3}D$, that satisfy condition (i) or (i)', respectively, as well as conditions (ii)-(iv).

(i) If $x \in D$ then ϕ satisfies the two-scaling relation

$$\phi(t) = h_0\phi(2t) + h_1\phi(2t-1) + h_2\phi(2t-2) + h_3\phi(2t-3);$$

(i)' If $x \in \frac{1}{3}D$ then ϕ satisfies the two-scaling relation

$$\tilde{\phi}(t) = \tilde{h}_0\phi(2t) + \tilde{h}_1\phi(2t-1) + \tilde{h}_2\phi(2t-2) + \tilde{h}_3\phi(2t-3);$$

(ii) $\sum_{\ell \in Z} \phi(t - \ell) = 1$ and $\sum_{\ell \in Z} \tilde{\phi}(t - \ell) = 1$;

(iii) $supp(\phi) = [0, 3]$ and $supp(\tilde{\phi}) = [0, 3]$;

(iv) $\langle \tilde{\phi}(t - \ell), \phi(t) \rangle = \delta_{\ell 0}$.

In addition, $t \in D$ implies $\phi(t) \in Q(\sqrt{3})$ and $t \in \frac{1}{3}D$ implies $\tilde{\phi}(t) \in Q(\sqrt{3})$.

It can also be proved that the values of ϕ and $\tilde{\phi}$ are conjugate symmetric about 3/2.

Similar to the example shown in Theorem 10, ϕ is both right and left-differentiable but is not differentiable at any point in $D \cup [0, 3)$ and $\tilde{\phi}$ is both right and left-differentiable but is not differentiable at any point in $\frac{1}{3}D \cup [0, 3)$.

In general, $m_0(z)$ and $\tilde{m}_0(z)$, the two-scale symbols of $\{h_j\}$ and $\{\tilde{h}_j\}$, respectively, can be written as

$$m_0(z) = \left(\frac{1+z}{2}\right)^n F(z), \quad \tilde{m}_0(z) = \left(\frac{1+z}{2}\right)^n \tilde{F}(z),$$

where $z = e^{-i\xi}$, $F(1) = \tilde{F}(1) = 1$ and $F(-1), \tilde{F}(-1) \neq 0$. Substituting $m_0(z)$ and $\tilde{m}_0(z)$ into Eq. (35), we have

$$|\frac{1+z}{2}|^{2n} F(z)\overline{\tilde{F}(z)} + |\frac{1-z}{2}|^{2n} F(-z)\overline{\tilde{F}(-z)} = 1.$$

Acknowledgments.

Many thanks to Ingrid Daubechies, Guy Battle, and the referees for suggesting improvements.

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