Fourier Transform of Bernstein-Bézier Polynomials

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Abstract

Explicit formulae, in terms of Bernstein-Bézier coefficients, of the Fourier transform of bivariate polynomials on a triangle and univariate polynomials on an interval are derived in this paper. Examples are given and discussed to illustrate the general theory. Finally, this consideration is related to the study of refinement masks of spline function vectors.

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1 Introduction

The objective of this paper is to present a compact formula of the Fourier transform of bivariate polynomials on a triangle and univariate polynomials on a bounded interval in terms of their Bernstein-Bézier (BB) coefficients (see, for example, [3, p. 58]). Of course the BB coefficients are formulated, as usual, in

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terms of the Barycentric coordinates, as opposed to the Cartesian coordinates $\mathbf{x} = (x_1, x_2)$ for $\mathbf{x} \in \mathbb{R}^2$ or $x \in \mathbb{R}$. We will focus on the bivariate setting and only consider the univariate formulae as simple consequences. In this regard, although our method of derivation can be extended to multivariate polynomials on simplexes, we have decided to present the detailed derivation only for bivariate polynomials, since the motivation of this research is the study of subdivision masks [1] for spline curves and surfaces.

Let $P_n(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} = (x_1, x_2)$, be a Bernstein-Bézier (or Bézier polynomial (see, for example, [3, p. 58])) on a triangle $\triangle A_1A_2A_3$ with vertices $A_i = (a_i, b_i)$, where $i = 1, 2, 3$. We write $P_n(\mathbf{x})$ in terms of the Barycentric coordinates (u, v, w) of $\triangle A_1A_2A_3$ as follows.

$$P_n(\mathbf{x}) \equiv P_n(x_1, x_2) = \sum_{0 \le j,k,l \le n,\ j+k+l=n} a_{j,k,l} \frac{n!}{j!k!l!} u^j v^k w^l, \tag{1.1}$$

where (u, v, w) is the Barycentric coordinate of $\mathbf{x} = (x_1, x_2) \in \triangle A_1A_2A_3$ (i.e., $(x_1, x_2) = (a_1u + a_2v + a_3w, b_1u + b_2v + b_3w)$, $0 \le u, v, w \le 1$ and $u + v + w = 1$), with Bernstein-Bézier (BB) coefficients $a_{j,k,l}$.

For $(m, s) \in \{(1, 2), (1, 3), (2, 3)\}$, the forward-backward and backward-forward operators, $\triangle_{m,s}$ and $\bigtriangledown_{m,s}$ respectively, are defined on the sequences $\{a_{j,k,l}\}_{j,k,l}$ with multi-indices j, k, l by:

$$\begin{aligned}
\triangle_{12} a_{j,k,l} &:= a_{j+1,k-1,l} - a_{j,k,l}, \quad & \bigtriangledown_{12} a_{j,k,l} &:= a_{j,k,l} - a_{j-1,k+1,l}, \\
\triangle_{13} a_{j,k,l} &:= a_{j+1,k,l-1} - a_{j,k,l}, \quad & \bigtriangledown_{13} a_{j,k,l} &:= a_{j,k,l} - a_{j-1,k,l+1}, \\
\triangle_{23} a_{j,k,l} &:= a_{j,k+1,l-1} - a_{j,k,l}, \quad & \bigtriangledown_{23} a_{j,k,l} &:= a_{j,k,l} - a_{j,k-1,l+1};
\end{aligned} \tag{1.2}$$

and in addition, we set $\triangle_{m,s}^k := \triangle_{m,s}\left(\triangle_{m,s}^{k-1}\right)$ and $\bigtriangledown_{m,s}^k := \bigtriangledown_{m,s}\left(\bigtriangledown_{m,s}^{k-1}\right)$ for $k = 1, 2, \cdots$.

For a triangle $T = \triangle A_1A_2A_3$, we use V_T to denote its area, given by

$$V_T := \frac{1}{2}\left|\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right)\right|.$$

The main result of this paper can be stated as follows.

Theorem 1.1 *The Fourier transform of $P_n(\mathbf{x})$ as in (1.1) over a triangle $\triangle A_1A_2A_3$ has the explicit formulation:*

$$
\begin{aligned}
\widehat{P}_n(\xi) &:= \int_{\triangle A_1A_2A_3} P_n(\mathbf{x}) e^{-i\xi\cdot\mathbf{x}} d\mathbf{x} \\
&= 2V_T \sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell} (-1)^{k+\ell} \frac{n!}{(n-k-\ell)!}\frac{1}{\gamma^{\ell+1}} \qquad (1.3)\\
&\times \Bigg\{ \frac{1}{\alpha^{k+1}} \left(\triangledown_{12}^{k} \triangledown_{23}^{\ell} a_{n-\ell,\ell,0}\ e^{-i(a_1\xi_1+b_1\xi_2)} - \triangle_{12}^{k} \triangledown_{23}^{\ell} a_{0,n,0}\ e^{-i(a_2\xi_1+b_2\xi_2)} \right) \\
&\quad - \frac{1}{\beta^{k+1}} \left(\triangledown_{13}^{k} \triangle_{23}^{\ell} a_{n-\ell,0,\ell}\ e^{-i(a_1\xi_1+b_1\xi_2)} - \triangle_{13}^{k} \triangle_{23}^{\ell} a_{0,0,n}\ e^{-i(a_3\xi_1+b_3\xi_2)} \right) \Bigg\},
\end{aligned}
$$

where $\xi = (\xi_1, \xi_2)$, *and* $\alpha := i\big((a_2-a_1)\xi_1 + (b_2-b_1)\xi_2\big), \beta := i\big((a_3-a_1)\xi_1 + (b_3-b_1)\xi_2\big), \gamma := i\big((a_3-a_2)\xi_1 + (b_3-b_2)\xi_2\big)$.

Of course the above formulation is valid for univariate polynomials, simply by setting $w = 0$, namely,

$$
p_n(x) = \sum_{0\le j,k\le n, j+k=n} b_{j,k} \frac{n!}{j!k!} u^j v^k, \qquad (1.4)
$$

with $u = (x-b)/(a-b)$, $v = (x-a)/(b-a)$, and $x \in [a,b]$. Then, by using $\triangle$ and $\triangledown$ to denote the forward-backward and backward-forward operators

$$
\triangle b_{j,k} := b_{j+1,k-1} - b_{j,k}; \quad \triangledown b_{j,k} := b_{j,k} - b_{j-1,k+1}, \qquad (1.5)
$$

and $\triangle^k := \triangle\left(\triangle^{k-1}\right)$; $\triangledown^k := \triangledown\left(\triangledown^{k-1}\right)$, for $k = 1, 2, \cdots$, we have, as an immediate consequence of Theorem 1.1, the following formulation of the Fourier transform of univariate polynomials.

Corollary 1.2 *The Fourier transform of $p_n(x)$ in (1.4) over a bounded interval $[a,b]$ has the explicit formulation:*

$$
\begin{aligned}
\hat{p}_n(\xi) &:= \int_a^b p_n(x) e^{-ix\xi} dx \\
&= (b-a)e^{-ib\xi} \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \frac{1}{(i(b-a)\xi)^{k+1}} \left(\triangledown^k b_{n,0} - \triangle^k b_{0,n} \right). \qquad (1.6)
\end{aligned}
$$

We will present the proof of Theorem 1.1 in the next section. In Section 3, we will compute the Fourier transforms of certain minimum-supported bivariate splines to illustrate the general theory and introduce the Fourier transform approach to computing subdivision (or refinement) masks.

2 Proof of Theorem

Certain properties of the hypergeometric functions, along with the notion of forward-backward and backward-forward operators defined as in (1.5), will be used to prove Theorem 1.1. More precisely, the integral of the Fourier transform $\widehat{P}_n$ of a bivariate polynomial P_n restricted to the triangle is related to the hypergeometric function ${}_1F_1$, called a confluent hypergeometric function (see, for example, [10]), defined by

$$ {}_1F_1(\alpha_1, \beta_1; z) := 1 + \frac{\alpha_1}{\beta_1}\frac{z}{1!} + \frac{\alpha_1(\alpha_1+1)}{\beta_1(\beta_1+1)}\frac{z^2}{2!} + \frac{\alpha_1(\alpha_1+1)(\alpha_1+2)}{\beta_1(\beta_1+1)(\beta_1+2)}\frac{z^3}{3!} + \cdots, \quad z \in \mathbb{C}, $$

where $\alpha_1, \beta_1 \in \mathbb{C}$ with $\beta_1 \notin \{0, -1, -2, \cdots\}$, so that

$$ {}_1F_1(0, \beta_1; z) = 1, \quad {}_1F_1(\beta_1, \beta_1; z) = e^z. $$

We need the following two properties of ${}_1F_1(\alpha_1, \beta_1; z)$ that are valid for all non-negative integers α_1 and β_1:

(i) For all integers $k \geq 0, m > 0$,

$$ \frac{z}{m}\,{}_1F_1(k+1, m+1; z) = {}_1F_1(k+1, m; z) - {}_1F_1(k, m; z), \quad z \in \mathbb{C}; \tag{2.1} $$

(ii) For all integers $k \geq 0, m \geq 0$, $0 \leq u \leq 1$, and $\rho \in \mathbb{C}$,

$$ \int_0^{1-u} \nu^k (1-u-\nu)^m e^{\rho\nu} d\nu = \frac{k!m!}{(k+m+1)!}\,{}_1F_1(k+1, k+m+2; (1-u)\rho). \tag{2.2} $$

The interested reader is referred to [10, p. 1013] and [10, p. 343] for more details.

To prove Theorem 1.1, we need the following result.

Lemma 2.1 *For any integer $s \geq 0$, real numbers $b_{m,j}$, and $\rho, z \in \mathbb{C}$,*

$$ \begin{aligned} &\sum_{m=0}^{s} b_{m,s-m} \frac{1}{(s+1)!} z^{s+1}\,{}_1F_1(m+1, s+2; \rho z) \\ &= \sum_{\ell=0}^{s} \frac{(-1)^\ell}{\rho^{\ell+1}} \nabla^\ell b_{s,0} \frac{z^{s-\ell}}{(s-\ell)!} e^{\rho z} - \sum_{\ell=0}^{s} \frac{(-1)^\ell}{\rho^{\ell+1}} \triangle^\ell b_{0,s} \frac{z^{s-\ell}}{(s-\ell)!}. \end{aligned} \tag{2.3} $$

Proof. By applying (2.1), we see that

$$\begin{aligned}
&\text{Left-hand side of (2.3)} = \sum_{m=0}^{s} b_{m,s-m}\frac{z^s}{\rho\cdot s!}\big({}_1F_1(m+1,s+1;\rho z) - {}_1F_1(m,s+1;\rho z)\big)\\
&= \sum_{m=0}^{s} b_{m,s-m}\frac{z^s}{\rho\cdot s!}{}_1F_1(m+1,s+1;\rho z) - \sum_{m=0}^{s} b_{m,s-m}\frac{z^s}{\rho\cdot s!}{}_1F_1(m,s+1;\rho z)\\
&= b_{s,0}\frac{z^s}{\rho\cdot s!}{}_1F_1(s+1,s+1;\rho z) + \frac{1}{\rho}\sum_{m=0}^{s-1} b_{m,s-m}\frac{z^s}{s!}{}_1F_1(m+1,s+1;\rho z)\\
&\quad -b_{0,s}\frac{z^s}{\rho\cdot s!}{}_1F_1(0,s+1;\rho z) - \sum_{m=1}^{s} b_{m,s-m}\frac{z^s}{\rho\cdot s!}{}_1F_1(m,s+1;\rho z)\\
&= b_{s,0}\frac{z^s}{\rho\cdot s!}e^{\rho z} + \frac{1}{\rho}\sum_{m=0}^{s-1} b_{m,s-m}\frac{z^s}{s!}{}_1F_1(m+1,s+1;\rho z)\\
&\quad -b_{0,s}\frac{z^s}{\rho\cdot s!} - \frac{1}{\rho}\sum_{m=0}^{s-1} b_{m+1,s-m-1}\frac{z^s}{s!}{}_1F_1(m+1,s+1;\rho z)\\
&= \frac{1}{\rho}b_{s,0}\frac{z^s}{s!}e^{\rho z} - \frac{1}{\rho}b_{0,s}\frac{z^s}{s!} + \frac{(-1)}{\rho}\sum_{m=0}^{s-1}\triangle b_{m,s-m}\frac{z^s}{s!}{}_1F_1(m+1,s+1;\rho z).
\end{aligned}$$

Now, repeating this process, we may conclude that the left-hand side of (2.3) is given by

$$\begin{aligned}
&\left(\frac{1}{\rho}b_{s,0}\frac{z^s}{s!} + \frac{(-1)}{\rho^2}\triangle b_{s-1,1}\frac{z^{s-1}}{(s-1)!} + \cdots + \frac{(-1)^\ell}{\rho^{\ell+1}}\triangle^\ell b_{s-\ell,\ell}\frac{z^{s-\ell}}{(s-\ell)!} + \cdots + \frac{(-1)^s}{\rho^{s+1}}\triangle^s b_{0,s}\right)e^{\rho z}\\
&\quad -\left(\frac{1}{\rho}b_{0,s}\frac{z^s}{s!} + \frac{(-1)}{\rho^2}\triangle b_{0,s}\frac{z^{s-1}}{(s-1)!} + \cdots + \frac{(-1)^\ell}{\rho^{\ell+1}}\triangle^\ell b_{0,s}\frac{z^{s-\ell}}{(s-\ell)!} + \cdots + \frac{(-1)^s}{\rho^{s+1}}\triangle^s b_{0,s}\right)\\
&= \sum_{\ell=0}^{s}\frac{(-1)^\ell}{\rho^{\ell+1}}\nabla^\ell b_{s,0}\frac{z^{s-\ell}}{(s-\ell)!}e^{\rho z} - \sum_{\ell=0}^{s}\frac{(-1)^\ell}{\rho^{\ell+1}}\triangle^\ell b_{0,s}\frac{z^{s-\ell}}{(s-\ell)!},
\end{aligned}$$

where the last equality follows from the identity $\triangle^\ell b_{s-\ell,\ell} = \nabla^\ell b_{s,0}$. Hence, we obtain (2.3). ∎

We are now ready to prove the main result of the paper.

Proof of Theorem 1.1. By simple calculations, we have

$$
\begin{aligned}
\widehat{P}_n(\xi) &= \int_{\Delta A_1 A_2 A_3} e^{-i\xi\cdot\mathbf{x}} P_n(\mathbf{x}) d\mathbf{x} = 2V_T \sum_{j=0}^{n}\sum_{k=0}^{n-j} a_{j,k,n-j-k} \frac{n!}{j!k!(n-j-k)!} \\
&\times \int_0^1 \int_0^{1-u} e^{-i\xi_1\left(a_3+(a_1-a_3)u+(a_2-a_3)v\right)} e^{-i\xi_2\left(b_3+(b_1-b_3)u+(b_2-b_3)v\right)} u^j v^k (1-u-v)^{n-j-k} dv du \\
&= 2V_T e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{j=0}^{n}\sum_{k=0}^{n-j} a_{j,k,n-j-k} \frac{n!}{j!k!(n-j-k)!} \int_0^1 e^{\beta u} u^j \int_0^{1-u} e^{\gamma v} v^k (1-u-v)^{n-j-k} dv du \\
&= 2V_T e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{j=0}^{n}\sum_{k=0}^{n-j} a_{j,k,n-j-k} \frac{n!}{j!(n-j+1)!} \\
&\times \int_0^1 e^{\beta u} u^j (1-u)^{n-j+1} {}_1F_1(k+1, n-j+2; (1-u)\gamma) du \\
&= 2V_T n! e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{j=0}^{n} \int_0^1 e^{\beta u} \frac{u^j}{j!} \\
&\times \left\{ \sum_{k=0}^{n-j} a_{j,k,n-j-k} \frac{1}{(n-j+1)!} (1-u)^{n-j+1} {}_1F_1(k+1, n-j+2; (1-u)\gamma) \right\} du,
\end{aligned}
$$

where the third equality follows from (2.2). Applying Lemma 2.1 to the summation inside the curly brackets of the rightmost equality with $z = 1-u, \rho = \gamma$, and $s = n-j, b_{m,\ell} = a_{j,m,\ell}$ (so that $\triangle b_{m,\ell} = \triangle_{23} a_{j,m,\ell}, \triangledown b_{m,\ell} = \triangledown_{23} a_{j,m,\ell}$), we see that

$$
\begin{aligned}
\widehat{P}_n(\xi) &= 2V_T n! e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{j=0}^{n} \int_0^1 e^{\beta u} \frac{u^j}{j!} \\
&\times \left\{ \sum_{\ell=0}^{n-j} \frac{(-1)^\ell}{\gamma^{\ell+1}} \triangledown_{23}^\ell a_{j,n-j,0} \frac{(1-u)^{n-j-\ell}}{(n-j-\ell)!} e^{(1-u)\gamma} - \sum_{\ell=0}^{n-j} \frac{(-1)^\ell}{\gamma^{\ell+1}} \triangle_{23}^\ell a_{j,0,n-j} \frac{(1-u)^{n-j-\ell}}{(n-j-\ell)!} \right\} du
\end{aligned}
$$

$$
\begin{aligned}
&= 2V_T n! e^{-i(a_2\xi_1+b_2\xi_2)} \sum_{j=0}^{n}\sum_{\ell=0}^{n-j} \frac{(-1)^\ell}{\gamma^{\ell+1}} \bigtriangledown_{23}^{\ell} a_{j,n-j,0} \frac{1}{j!(n-j-\ell)!}\int_0^1 u^j(1-u)^{n-j-\ell}e^{\alpha u}du \\
&\quad -2V_T n! e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{j=0}^{n}\sum_{\ell=0}^{n-j} \frac{(-1)^\ell}{\gamma^{\ell+1}} \triangle_{23}^{\ell} a_{j,0,n-j} \frac{1}{j!(n-j-\ell)!}\int_0^1 u^j(1-u)^{n-j-\ell}e^{\beta u}du \\
&= 2V_T n! e^{-i(a_2\xi_1+b_2\xi_2)} \sum_{j=0}^{n}\sum_{\ell=0}^{n-j} \frac{(-1)^\ell}{\gamma^{\ell+1}} \bigtriangledown_{23}^{\ell} a_{j,n-j,0} \frac{1}{(n-\ell+1)!}\, {}_1F_1(j+1, n-\ell+2;\alpha) \\
&\quad -2V_T n! e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{j=0}^{n}\sum_{\ell=0}^{n-j} \frac{(-1)^\ell}{\gamma^{\ell+1}} \triangle_{23}^{\ell} a_{j,0,n-j} \frac{1}{(n-\ell+1)!}\, {}_1F_1(j+1, n-\ell+2;\beta) \\
&= 2V_T n! e^{-i(a_2\xi_1+b_2\xi_2)} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\gamma^{\ell+1}} \sum_{j=0}^{n-\ell} \bigtriangledown_{23}^{\ell} a_{j,n-j,0} \frac{1}{(n-\ell+1)!}\, {}_1F_1(j+1, n-\ell+2;\alpha) \\
&\quad -2V_T n! e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\gamma^{\ell+1}} \sum_{j=0}^{n-\ell} \triangle_{23}^{\ell} a_{j,0,n-j} \frac{1}{(n-\ell+1)!}\, {}_1F_1(j+1, n-\ell+2;\beta).
\end{aligned}
$$

Finally, applying Lemma 2.1 to the first and second terms in the above equation with $s = n-\ell, b_{m,k} = \bigtriangledown_{23}^{\ell} a_{m,k+\ell,0}$, $z = 1, \rho = \alpha$ and $s = n-\ell, b_{m,k} = \bigtriangledown_{23}^{\ell} a_{m,0,k+\ell}$, $z=1, \rho = \beta$, respectively, we have

$$
\begin{aligned}
\widehat{P}_n(\xi) &= 2V_T n! e^{-i(a_2\xi_1+b_2\xi_2)} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\gamma^{\ell+1}} \\
&\quad \times \left\{ \sum_{k=0}^{n-\ell} \frac{(-1)^k}{\alpha^{k+1}} \bigtriangledown_{12}^{k} \bigtriangledown_{23}^{\ell} a_{n-\ell,\ell,0} \frac{e^\alpha}{(n-\ell-k)!} - \sum_{k=0}^{n-\ell} \frac{(-1)^k}{\alpha^{k+1}} \triangle_{12}^{k} \bigtriangledown_{23}^{\ell} a_{0,n,0} \frac{1}{(n-\ell-k)!} \right\} \\
&\quad -2V_T n! e^{-i(a_3\xi_1+b_3\xi_2)} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\gamma^{\ell+1}} \\
&\quad \times \left\{ \sum_{k=0}^{n-\ell} \frac{(-1)^k}{\beta^{k+1}} \bigtriangledown_{13}^{k} \triangle_{23}^{\ell} a_{n-\ell,0,\ell} \frac{e^\beta}{(n-\ell-k)!} - \sum_{k=0}^{n-\ell} \frac{(-1)^k}{\beta^{k+1}} \triangle_{13}^{k} \triangle_{23}^{\ell} a_{0,0,n} \frac{1}{(n-\ell-k)!} \right\} \\
&= 2V_T \sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell} (-1)^{k+\ell} \frac{n!}{(n-k-\ell)!} \frac{1}{\gamma^{\ell+1}} \\
&\quad \times \left\{ \frac{1}{\alpha^{k+1}} \left(\bigtriangledown_{12}^{k} \bigtriangledown_{23}^{\ell} a_{n-\ell,\ell,0}\, e^{-i(a_1\xi_1+b_1\xi_2)} - \triangle_{12}^{k} \bigtriangledown_{23}^{\ell} a_{0,n,0}\, e^{-i(a_2\xi_1+b_2\xi_2)} \right) \right. \\
&\quad \left. - \frac{1}{\beta^{k+1}} \left(\bigtriangledown_{13}^{k} \triangle_{23}^{\ell} a_{n-\ell,0,\ell}\, e^{-i(a_1\xi_1+b_1\xi_2)} - \triangle_{13}^{k} \triangle_{23}^{\ell} a_{0,0,n}\, e^{-i(a_3\xi_1+b_3\xi_2)} \right) \right\},
\end{aligned}
$$

as desired. ∎

Remark 1. By applying Lemma 2.1 and following the above procedure, the main result in this paper can be extended to higher dimensions. That is, explicit formulae of the Fourier transform of Bernstein-Bézier representations of multivariate polynomials on simplexes can be derived in a similar way.

3 Application to refinable bivariate splines on triangles

Refinable spline functions (see, for example, [1] for a precise definition) are instrumental to surface subdivisions. For example, the bi-cubic B-spline is used in the Catmull-Clark scheme [2] and the three-direction box-spline B_{222} is used in the Loop scheme [11]. (See, for example, [3, pp. 15-18] for the definition of box splines and their direction sets.) The simple reasons are that firstly, the refinement masks of such spline functions immediately give the so-called "local averaging rules" for the subdivision schemes; and secondly, the spline representations are precisely the subdivision surfaces. In the Fourier domain, the refinement mask P of a refinable function, or function vector Φ, is defined by $\hat{\Phi}(\cdot) = P(\cdot/2)\hat{\Phi}(\cdot/2)$ provided that the Fourier transform $\hat{\Phi}$ exists. While the refinement masks of the bi-cubic B-spline and the box-spline B_{222}, being defined by convolutions of the characteristic function of the unit square along the appropriate directions, are readily computable, computation of the refinement masks for others, such as basis functions with minimum and quasi-minimum supports, is usually very tedious. Examples of the recent development in this direction are the refinable bivariate C^2-cubic, C^2-quartic, and C^2-quintic spline functions in [7, 8, 9], introduced for matrix-valued surface subdivisions to gain such desirable properties as surface geometric shape control parameters, smaller subdivision template size (to better address the often unavoidable extraordinary vertices), and interpolation of the position components of the (initial) control vertices. Computation of the refinement masks of these bivariate spline functions requires formulating and solving large linear systems in terms of Bernstein-Bézier coefficients, see [8, 9]. For this reason, the original motivation for this research was to extend the standard Fourier approach to computing the (scalar-valued) refinement masks of refinable spline functions to computing the matrix-valued refinement masks of refinable spline function vectors.

As an application, let us consider the Fourier transform of the minimum-supported (ms) and quasi-minimum-supported (qms) bivariate spline functions in $S^n_{m(n)}(\Delta^{(i)})$, $i = 1$ and 2, where $\Delta^{(1)}$ and $\Delta^{(2)}$ denote the 3-direction mesh and 4-direction mesh, respectively, in $\mathbb{R}^2$ with integer grid points, obtained by partitioning $\mathbb{R}^2$ with the two sets of grid lines $x = j$, $y = j$, $x - y = j$ and $x = j$, $y = j$, $x - y = j$, $x + y = j$, respectively, where $j = \ldots, -1, 0, 1 \ldots$. Here, the bivariate spline spaces $S^n_{m(n)}(\Delta^{(i)})$, $i = 1$ and 2, are the spaces of functions in C^n

whose restrictions on the triangular cells of $\Delta^{(i)}$ are polynomials of (total) degree $m(n)$, where $m(n)$ denotes the smallest nonnegative integer for which $S^n_{m(n)}(\Delta^{(i)})$ contains at least one nontrivial locally supported function. It is well-known that for $\Delta^{(1)}$, we have $m(2n-1) = 3n$ and $m(2n) = 3n+1$; and for $\Delta^{(2)}$, we have $m(3n-1) = 4n$, $m(3n) = 4n+1$, and $m(3n+1) = 4n+2$ (see [3, 4] for details).

In the following, B_{jkl} and B_{jklm} denote the box splines in the spaces $S^n_{m(n)}(\Delta^{(1)})$ and $S^n_{m(n)}(\Delta^{(2)})$ for $m(n) = j+k+l-2$ or $m(n) = j+k+l+m-2$, respectively. Here, j, k, l and m denote the numbers of repetition of the vectors $(1,0)$, $(0,1)$, $(1,1)$, and $(1,-1)$, respectively, in the direction sets that determine $\Delta^{(i)}$. (See, for example, [3, pp. 17-18] for details). Bivariate splines with local supports that are ms and qms, as studied in [3, 4, 5], are defined as follows. The support of a locally supported function f in a spline space is the closure of the set on which f does not vanish and is denoted by *supp*(f). A set S is called a minimal support of a spline space if there is some f, called an ms spline, in the space with *supp*$(f) = S$, but there does not exist a nontrivial g in the space with *supp*(g) properly contained in S. A function f in a spline space is called a qms spline if (i) f cannot be written as a (finite) linear combination of ms splines in the space, and (ii) for any h in the space properly contained in *supp*(f), h is some (finite) linear combination of ms spines in the space.

For example, for the 3-direction mesh $\Delta^{(1)}$, while the spline space $S^0_1(\Delta^{(1)})$ has only one ms spline g^0_1, which is the box spline B_{111} with direction set $\{(1,0)$, $(0,1)$, $(1,1)\}$, the space $S^{-1}_0(\Delta^{(1)})$ has two ms splines g^0_2 and g^0_3 which are not box-splines, but are characteristic functions of the sets χ_A and χ_B, respectively, where A is the triangle bounded by the 3 lines: $x = 0$, $y = 1$, and $y = x$, and B the triangle bounded by the 3 lines: $x = 1$, $y = 0$ and $y = x$.

On the other hand, for the 4-direction mesh $\Delta^{(2)}$, the space $S^1_2(\Delta^{(2)})$ has only one ms spline $f^0_1 = B_{1111}$, the box spline with direction set $\{(1,0), (0,1), (1,1), (1,-1)\}$; but there are two ms splines f^0_2 and f^0_3 in the space $S^0_1(\Delta^{(2)})$, where f^0_2 is the Courant hat function with support given by the diamond having vertices at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, and with the value of 1 at the center $(0,0)$ of the support; and f^0_3 is the other Courant hat function supported on the unit square $[0,1]^2$ with the value 1 at its center $(1/2, 1/2)$. Furthermore, there are two ms splines f^0_4 and f^0_5 and one qms spline f^0_6 in $S^2_4(\Delta^{(2)})$ (see [3, 4, 5]).

In the following, let us consider the k-fold 2-dimensional convolutions: $g^k_1 = \underbrace{g^0_1 * g^0_1 * \cdots * g^0_1}_{k+1}$ and $f^k_1 = \underbrace{f^0_1 * f^0_1 * \cdots * f^0_1}_{k+1}$ of g^0_1 and f^0_1, respectively. Now, the spline function vectors of interest are:

$$G^k \equiv \begin{bmatrix} g^k_2 \\ g^k_3 \end{bmatrix} := g^{k-1}_1 * \begin{bmatrix} g^0_2 \\ g^0_3 \end{bmatrix}, \quad F^k \equiv \begin{bmatrix} f^k_2 \\ f^k_3 \end{bmatrix} := f^{k-1}_1 * \begin{bmatrix} f^0_2 \\ f^0_3 \end{bmatrix}, \qquad (3.1)$$

where $k \geq 1$ and $f * \begin{bmatrix} g \\ h \end{bmatrix} := \begin{bmatrix} f * g \\ f * h \end{bmatrix}$.

We remark that among the functions g_i^k, f_i^k, $i = 1, 2, 3$, only g_1^k and f_1^k are box splines, while all of them are the unique ms and qms splines in the corresponding spline spaces (where the notion of uniqueness is according to the statement of Theorem 3.2 in [3]).

The Fourier transforms of the "initial" ms splines g_i^0, f_i^0, $i = 2, 3$, can be evaluated by using the formula provided in this paper, and the Fourier transforms of the other splines are given by the corresponding products with those of g_1^k or f_1^k.

Finally, the refinement masks of G^k, F^k can be easily computed by making use of (3.1) from the refinements of the "initial" G^0 and F^0.

Example 1. The Fourier transform of G^k is given by $\widehat{G^k}(\xi_1, \xi_2) = [\widehat{g_2^k}, \widehat{g_3^k}]^T(\xi_1, \xi_2)$, where $\widehat{g_2^k} = \widehat{g_2^0}(\widehat{g_1^0})^k$ and $\widehat{g_3^k} = \widehat{g_3^0}(\widehat{g_1^0})^k$. Here, we have

$$\widehat{g_1^0}(\xi_1, \xi_2) = \widehat{B_{111}}(\xi_1, \xi_2) = \frac{1 - e^{-i\xi_1}}{i\xi_1} \frac{1 - e^{-i\xi_2}}{i\xi_2} \frac{1 - e^{i(\xi_1+\xi_2)}}{i(\xi_1 + \xi_2)},$$

$$\widehat{g_2^0}(\xi_1, \xi_2) = \frac{1 - e^{-i(\xi_1+\xi_2)}}{\xi_1(\xi_1 + \xi_2)} - \frac{1 - e^{-i\xi_2}}{\xi_1\xi_2}, \tag{3.2}$$

and

$$\widehat{g_3^0}(\xi_1, \xi_2) = \frac{1 - e^{-i(\xi_1+\xi_2)}}{\xi_2(\xi_1 + \xi_2)} - \frac{1 - e^{-i\xi_1}}{\xi_1\xi_2}. \tag{3.3}$$

■

Next, let us compute the Fourier transform of F^k. For this purpose, recall that

$$\left(\phi(A \cdot -\mathbf{k})\right)^\wedge(\xi) = |\det(A)|^{-1} e^{-i\xi^T \cdot (A^{-1}\mathbf{k})} \widehat{\phi}\left((A^{-1})^T \xi\right), \tag{3.4}$$

for any invertible matrix A of dimension s.

Example 2. Let f_2^0, f_3^0 be the two Courant hat functions in $S_1^0(\triangle^{(2)})$ as introduced previously. To compute the Fourier transform of f_2^0, we use the x-y axes to partition its support into four triangles: Δ_1, Δ_2, Δ_3, and Δ_4 in the the first, second, third, and fourth quadrants, respectively. Then f_2^0 can be written as the sum of four functions: ϕ_1, ϕ_2, ϕ_3, and ϕ_4, with supports given by Δ_1, Δ_2, Δ_3, and Δ_4, respectively.

By the formula (1.3), the Fourier transform of ϕ_1 is given by

$$\widehat{\phi_1}(\xi_1, \xi_2) = -\frac{1}{\xi_1\xi_2} + i\frac{1 - e^{-i\xi_1}}{\xi_1^2(\xi_1 - \xi_2)} + i\frac{1 - e^{-i\xi_2}}{\xi_2^2(\xi_2 - \xi_1)}. \tag{3.5}$$

Since $\phi_2(x,y) = \phi_1(-x,y)$, it follows from (3.5) and (3.4) that

$$\widehat{\phi_2}(\xi_1,\xi_2) = \widehat{\phi_1}(-\xi_1,\xi_2) = \frac{1}{\xi_1\xi_2} - i\frac{1-e^{i\xi_1}}{\xi_1^2(\xi_1+\xi_2)} + i\frac{1-e^{-i\xi_2}}{\xi_2^2(\xi_1+\xi_2)}.$$

Similarly, since $\phi_3(x,y) = \phi_1(-x,-y)$ and $\phi_4(x,y) = \phi_1(x,-y)$, we have

$$\begin{aligned}\widehat{\phi_3}(\xi_1,\xi_2) &= -\frac{1}{\xi_1\xi_2} + i\frac{1-e^{i\xi_1}}{\xi_1^2(\xi_2-\xi_1)} + i\frac{1-e^{i\xi_2}}{\xi_2^2(\xi_1-\xi_2)};\\ \widehat{\phi_4}(\xi_1,\xi_2) &= \frac{1}{\xi_1\xi_2} + i\frac{1-e^{-i\xi_1}}{\xi_1^2(\xi_1+\xi_2)} - i\frac{1-e^{i\xi_2}}{\xi_2^2(\xi_1+\xi_2)}.\end{aligned}$$

Consequently, we arrive at

$$\widehat{f_2^0}(\xi_1,\xi_2) = \sum_{j=1}^{4}\widehat{\phi}_j(\xi_1,\xi_2) = 2i\frac{e^{i\xi_2}-e^{-i\xi_2}}{\xi_2(\xi_1+\xi_2)(\xi_2-\xi_1)} + 2i\frac{e^{i\xi_1}-e^{-i\xi_1}}{\xi_1(\xi_1+\xi_2)(\xi_1-\xi_2)}. \tag{3.6}$$

Next, observe that the linear transformation $B\begin{bmatrix} x \\ y\end{bmatrix} - \begin{bmatrix} 1 \\ 0\end{bmatrix}$, with

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix},$$

maps the *supp*(f_3^0) to *supp*(f_2^0), with the vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ of *supp*(f_3^0) corresponding to the vertices $(-1,0)$, $(0,1)$, $(1,0)$, and $(0,-1)$ of *supp*(f_2^0), respectively. Hence, we may write

$$f_3^0(x,y) = f_2^0(B\begin{bmatrix} x \\ y\end{bmatrix} - \begin{bmatrix} 1 \\ 0\end{bmatrix}),$$

and apply (3.4) to obtain

$$\begin{aligned}\widehat{f_3^0}(\xi_1,\xi_2) &= \frac{1}{2}e^{-i\xi\cdot\frac{1}{2}B[1,0]^T}\widehat{f_2^0}(\frac{1}{2}B\xi) = \frac{1}{2}e^{-i\frac{\xi_1+\xi_2}{2}}\widehat{f_2^0}(\frac{\xi_1+\xi_2}{2},\frac{\xi_1-\xi_2}{2})\\ &= \frac{2i}{\xi_1\xi_2}\left(\frac{e^{-i\xi_2}-e^{-i\xi_1}}{\xi_2-\xi_1} + \frac{1-e^{-i(\xi_1+\xi_2)}}{\xi_1+\xi_2}\right).\end{aligned} \tag{3.7}$$

■

Finally, to compute the refinement masks of G^k and F^k in (3.1), we observe that the convolution of two finitely refinable functions (i.e., refinable functions whose refinement masks consist of finitely many terms) remains to be finitely refinable, and that G^k is the convolution of G^0 with the refinable box spline g_1^k, and F^k the convolution of F^0 with the refinable box spline f_1^k. Hence, we only need to compute the refinement masks of the initial ms splines. Precisely, from $\widehat{g}_2^0$ and $\widehat{g}_3^0$ given by (3.2) and (3.3), we have

$$\widehat{G}^0(2\xi_1, 2\xi_2) = Q_0(\xi_1, \xi_2)\widehat{G}^0(\xi_1, \xi_2)$$

where

$$Q_0(\xi_1, \xi_2) = \begin{bmatrix} 1 + z_2 + z_1 z_2 & z_2 \\ z_1 & 1 + z_1 + z_1 z_2 \end{bmatrix}, \quad z_1 = e^{-i\xi_1},\ z_2 = e^{-i\xi_2};$$

and from $\widehat{f}_2^0$ and $\widehat{f}_3^0$ given by (3.6) and (3.7), we have

$$\widehat{F}^0(2\xi_1, 2\xi_2) = R_0(\xi_1, \xi_2)\widehat{F}^0(\xi_1, \xi_2)$$

where

$$R_0(\xi_1, \xi_2) = \frac{1}{4}\begin{bmatrix} 1 + \frac{1}{2}(z_1 + \frac{1}{z_2})(z_1 + z_2) & \frac{1}{2}(1 + \frac{1}{z_1})(1 + \frac{1}{z_2}) \\ z_1 z_2 & \frac{1}{2}(1 + z_1)(1 + z_2) \end{bmatrix}, \ z_1 = e^{-i\xi_1}, z_2 = e^{-i\xi_2}.$$

The interested reader is referred to [6] for computing the refinement mask for $F^0(\mathbf{x})$ by calculating the BB coefficients of $F^0(B^{-1}\mathbf{x})$ directly.

Therefore, we conclude, from the definitions in (3.1), that

$$\widehat{G}^k(2\xi_1, 2\xi_2) = Q_k(\xi_1, \xi_2)\widehat{G}^k(\xi_1, \xi_2), \quad \widehat{F}^k(2\xi_1, 2\xi_2) = R_k(\xi_1, \xi_2)\widehat{F}^k(\xi_1, \xi_2)$$

with

$$Q_k(\xi_1, \xi_2) = \big(q(\xi_1, \xi_2)\big)^k Q_0(\xi_1, \xi_2), \quad R_k(\xi_1, \xi_2) = \big(r(\xi_1, \xi_2)\big)^k R_0(\xi_1, \xi_2) \quad (3.8)$$

where $q(\xi_1, \xi_2) = \frac{1}{8}(1 + e^{-i\xi_1})(1 + e^{-i\xi_2})(1 + e^{-i(\xi_1+\xi_2)})$ is the mask (or two-scale symbol) of g_1^0, and $r(\xi_1, \xi_2) = \frac{1}{16}(1+e^{-i\xi_1})(1+e^{-i\xi_2})(1+e^{-i(\xi_1+\xi_2)})(1+e^{-i(\xi_1-\xi_2)})$ the mask (or two-scale symbol) of f_1^0.

That is, we have the following result.

Theorem 3.1 *The vector-valued functions G^k and F^k are finitely refinable with refinement masks given by (3.8).*

Similarly,

$$f_1^{k-1} * \begin{bmatrix} f_4^0 \\ f_5^0 \\ f_6^0 \end{bmatrix}$$

is also finitely refinable, with refinement mask given by

$$\big(r(\xi_1, \xi_2)\big)^k S_0(\xi_1, \xi_2),$$

where S_0 is the refinement mask of $\begin{bmatrix} f_4^0 \\ f_5^0 \\ f_6^0 \end{bmatrix}$. Computation of S_0 as well as the refinement masks of other initial ms and qms bivariate splines in general is usually nontrivial and requires further investigation.

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