A Unified Approach to Generalized Stirling Functions

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Abstract

Here presented is a unified approach to generalized Stirling functions by using generalized factorial functions, k-Gamma functions, generalized divided difference, and the unified expression of Stirling numbers defined in [17]. Previous well-known Stirling functions introduced by Butzer and Hauss [4], Butzer, Kilbas, and Trujilloet [6] and others are included as particular cases of our generalization. Some basic properties related to our general pattern such as their recursive relations, generating functions, and asymptotic properties are discussed, which extend the corresponding results about the Stirling numbers shown in [19] to the defined Stirling functions.

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1 Introduction

The classical Stirling numbers of the first kind and the second kind, denoted by $s(n,k)$ and $S(n,k)$, respectively, can be defined via a pair of inverse relations

$$[z]_n = \sum_{k=0}^{n} s(n,k) z^k, \quad z^n = \sum_{k=0}^{n} S(n,k)[z]_k, \tag{1.1}$$

with the convention $s(n,0) = S(n,0) = \delta_{n,0}$, the Kronecker symbol, where $z \in \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the falling factorial polynomials $[z]_n = z(z-1)\cdots(z-n+1)$. $|s(n,k)|$ presents the number of permutations of n elements with k disjoint cycles while $S(n,k)$ gives the number of ways to partition n elements into k nonempty subsets. The simplest way to compute $s(n,k)$ is finding the coefficients of the expansion of $[z]_n$.

Another way of introducing classical Stirling numbers is via their exponential generating functions

$$\frac{(\log(1+x))^k}{k!}=\sum_{n\geq k}s(n,k)\frac{x^n}{n!},\quad \frac{(e^x-1)^k}{k!}=\sum_{n\geq k}S(n,k)\frac{x^n}{n!},\tag{1.2}$$

where $|x|<1$ and $k\in\mathbb{N}_0$. In [21], Jordan said that, "Stirling's numbers are of the greatest utility. This however has not been fully recognized." He also thinks that, "Stirling's numbers are as important or even more so than Bernoulli's numbers."

Besides the above two expressions, the Stirling numbers of the second kind has the following third definition (see [11] and [21]), which is equivalent to the above two definitions but makes a more important rule in computation and generalization.

$$\begin{aligned} S(n,k) &:= \frac{1}{k!}\,\Delta^k z^n\big|_{z=0}=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n \\ &= \frac{1}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(k-j)^n. \end{aligned}\tag{1.3}$$

Expressions (1.1) - (1.3) are starting points in [17] to extend the classical Stirling number pair and the Stirling numbers to the defined generalized Stirling numbers.

Denote $\langle z\rangle_{n,\alpha}:=z(z+\alpha)\cdots(z+(n-1)\alpha)$ for $n=1,2,\ldots$, and $\langle z\rangle_{0,\alpha}=1$, where $\langle z\rangle_{n,\alpha}$ is called the generalized factorial of z with increment α. Thus, $\langle z\rangle_{n,-1}=[z]_n$ is the classical falling factorial with $[z]_0=1$, and $\langle z\rangle_{n,0}=z^n$. More properties of $\langle z\rangle_{n,\alpha}$ are shown in [17]. For the sake of convenience, we give a brief survey in the following .

With a closed observation, Stirling numbers of two kinds defined in (1.1) can be written as a unified Newton form:

$$\langle z\rangle_{n,-\alpha}=\sum_{k=0}^{n}S(n,k,\alpha,\beta)\langle z\rangle_{n,-\beta},\tag{1.4}$$

with $S(n,k,1,0)=s(n,k)$, the Stirling numbers of the first kind and $S(n,k,0,1)=S(n,k)$. the Stirling numbers of the second kind. Inspired by (1.4) and many extensions of classical Stirling numbers or Stirling number pairs introduced by [3], [7], [8], [9], [10], [16], [**?**], [20], [25], [26], [28], [31], [30], in particular, [19]. [17] define a unified *generalized Stirling numbers* $S(n,k,\alpha,\beta,r)$ as follows.

Definition 1.1 *[17] Let $n\in\mathbb{N}$ and $\alpha,\beta,r\in\mathbb{R}$. A generalized Stirling number denoted by $S(n,k,\alpha,\beta,r)$ is defined by*

$$\langle z\rangle_{n,-\alpha}=\sum_{k=0}^{n}S(n,k,\alpha,\beta,r)\langle z-r\rangle_{k,-\beta}.\tag{1.5}$$

In particular, if $(\alpha,\beta,r)=(1,0,0)$, $S(n,k,1,0,0)$ is reduced to the unified form of Classical Stirling numbers defined by (1.4).

From [2], each $\langle z\rangle_{n,-\alpha}$ does have exactly one such expansion (1.5) for any given z. Since $deg\ \langle z-r\rangle_{k,-\beta}=k$ for all k, which generates a graded basis for $\Pi\subset\mathbb{F}\to\mathbb{F}$, the linear spaces of polynomials in one real (when $\mathbb{F}=\mathbb{R}$) or complex (when $\mathbb{F}=\mathbb{C}$), in the sense that, for each n, $\{\langle z-r\rangle_{n,-\beta}\}$ is a basis for $\Pi_n\subset\Pi$, the subspace of all polynomials of degree $<n$. In other wards, the column map

$$W_z:\mathbb{F}_0^N\to\Pi:s\mapsto\sum_{k\geq 0}S(n,k,\alpha,\beta,r)\langle z\rangle_{k,-\beta},$$

from the space $\mathbb{F}_0^N$ of scalar sequences with finitely many nonzero entries to the space Π is one-to-one and onto, hence invertible. In particular, for each $n\in\mathbb{N}$, the coefficient $c(n)$ in the Newton form (1.5) for $\langle z\rangle_{n,-\alpha}$ depends linearly on $\langle z\rangle_{n,-\alpha}$, i.e., $\langle z\rangle_{n,-\alpha}\mapsto s(n)=(W_z^{-1}\langle z\rangle_{n,-\alpha})(n)$, the set of $S(n,k,\alpha,\beta,r)$, is a well-defined linear functional on Π, and vanishes on $\Pi_{<n-1}$.

Similarly to (1.1), from Definition 1.1 a Stirling-type pair $\{S^1,S^2\}=\{S^1(n,k),S^2(n,k)\}\equiv\{S(n,k;\ \alpha,\beta,r),\ S(n,k;\beta,\alpha,-r)\}$ (see also in [19]) can be defined by the inverse relations

$$\begin{aligned}\langle z\rangle_{n,-\alpha} &= \sum_{k=0}^{n}S^1(n,k)\langle z-r\rangle_{k,-\beta}\\ \langle z\rangle_{n,-\beta} &= \sum_{k=0}^{n}S^2(n,k)\langle z+r\rangle_{k,-\alpha},\end{aligned}\tag{1.6}$$

where $n\in\mathbb{N}$ and the parameter triple $(\alpha,\beta,r)\neq(0,0,0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. Hence, we may call S^1 and S^2 an (α,β,r) and a $(\beta,\alpha,-r)-$ pair. Obviously,

$$S(n,k;0,0,1)=\binom{n}{k}$$

because $z^n=\sum_{k=0}^{n}\binom{n}{k}(z-1)^k$. In addition, the classical Stirling number pair $\{s(n,k),S(n,k)\}$ is the $(1,0,0)-$ pair $\{S^1,S^2\}$, namely,

$$s(n,k)=S^1(n,k;1,0,0)\quad S(n,k)=S^2(n,k;1,0,0).$$

For brevity, we will use $S(n,k)$ to denote $S(n,k,\alpha,\beta,r)$ if there is no need to indicate α, β, and r explicitly. From (1.5), one may find

$$S(0,0)=1,\quad S(n,n)=1,\quad S(1,0)=r,\quad \textit{and}\quad S(n,0)=\langle r\rangle_{n,-\alpha}.\tag{1.7}$$

Evidently, substituting $n=k=0$ into (1.5) yields the first formula of (1.7). Comparing the coefficients of the highest power terms on the both sides of (1.5), we obtain the second formula of (1.7). Let $n=1$ in (1.5) and noting $S(1,1)=1$, we have the third formula. Finally, substituting $z=r$ in (1.5), one can establish the last formula of (1.7). The numbers $\sigma(n,k)$ discussed by Doubilet et al. in [14] and by Wagner in [32] is $k!S(n,k;0,1,0)$. More special cases of the generalized Stirling numbers and Stirling-type pairs defined by (1.5) or (1.6) are surveyed in Table 1 of [17].

The classical falling factorial polynomials $[z]_n = z(z-1)\cdots(z-n+1)$ and classical rising factorial polynomials $[z]^n = z(z+1)\cdots(z+n-1)$, $z \in \mathbb{C}$ and $n \in \mathbb{N}$, can be unified to the expression

$$\langle z\rangle_{n,\pm 1} := z(z\pm 1)\cdots(z\pm(n-1)),$$

using the *generalized factorial polynomial* expression

$$\langle z\rangle_{n,k} := z(z+k)\cdots(z+(n-1)k) = \langle z+(n-1)k\rangle_{n,-k} \quad (z\in\mathbb{C}, n\in\mathbb{N}). \tag{1.8}$$

Thus $\langle z\rangle_{n,1} = [z]^n$ and $\langle z\rangle_{n,-1} = [z]_n$. In addition, we immediately have the relationship between $[z]^n$ and $\langle z\rangle_{n,k}$ as

$$\langle z\rangle_{n,k} = k^n [z/k]^n \quad (z\in\mathbb{C}, n\in\mathbb{N}, k>0). \tag{1.9}$$

Similarly, we obtain

$$\langle z\rangle_{n,-k} = z(z-k)\cdots(z-(n-1)k) = k^n[z/k]_n \quad (z\in\mathbb{C}, n\in\mathbb{N}, k>0). \tag{1.10}$$

The history as well as some important basic results of the generalized factorials can be found in Chapter II of [21], and an application of the generalized factorials in the Lagrange interpolation is shown on Page 31 of [15].

It is known that the falling factorial polynomials and rising factorial polynomials can be presented in terms of Gamma functions: $[z]_n = \Gamma(z+1)/\Gamma(z-n+1)$ and $[z]^n = \Gamma(z+n)/\Gamma(z)$, and the gamma function $\Gamma(z)$ can be defined in terms of factorial functions by (see, for example, [23])

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^{z-1}}{[z]^n} \quad (z\in\mathbb{C} - k\mathbb{Z}_-). \tag{1.11}$$

As an analogy, the *k-gamma function* Γ_k, a one parameter deformation of the classical gamma function, is defined by (see, for example [13])

$$\Gamma_k(z) := \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{z}{k}-1}}{\langle z\rangle_{n,k}} \quad (k>0, z\in\mathbb{C} - k\mathbb{Z}_-). \tag{1.12}$$

$[z]^n$ and $\langle z\rangle_{n,k}$ $(k>0)$ are also called the *Pochhammer symbol* and *k-Pochhammer symbol,* respectively. Even the parameter k is replaced by other parameters, we still call the corresponding Pochhammer symbol the k-Pochhammer.

For $k>0$, from (1.9), (1.11) and (1.12) (see also [22]) we have

$$\Gamma_k(z) = k^{(z/k)-1}\Gamma\left(\frac{z}{k}\right). \tag{1.13}$$

Since $[z]^n = \Gamma(z+n)/\Gamma(z)$, [4] extends the classical raising and falling factorial polynomials to generalized raising and falling functions associated with real number γ by setting

$$[z]^\gamma := \frac{\Gamma(z+\gamma)}{\Gamma(z)} \quad [z]_\gamma := \frac{\Gamma(z+1)}{\Gamma(z-\gamma+1)}, \tag{1.14}$$

respectively. We now extend $\langle z\rangle_{n,k}$ defined by (1.8) to a generalized form associated with $\gamma \in \mathbb{C}$ using the relationship (1.9), namely,

$$\langle z\rangle_{\gamma,k} = k^{\gamma}[z/k]^{\gamma}, \quad \langle z\rangle_{\gamma,-k} = k^{\gamma}[z/k]_{\gamma} \quad (z\in\mathbb{C}, \gamma\in\mathbb{C}, k>0), \tag{1.15}$$

which are called the generalized raising and falling factorial functions associated with complex number γ, respectively. Using (1.13)-(1.15), we establish the following result.

Theorem 1.2 *If $k>0$ and $\langle z\rangle_{\gamma,k}$ is defined by (1.15), then*

$$\langle z\rangle_{\gamma,k} = \frac{\Gamma_k(z+\gamma k)}{\Gamma_k(z)} \qquad \langle z\rangle_{\gamma,-k} = \frac{\Gamma_k(z+k)}{\Gamma_k(z-(\gamma-1)k)}. \tag{1.16}$$

Proof. For $k>0$

$$\begin{aligned}\langle z\rangle_{\gamma,k} &= k^{\gamma}\left[\frac{z}{k}\right]^{\gamma} = k^{\gamma}\frac{\Gamma\left(\frac{z}{k}+\gamma\right)}{\Gamma\left(\frac{z}{k}\right)}\\ &= \frac{k^{\frac{z+\gamma k}{k}-1}\Gamma\left(\frac{z}{k}+\gamma\right)}{k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)},\end{aligned}$$

which implies (1.16) because of (1.13). Similarly, for $k>0$,

$$\begin{aligned}\langle z\rangle_{\gamma,-k} &= k^{\gamma}\left[\frac{z}{k}\right]_{\gamma} = k^{\gamma}\frac{\Gamma\left(\frac{z}{k}+1\right)}{\Gamma\left(\frac{z}{k}-\gamma+1\right)}\\ &= \frac{k^{\frac{z+k}{k}-1}\Gamma\left(\frac{z}{k}+1\right)}{k^{\frac{z+k}{k}-\gamma-1}\Gamma\left(\frac{z}{k}-\gamma+1\right)}\\ &= \frac{\Gamma_k(z+k)}{\Gamma_k(z-(\gamma-1)k)}.\end{aligned}$$

■

There hold the following recurrence relations of the generalized raising and falling factorial functions.

Proposition 1.3 *If $k>0$ and $\langle z\rangle_{\gamma,k}$ is defined by (1.15), then there hold*

$$\langle z\rangle_{\gamma,k} = (z+(\gamma-1)k)\langle z\rangle_{\gamma-1,k}, \quad \langle z\rangle_{\gamma,-k} = (z-(\gamma-1)k)\langle z\rangle_{\gamma-1,-k}. \tag{1.17}$$

Proof. It is easy to show that $\Gamma_k(z+k) = z\Gamma_k(z)$ from (1.13), which is an extension of the classical formula $\Gamma(z+1) = z\Gamma(z)$. More precisely,

$$\begin{aligned}\Gamma_k(z+k) &= k^{((z+k)/k)-1}\Gamma\left(\frac{z+k}{k}\right) = k^{z/k}\Gamma\left(\frac{z}{k}+1\right)\\ &= k^{z/k}\frac{z}{k}\Gamma\left(\frac{z}{k}\right) = zk^{(z/k)-1}\Gamma\left(\frac{z}{k}\right) = z\Gamma_k(z).\end{aligned}$$

Hence, from Theorem 1.2 we have

$$\langle z\rangle_{\gamma,k}=\frac{\Gamma_k(z+\gamma k)}{\Gamma_k(z)}=(z+(\gamma-1)k)\frac{\Gamma_k(z+(\gamma-1)k)}{\Gamma_k(z)}=(z+(\gamma-1)k)\langle z\rangle_{\gamma-1,k}.$$

Similarly, we have

$$\begin{aligned}\langle z\rangle_{\gamma,-k} &= \frac{\Gamma_k(z+k)}{\Gamma_k(z-(\gamma-1)k)}=(z-(\gamma-1)k)\frac{\Gamma_k(z+k)}{(z-(\gamma-1)k)\Gamma_k(z-(\gamma-1)k)}\\ &= (z-(\gamma-1)k)\frac{\Gamma_k(z_k)}{\Gamma_k(z-(\gamma-2)k)}=(z-(\gamma-1)k)\langle z\rangle_{\gamma-1,-k},\end{aligned}$$

which completes the proof.

■

In next section, we use the k-Pochhammer symbol and k-Gamma functions to extend the generalized Stirling numbers of integer orders to the complex number orders, which are called the *generalized Stirling functions*. The convergence and the recurrence relation of the generalized Stirling functions as well as their generating functions will also be presented. Finally, in Section 3 we will give more properties of generalized Stirling numbers and functions using the generating functions of generalized Stirling function sequences shown in Section 2, which include the asymptotic expansions of generalized Stirling functions.

2 Generalized Stirling functions

In [17], the author gives an equivalent form of the generalized Stirling numbers $S(n,k)$ defined by (1.5) by using the *generalized difference operator* in terms of β ($\beta\neq 0$) defined by

$$\Delta_\beta^k f=\Delta_\beta(\Delta_\beta^{k-1}f)\quad (k\geq 2)\quad \textit{and}\quad \Delta_\beta f(t):=f(t+\beta)-f(t). \tag{2.1}$$

It can be seen that $\Delta_\beta^k\langle z\rangle_{j,-\beta}\Big|_{z=0}=\beta^k k!\delta_{k,j}$, where $\delta_{k,j}$ is the Kronecker delta symbol; i.e., $\delta_{k,j}=1$ when $k=j$ and 0 otherwise. Evidently, from (1.10) there holds

$$\Delta_\beta^k\langle z\rangle_{j,-\beta}\big|_{z=0}=\Delta_\beta^k\beta^j\left[\frac{t}{\beta}\right]_j\Bigg|_{z=0}=\beta^j\Delta^k[t]_j\big|_{z=0}=\beta^k k!\delta_{k,j}. \tag{2.2}$$

Denote the divided difference of $f(t)$ at $t+i$, $i=0,1,\ldots,k$, by $f[t,t+1,\ldots,t+k]$, or $[t,t+1,\ldots,t+k]f(t)$. Using the well-known forward difference formula, it is easy to check that

$$\frac{1}{k!}\Delta^k f(t)=f[t,t+1,\ldots,t+k]=[t,t+1,\ldots,t+k]f(t)$$

and

$$\frac{1}{\beta^k k!}\Delta_\beta^k f(t) = f[t, t+\beta, t+2\beta, \ldots, t+k\beta] = [t, t+\beta, \ldots, t+k\beta]f(t).$$

[17] gives the following definition of the *generalized divided differences.*

Definition 2.1 *[17] We define* $\underline{\triangle}_\beta^k f(t)$ *by*

$$\underline{\triangle}_\beta^k f(t) = \begin{cases} \frac{1}{\beta^k k!}\Delta_\beta^k f(t) = f[t, t+\beta, \ldots, t+k\beta] & if\ \beta \neq 0 \\ \frac{1}{k!}D^k f(t) & if\ \beta = 0 \end{cases}, \tag{2.3}$$

where $\Delta_\beta^k f(t)$ *is shown in (2.1),* $f[t, t+\beta, \ldots, t+k\beta] \equiv [t, t+\beta, \ldots, t+k\beta]f$ *is the* k*th divided difference of* f *in terms of* $\{t, t+\beta, \ldots, t+k\beta\}$*, and* $D^k f(t)$ *is the* k*th derivative of* $f(t)$*.*

From the well-known formula

$$f[t, t+\beta, t+2\beta, \ldots, t+k\beta] = \frac{D^k f(\xi)}{k!},$$

where ξ is between t and $t+k\beta$, it is clear that

$$D^k f(t) = \lim_{\beta\to 0}\frac{1}{\beta^k}\Delta_\beta^k f(t), \tag{2.4}$$

which shows the generalized divided difference is well defined.

[17] gives a unified expression of the generalized Stirling numbers in terms of the the generalized divided differences.

Theorem 2.2 *[17] Let* $n, k \in \mathbb{N}_0$ *and the parameter triple* $(\alpha, \beta, r) \neq (0, 0, 0)$ *is in* $\mathbb{R}^3$ *or* $\mathbb{C}^3$*. For the generalized Stirling numbers defined by (1.5), there holds*

$$\begin{aligned} & S(n, k, \alpha, \beta, r) = \underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} \\ = & \begin{cases} \frac{1}{\beta^k k!}\ \Delta_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} = [r, r+\beta, \ldots, r+k\beta]\langle z\rangle_{n,-\alpha} & if\ \beta \neq 0 \\ \frac{1}{k!}\ D^k\langle z\rangle_{n,-\alpha}\big|_{z=r} & if\ \beta = 0. \end{cases} \end{aligned} \tag{2.5}$$

In particular, for the generalized Stirling number pair defined by (1.6), we have the expressions

$$\begin{aligned} & S^1(n, k) \equiv S^1(n, k, \alpha, \beta, r) = \underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} \\ = & \begin{cases} \frac{1}{\beta^k k!}\ \Delta_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} = [r, r+\beta, \ldots, r+k\beta]\langle z\rangle_{n,-\alpha}, & if\ \beta \neq 0 \\ \frac{1}{k!}D^k\langle z\rangle_{n,-\alpha}\big|_{z=r}, & if\ \beta = 0 \end{cases} \end{aligned} \tag{2.6}$$

$$\begin{aligned} & S^2(n, k) \equiv S^2(n, k, \beta, \alpha, -r) = \underline{\triangle}_\alpha^k \langle z\rangle_{n,-\beta}\Big|_{z=-r} \\ = & \begin{cases} \frac{1}{\alpha^k k!}\ \Delta_\alpha^k \langle z\rangle_{n,-\beta}\big|_{z=-r} = [-r, -r+\alpha, \ldots, -r+k\alpha]\langle z\rangle_{n,-\beta}, & if\ \alpha \neq 0 \\ \frac{1}{k!}D^k\langle z\rangle_{n,-\beta}\big|_{z=-r}, & if\ \alpha = 0 \end{cases} \end{aligned} \tag{2.7}$$

Furthermore, if $(\alpha, \beta, r) = (1, 0, 0)$, then (2.5) is reduced to the classical Stirling numbers of the first kind defined by (1.1) with the expression

$$s(n, k) = S(n, k, 1, 0, 0) = \frac{1}{k!} D^k [z]_n \big|_{z=0} .$$

If $(\alpha, \beta, r) = (0, 1, 0)$, then (2.5) is reduced to the classical Stirling numbers of the second kind shown in (1.3) with the following divided difference expression form:

$$S(n, k) = S(n, k, 0, 1, 0) = [0, 1, 2, \ldots, k] z^n \big|_{z=0} . \tag{2.8}$$

The following corollary is obvious due to the expansion formula of the divided differences generated from their definition.

Corollary 2.3 *[17] Let $n, k \in \mathbb{N}_0$ and the parameter triple $(\alpha, \beta, r) \neq (0, 0, 0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. If $\beta \neq 0$, for the generalized Stirling numbers defined by (1.5), there holds*

$$S(n, k) \equiv S(n, k, \alpha, \beta, r) = \frac{1}{\beta^k k!} \sum_{j=0}^{n} (-1)^j \binom{k}{j} \langle r + (k - j)\beta \rangle_{n, -\alpha} \quad (n \neq 0), \tag{2.9}$$

and $S(0, k) = \delta_{0k}$.

[17] gives four algorithms for calculating the Stirling numbers and their generalizations based on their unified expression, which include two comprehensive algorithms using the characterization of Riordan arrays.

We now extend the Stirling numbers $S(n, k))$ expressed by (2.5) to a more wider generation form using the idea of [6]. First, in order to cover as large a function class as possible, we recall that the *generalized fractional difference operator* $\Delta_\beta^{\eta,\epsilon}$ with an exponential factor, which is introduced in [6]. More precisely, for $\eta \in \mathbb{C}$, $\beta \in \mathbb{R}_+$, $\epsilon \geq 0$, the generalized fractional difference operator $\Delta_\beta^{\eta,\epsilon}$ is defined for "sufficient good" functions f by

$$\Delta_\beta^{\eta,\epsilon} f(z) := \sum_{j \geq 0} (-1)^j \binom{\eta}{j} e^{(\eta - j)\epsilon} f(z + (\eta - j)\beta) \quad (z \in \mathbb{C}), \tag{2.10}$$

where $\binom{\eta}{j}$ are the general binomial coefficients given by

$$\binom{\eta}{j} = \frac{[\eta]_j}{j!} := \frac{\eta(\eta - 1) \cdots (\eta - j + 1)}{j!} \quad (j \in \mathbb{N}), \tag{2.11}$$

with $[\beta]_0 = 1$. Noting the generalized Stirling numbers $S(n, k)$ can be represented by (2.6), or equivalently,

$$S(n, k) = \frac{1}{\beta^k k!} \lim_{z \to r} \Delta_\beta^k \langle z \rangle_{n, -\alpha},$$

which has an extension shown in (2.9). We now extend (2.9) to a more generalized form shown in the following definition.

Definition 2.4 *The generalized Stirling functions, $S(\gamma, \eta, \alpha, \beta, r; \epsilon)$ for any complex numbers γ and η are given by*

$$S(\gamma,\eta;\epsilon) \equiv S(\gamma,\eta,\alpha,\beta,r;\epsilon) := \frac{1}{\beta^\eta \Gamma(\eta+1)} \lim_{z\to r} \Delta_\beta^{\eta,\epsilon}(\langle z\rangle_{\gamma,-\alpha}) \quad (\epsilon \geq 0), \tag{2.12}$$

provided the limit exists; or equivalently, by

$$S(\gamma,\eta;\epsilon) \equiv S(\gamma,\eta,\alpha,\beta,r;\epsilon) = \frac{1}{\beta^\eta \Gamma(\eta+1)} \sum_{j\geq 0} (-1)^j \binom{\eta}{j} e^{(\eta-j)\epsilon} \langle r+(\eta-j)\beta\rangle_{\gamma,-\alpha} \quad (\gamma \neq 0), \tag{2.13}$$

provided the series converges absolutely. and

$$S(0,\eta) = \frac{(e^\epsilon - 1)^\eta}{\beta^\eta \Gamma(\eta+1)}. \tag{2.14}$$

From (2.13), we immediately have

$$S(\gamma, 0; \epsilon) = \langle r\rangle_{\gamma,-\alpha} \quad (\gamma \neq 0). \tag{2.15}$$

Now, an explicit expression of $S(\gamma, \eta; \epsilon)$ can be given by the following result.

Theorem 2.5 *If $\gamma \in \mathbb{C}$ and either of the conditions $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}_-$), $\epsilon > 0$, or $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}_-$, $Re(\eta) > Re(\gamma)$), $\epsilon = 0$ hold, then the generalized Stirling functions $S(\gamma, \eta; \epsilon)$ can be represented in the form (2.13) and $S(0, \eta; \epsilon) = \delta_{\eta,0}$. In particular, if $\gamma = n \in \mathbb{N}_0$, $\eta = k \in \mathbb{N}$, and $\epsilon \geq 0$, then the corresponding generalized Stirling functions $S(n, k; \epsilon)$ has the representation (2.13).*

Proof. First, from equation (1.51) in [29] we have the estimation

$$\left| \binom{\eta}{j} \right| \leq \frac{A}{j^{Re(\eta)} + 1} \tag{2.16}$$

for any $\eta \in \mathbb{C}$, $\eta \neq -1, -2, \ldots$, and sufficiently large $j \in \mathbb{N}$, where $A > 0$ is a constant.

Secondly, from the second formula of (1.16) and expression (1.13), we obtain

$$\begin{aligned} \langle r + (\eta-j)\beta\rangle_{\gamma,-\alpha} &= \frac{\Gamma_\alpha(r+(\eta-j)\beta+\alpha)}{\Gamma_\alpha(r+(\eta-j)\beta-(\gamma-1)\alpha)} \\ = \quad & \frac{\alpha^{((r+(\eta-j)\beta+\alpha)/\alpha)-1}\Gamma\left(\frac{r+(\eta-j)\beta+\alpha}{\alpha}\right)}{\alpha^{((r+(\eta-j)\beta-(\gamma-1)h)/h)-1}\Gamma\left(\frac{\beta-j-(\gamma-1)\alpha}{\alpha}\right)} = \alpha^\gamma \frac{\Gamma\left(\frac{r+(\eta-j)\beta+\alpha}{\alpha}\right)}{\Gamma\left(\frac{r+(\eta-j)\beta-(\gamma-1)\alpha}{\alpha}\right)}. \end{aligned}$$

From [5] or [29], we obtain

$$|\langle r + (\eta-j)\beta\rangle_{\gamma,-\alpha}| \leq Bj^{Re(\gamma)}$$

with a certain constant B. Thus,

$$\left|(-1)^j \binom{\eta}{j} e^{(\eta-j)\epsilon} \langle r + (\eta - j)\beta \rangle_{\gamma,-\alpha}\right| \leq C \frac{e^{-\epsilon j}}{j^{Re(\eta-\gamma)+1}},$$

where $C = ABe^{Re(\beta)\epsilon}$. Hence, the series on the right-hand side of (2.13) is absolutely convergent if either $\epsilon > 0$ or $\epsilon = 0$ with $Re(\beta) > Re(\gamma)$. Similarly, when $n \in \mathbb{N}_0$, $\eta = k \in \mathbb{N}$, and $\epsilon \geq 0$, the corresponding generalized Stirling functions $S(n, k; \epsilon)$ has the representation (2.13), which completes the proof of the theorem.

■

We now present the recurrence relation of the generalized Stirling functions defined by (2.13) by using the recurrence relations of the generalized raising and falling factorial functions shown in Proposition 1.3.

Theorem 2.6 *There hold the following three results.*

(a) For $\gamma \in \mathbb{C}$, $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}_-$), and $\epsilon > 0$, the generalized Stirling functions $S(\gamma, \eta; \epsilon)$ defined by (2.13) satisfy

$$S(\gamma, \eta; \epsilon) = (r + \eta\beta - (\gamma - 1)\alpha)S(\gamma - 1, \eta; \epsilon) + S(\gamma - 1, \eta - 1; \epsilon). \tag{2.17}$$

(b) Let $\gamma \in \mathbb{C}$, $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}_-$), and $Re(\eta) > Re(\gamma)$). The generalized Stirling functions $S(\gamma, \eta) \equiv S(\gamma, \eta; 0)$ satisfy

$$S(\gamma, \eta) = (r + \eta\beta - (\gamma - 1)\alpha)S(\gamma - 1, \eta) + S(\gamma - 1, \eta - 1). \tag{2.18}$$

(c) For $\gamma \in \mathbb{C}$, $k \in \mathbb{N}$, and $\epsilon \geq 0$, the generalized Stirling functions $S(\gamma, k; \epsilon; h)$ defined by (2.13) satisfy

$$S(\gamma, k; \epsilon) = (r + k\beta - (\gamma - 1)\alpha)S(\gamma - 1, k; \epsilon) + S(\gamma - 1, k - 1; \epsilon). \tag{2.19}$$

In particular,

$$S(\gamma, k) = (r + k\beta - (\gamma - 1)\alpha)S(\gamma - 1, k) + S(\gamma - 1, k - 1).$$

Proof. In accordance with Theorem 2.5, all terms on both sides of equations (2.17)-(2.19) are well defined for the given ranges of parameters γ, η, n, and ϵ. From (2.13), we can write the right-hand side of (2.17) as

$$
\begin{aligned}
& (r+\eta\beta-(\gamma-1)\alpha)S(\gamma-1,\eta;\epsilon)+S(\gamma-1,\eta-1;\epsilon) \\
= \quad & \frac{r+\eta\beta-(\gamma-1)\alpha}{\beta^{\eta}\Gamma(\eta+1)}\sum_{j\geq 0}(-1)^{j}\binom{\eta}{j}e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma-1,-\alpha} \\
& +\frac{1}{\beta^{\eta-1}\Gamma(\eta)}\sum_{j\geq 0}(-1)^{j}\binom{\eta-1}{j}e^{(\eta-1-j)\epsilon}\langle \beta-1-j\rangle_{\gamma-1,-\alpha} \\
= \quad & \frac{r+\eta\beta-(\gamma-1)\alpha}{\beta^{\eta}\Gamma(\eta+1)}e^{\eta\epsilon}\langle r+\eta\beta\rangle_{\gamma-1,-\alpha} \\
& +\frac{r+\eta\beta-(\gamma-1)\alpha}{\beta^{\eta}\Gamma(\eta+1)}\sum_{j\geq 1}(-1)^{j}\binom{\eta}{j}e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma-1,-\alpha} \\
& +\frac{\eta}{\beta^{\eta-1}\Gamma(\eta+1)}\sum_{j\geq 1}(-1)^{j+1}\binom{\eta-1}{j-1}e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma-1,-\alpha} \\
= \quad & \frac{r+\eta\beta-(\gamma-1)\alpha}{\beta^{\eta}\Gamma(\eta+1)}e^{\eta\epsilon}\langle r+\eta\beta\rangle_{\gamma-1,-\alpha}+\frac{1}{\beta^{\eta}\Gamma(\eta+1)}\sum_{j\geq 1}(-1)^{j}\left[(r+\eta\beta-(\gamma-1)\alpha)\binom{\beta}{j}\right. \\
& \left.-\eta\beta\binom{\eta-1}{j-1}\right]e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma-1,-\alpha} \\
= \quad & \frac{1}{\beta^{\eta}\Gamma(\eta+1)}e^{\eta\epsilon}\langle r+\eta\beta\rangle_{\gamma,-\alpha}+\frac{1}{\beta^{\eta}\Gamma(\eta+1)}\sum_{j\geq 1}(-1)^{j}\left[(r+\eta\beta-(\gamma-1)\alpha)\binom{\eta}{j}\right. \\
& \left.-j\beta\binom{\eta}{j}\right]e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma-1,-\alpha} \\
= \quad & \frac{1}{\beta^{\eta}\Gamma(\eta+1)}e^{\eta\epsilon}\langle r+\eta\beta\rangle_{\gamma,-\alpha} \\
& +\frac{1}{\beta^{\eta}\Gamma(\eta+1)}\sum_{j\geq 1}(-1)^{j}(r+(\eta-j)\beta-(\gamma-1)\alpha)\binom{\eta}{j}e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma-1,-\alpha} \\
= \quad & \frac{1}{\beta^{\eta}\Gamma(\eta+1)}\sum_{j\geq 0}(-1)^{j}\binom{\eta}{j}e^{(\eta-j)\epsilon}\langle r+(\eta-j)\beta\rangle_{\gamma,-\alpha},
\end{aligned}
$$

which is the right-hand side of (2.17). In the last step, we used the second recurrence formula of (1.17) in Proposition 1.3. (2.18) and (2.19) can be proved similarly.

■

Clearly, Theorem 6 in [6] is a special case of Theorem 2.6 for $\alpha,\beta=0$. And Theorem 3, Corollaries 3.1 and 3.2 in [6] are special cases of Theorem 2.6 for $\alpha,\beta=0$ and $\gamma=n\in\mathbb{N}$.

Now we construct the exponential generating function for the generalized Stirling functions $S(n,\eta;\epsilon)$.

Theorem 2.7 *Let $z\in\mathbb{C}$, $\eta\in\mathbb{C}$, and $\epsilon\geq 0$. The generating function for the generalized Stirling functions $S(\gamma,\eta;\epsilon)$ defined by (2.13) with $\gamma=n$ and $\alpha\beta\neq 0$ is*

$$\frac{1}{\Gamma(\eta+1)}(1+\alpha z)^{r/\alpha}\left(\frac{e^{\epsilon}(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{\eta}=\sum_{n\geq 0}S(n,\eta;\epsilon)\frac{z^n}{n!} \tag{2.20}$$

for $\eta\notin\mathbb{Z}_-$ *and* $\epsilon>0$, *and*

$$\frac{1}{k!}(1+\alpha z)^{r/\alpha}\left(\frac{e^{\epsilon}(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{k}=\sum_{n\geq 0}S(n,k;\epsilon)\frac{z^n}{n!} \tag{2.21}$$

for $\eta=k\in\mathbb{N}_0$ *and* $\epsilon\geq 0$.

Proof. Denote the generating function for the generalized Stirling functions $S(\gamma,\eta;\epsilon)$ defined by (2.13) with $\alpha\beta\neq 0$ by

$$x_\eta(z)=\sum_{n\geq 0}S(n,\eta;\epsilon)\frac{z^n}{n!}. \tag{2.22}$$

It can be seen that for $\eta\neq 0$, $x_\eta(z)$ satisfies the differential equation

$$(1+\alpha z)\frac{d}{dz}x_\eta(z)-(r+\eta\beta)x_\eta(z)=x_{\eta-1}(z) \tag{2.23}$$

with initial condition

$$x_\eta(0)=S(0,\eta;\epsilon)=\frac{(e^{\epsilon}-1)^{\eta}}{\beta^{\eta}\Gamma(\eta+1)}, \tag{2.24}$$

and

$$x_0(z)=(1+\alpha z)^{r/\alpha}. \tag{2.25}$$

Evidently, using (2.17) one may write the left-hand side of (2.22) as

$$\begin{aligned}
&(1+\alpha z)\frac{d}{dz}x_\eta(z)-(r+\eta\beta)x_\eta(z)\\
=\;&(1+\alpha z)\sum_{n\geq 1}S(n,\eta;\epsilon)\frac{z^{n-1}}{(n-1)!}-\sum_{n\geq 0}(r+\eta\beta)S(n,\eta;\epsilon)\frac{z^n}{n!}\\
=\;&\sum_{n\geq 0}\frac{z^n}{n!}\left(S(n+1,\eta;\epsilon)-(r+\eta\beta-n\alpha)S(n,\eta;\epsilon)\right)\\
=\;&\sum_{n\geq 0}S(n,\eta-1;\epsilon)\frac{z^n}{n!}=x_{\eta-1}(z).
\end{aligned}$$

Substituting $z=0$ into (2.22) yields

$$x_\eta(0)=S(0,\eta;\epsilon),$$

which implies (2.24) by making use of (2.14). Finally, from (2.15) and (1.15) there holds

$$\begin{aligned} x_0(z) &= \sum_{n\geq 0} S(n,0;\epsilon)\frac{z^n}{n!} = \sum_{n\geq 0}\langle r\rangle_{n,-\alpha}\frac{z^n}{n!} \\ &= \sum_{n\geq 0}\alpha^n\left[\frac{r}{\alpha}\right]_n\frac{z^n}{n!} = \sum_{n\geq 0}\binom{r/\alpha}{n}\frac{(\alpha z)^n}{n!}, \end{aligned}$$

which implies (2.25).

Denote the left-hand side of Equation (2.20) by $y_\eta(z)$. It can be checked that $y_\eta(z)$ is also the solution of initial-value problem (2.23) and (2.24) that satisfies (2.25). Indeed,

$$\begin{aligned} &(1+\alpha z)\frac{d}{dz}y_\eta(z) - (r+\eta\beta)y_\eta(z) \\ = \; &\frac{1+\alpha z}{\Gamma(\eta+1)}\left[r(1+\alpha z)^{r/\alpha-1}\left(\frac{e^\epsilon(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^\eta + \eta e^\epsilon(1+\alpha z)^{r/\alpha+\beta/\alpha-1}\right. \\ &\left.\left(\frac{e^\epsilon(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{\eta-1}\right] - \frac{r+\eta\beta}{\Gamma(\eta+1)}(1+\alpha z)^{r/\alpha}\left(\frac{e^\epsilon(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^\eta \\ = \; &\frac{1}{\Gamma(\eta)}(1+\alpha z)^{r/\alpha}\left(\frac{e^\epsilon(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{\eta-1} = y_{\eta-1}(z). \end{aligned}$$

It is easy to see that

$$y_0(z) = (1+\alpha z)^{r/\alpha} \quad \textit{and} \quad y_k(0) = \frac{(e^\epsilon-1)^\eta}{\beta^\eta\Gamma(\eta+1)}.$$

Since the solution of the initial-value problem (2.23)-(2.25) is unique, we have $y_\eta(z) = x_\eta(z)$. Thus, from the definition (2.22), we obtain (2.20). A similar argument can be used to prove (2.21).

■

Remark 3.1 The condition $\alpha\beta \neq 0$ is not necessary for the left-hand side of (2.20). In fact, taking $r = 0$, $\beta = 1$, and letting $\alpha \to 0^+$, we see that (2.20) yields the generating function for the generalized Stirling functions of the second kind:

$$\frac{1}{\Gamma(\eta+1)}(e^{z+s}-1)^\eta = \sum_{n\geq 0} S(n,\eta,0,1,0;\epsilon)\frac{z^n}{n!},$$

which was studied in Theorem 4 of [6], and it can be considered as a particular case of our Theorem 2.7.

Similarly, taking $\epsilon, r = 0$, $\alpha = 1$ and letting $\beta \to 0^+$ yields the generating function of the generalized Stirling functions of the first kind:

$$\frac{1}{\Gamma(\eta+1)}(\ln(1+z))^\eta = \sum_{n\geq 0} S(n,\eta,1,0,0)\frac{z^n}{n!}.$$

3 More properties of the generalized Stirling functions

let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[[t;\{c_k\}]]$ or $\mathbb{C}[[t;\{c\}]]$ (where $c = (c_0, c_1, c_2, \ldots)$ satisfies $c_0 = 1$, $c_k > 0$ for all $k = 1, 2, \ldots$); the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k / c_k$, is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s.'s $f(t)$ for which the compositional inverse $\overline{f}(t)$ exists such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$. We call the element $g \in \mathcal{F}$ with the form $g(x) = \sum_{k\geq 0} \frac{x^k}{c_k}$ a generalized power series (GPS) associated with $\{c_n\}$ or, simply, a (c)-GPS, and $\mathcal{F}$ the GPS set associated with $\{c_n\}$. In particular, when $c = (1, 1, \ldots)$, the corresponding $\mathcal{F}$ and $\mathcal{F}_r$ denote the classical formal power series and the classical formal power series of order r, respectively.

We now develop a kind of asymptotic expansions for the generalized Stirling functions $S(n,\mu,r;\epsilon) \equiv S(n,\mu,\alpha,\beta,r;\epsilon)$ and $S(n,\mu,\mu r;\epsilon) \equiv S(n,\mu,\alpha,\beta,\mu r;\epsilon)$ and generalized Stirling numbers $S(n+\mu,\mu,r) \equiv S(n+\mu,\mu,\alpha,\beta,r)$ and $S(n+\mu,\mu,\mu r) \equiv S(n+\mu,\mu,\alpha,\beta,\mu r)$ for large μ and n with the condition $n = 0(\mu^{1/2})$ $(\mu \to \infty)$. The asymptotic expansions of Hsu and Shiue Stirling numbers in [19] and Tsylova Stirling numbers in [31], involving a generalization of Moser and Wyman's result [24], are included as particular cases.

The major tool of construction of the asymptotic expansion is the known result about the asymptotic formula for the coefficients of power-type generating functions involving large parameters shown in [18]. Let $\sigma(n)$ be the set of partition of n $(n \in \mathbb{N})$, which can be represented by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $1k_1 + 2k_2 + \cdots nk_n = n$, $k_j \geq 0$ $(j = 1, 2, \ldots, n)$, and with $k = k_1 + k_2 + \cdots + k_n$ expressing the number of the parts of the partition. For given k $(1 \leq k \leq n)$, we denote by $\sigma(n,k)$ the subset of $\sigma(n)$ consisting of partitions of n having k parts.

Let $\phi(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series over the complex field $\mathbb{C}$ in $\mathcal{F}_0$, with $a_0 = g(0) = 1$. For every j $(0 \leq j < n)$ define

$$W(n,j) = \sum_{\sigma(n,n-j)} \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!}, \tag{3.1}$$

where the summation is taken over all such partition $1^{k_1}2^{k_2}\cdots n^{k_n}$ of n that have $n - j$ parts. We have the following known result (see for instance [18]):

For a fixed $m \in \mathbb{N}$ and for large μ and n such that $n = o(\mu^{1/2})$ $(\mu \to \infty)$, we have the asymptotic expansion

$$\frac{1}{[\mu]_n}[z^n](\phi(z))^{\mu} = \sum_{j=0}^{m} \frac{W(n,j)}{[\mu - n + j]_j} + o\left(\frac{W(n,m)}{[\mu - n + m]_m}\right), \tag{3.2}$$

where $W(n,j)$ are given by (3.1). (3.2) is used to derive the Hsu-Shiue Stirling numbers in [19]. We now generalize (3.2) and the corresponding argument to give asymptotic expansion formulas of generalized Stirling functions $S(n,\mu,r;\epsilon) \equiv S(n,\mu,\alpha,\beta, r;\epsilon)$, $S(n,\mu,\mu r;\epsilon) \equiv S(n,\mu,\alpha,\beta,\mu r;\epsilon)$, $S(n+\mu,\mu,r) \equiv S(n+\mu,\mu,\alpha,\beta,r)$ and $S(n+\mu,\mu,\mu r) \equiv S(n+\mu,\mu,\alpha,\beta,\mu r)$ for large μ and n with the condition $n = 0(\mu^{1/2})$ as $\mu \to \infty$.

Let $g(z)=\sum_{n\geq 0} a_n z^n$ be a formal power series over the complex field $\mathbb{C}$ in $\mathcal{F}_0$, with $a_0=g(0)\neq 0$. We may write

$$g(z)=a_0\sum_{n\geq 0}\frac{a_n}{a_0}z^n.$$

For a fixed $m\in\mathbb{N}$ and for large μ and n such that $n=o(\mu^{1/2})$ $(\mu\to\infty)$, From formulas (3.1) and (3.2) we have the asymptotic expansion

$$\frac{1}{[\mu]_n}[z^n](g(z))^\mu=\sum_{j=0}^{m}\frac{W(n,j)}{a_0^{n-\mu-j}[\mu-n+j]_j}+o\left(\frac{W(n,m)}{a_0^{m-\mu}[\mu-n+m]_m}\right), \tag{3.3}$$

where $W(n,j)$ are given by (3.1). In particular, when n is fixed, the remainder estimate becomes $O(\mu^{-m-1})$.

To apply (3.2) to the generalized Stirling numbers $S(\gamma,\eta;\epsilon)$ defined by (2.13) with $\gamma=n$, $\eta=\mu$ and $\alpha\beta\neq 0$, let us use (2.21) to take

$$g(z)=(1+\alpha z)^{r/\alpha}\frac{e^\epsilon(1+\alpha z)^{\beta/\alpha}-1}{\beta}=\sum_{n\geq 0}\frac{S(n,1;\epsilon)}{n!}z^n \tag{3.4}$$

when $\epsilon\neq 0$, and

$$\bar{g}(z)=(1+\alpha z)^{r/\alpha}\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta z}=\sum_{n\geq 0}\frac{S(n+1,1)}{(n+1)!}z^n \tag{3.5}$$

when $\epsilon=0$, so that $g(0)=(e^\epsilon-1)/\beta$ $(\epsilon\neq 0)$ and $\bar{g}(0)=1$ $(\epsilon=0)$ not being zero in both cases, where $S(n,1;\epsilon)\equiv S(n,1,\alpha,\beta,r;\epsilon)$ and $S(n+1,1)\equiv S(n+1,1,\alpha,\beta,r)$, $g(0)=(e^\epsilon-1)/\beta$. Consequently, from (2.21) we have

$$\begin{aligned}(g(z))^\mu &= (1+\alpha z)^{\mu r/\alpha}\left(\frac{e^\epsilon(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^\mu\\ &= \mu!\sum_{n\geq 0}\frac{S(n,\mu,\alpha,\beta,\mu r;\epsilon)}{n!}z^n \end{aligned}\tag{3.6}$$

for $\epsilon\neq 0$, and

$$\begin{aligned}(\bar{g}(z))^\mu &= (1+\alpha z)^{\mu r/\alpha}\left(\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta z}\right)^\mu\\ &= \mu!\sum_{n\geq 0}\frac{S(n+\mu,\mu,\alpha,\beta,\mu r)}{(n+\mu)!}z^n \end{aligned}\tag{3.7}$$

for $\epsilon=0$. Therefore, making use of (3.3) yields

$$\begin{aligned}
&\frac{S(n,\mu,\alpha,\beta,\mu r;\epsilon)}{[\mu]_n[n]_\mu}\\
=\ & \left(\frac{\beta}{e^\epsilon-1}\right)^{n-\mu}\sum_{j=0}^m\left(\frac{e^\epsilon-1}{\beta}\right)^j\frac{W(n,j)}{[\mu-n+j]_j}+o\left(\left(\frac{\beta}{e^\epsilon-1}\right)^{n-\mu}\frac{W(n,m)}{[\mu-n+m]_m}\right)
\end{aligned}\tag{3.8}$$

for $\epsilon\neq 0$, and

$$\frac{S(n+\mu,\mu,\alpha,\beta,\mu r)}{[\mu]_n[n+\mu]_\mu}=\sum_{j=0}^m\frac{W(n,j)}{[\mu-n+j]_j}+o\left(\frac{W(n,m)}{[\mu-n+m]_m}\right)\tag{3.9}$$

for $\epsilon=0$, where $n=o(\mu^{1/2})$ as $\mu\to\infty$ and $W(n,j)$ $(j=0,1,2,\ldots)$ are given by (3.1) with a_j being determined by (3.4); namely, for $\epsilon\neq 0$, $a_0=(e^\epsilon-1)/\beta$ and

$$a_j=[z^j]g(z)=\frac{S(j,1;\epsilon)}{j!},\tag{3.10}$$

while for $\epsilon=0$, $a_0=1$ and

$$a_j=[z^j]\bar{g}(z)=\frac{S(j+1,1)}{(j+1)!}.\tag{3.11}$$

The coefficients defined by (3.10) and (3.11) can be evaluated by using the Vandermonde-Chu formula as follows. From (3.4), for $j=1,2,\ldots,$ we have

$$\begin{aligned}
&[z^j]g(z)=[z^j](1+\alpha z)^{r/\alpha}\left[\frac{e^\epsilon-1}{\beta}+\frac{e^\epsilon}{\beta}\sum_{k\geq 1}\binom{\beta/\alpha}{k}(\alpha z)^k\right]\\
=\ &[z^j]\left[\frac{e^\epsilon-1}{\beta}\sum_{\ell\geq 0}\binom{r/\alpha}{\ell}(\alpha z)^\ell+\frac{e^\epsilon}{\beta}\sum_{\ell\geq 0}\sum_{k\geq 1}\binom{r/\alpha}{\ell}\binom{\beta/\alpha}{k}(\alpha z)^{\ell+k}\right]\\
=\ &\frac{e^\epsilon-1}{\beta}\alpha^j\binom{r/\alpha}{j}+\frac{e^\epsilon}{\beta}\alpha^j\sum_{k=1}^j\binom{r/\alpha}{j-k}\binom{\beta/\alpha}{k}\\
=\ &\frac{e^\epsilon-1}{j!\beta}\langle r\rangle_{j,-\alpha}+\frac{e^\epsilon}{\beta}\alpha^j\left[\binom{r/\alpha+\beta/\alpha}{j}-\binom{r/\alpha}{j}\right]\\
=\ &\frac{1}{j!\beta}\left[\langle r+\beta\rangle_{j,-\alpha}+(e^\epsilon-2)\langle r\rangle_{j,-\alpha}\right].
\end{aligned}$$

Here, the classical Vandermonde-Chu convolution formula we used above, regarded as "perhaps the most widely used combinatorial identity" (see P. 8 in [27] by Riordan and PP. 51, 61, 64, and 227 in [1] by Andrews), which can be written as

$$\sum_{k=0}^n\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}\quad (x,y\in\mathbb{R},n\in\mathbb{N}_0).$$

Similarly, we obtain

$$[z^j]\bar{g}(z) = \frac{1}{(j+1)!\beta}\left[\langle r+\beta\rangle_{j+1,-\alpha} - \langle r\rangle_{j+1,-\alpha}\right]$$

for $j = 0, 1, 2, \ldots$. Hence, we may survey the above into the following theorem.

Theorem 3.1 *For $\epsilon \neq 0$, there holds the asymptotic expansion (3.8) of $S(n, \mu, \mu r; \epsilon) \equiv S(n, \mu, \alpha, \beta, \mu r; \epsilon)$ for n with $n = o(\mu^{1/2})$ $(\mu \to \infty)$, where $W(n, j)$ is defined by (3.1) with $a_0 = (e^\epsilon - 1)/\beta$ and*

$$a_j = \frac{1}{j!\beta}\left[\langle r+\beta\rangle_{j,-\alpha} + (e^\epsilon - 2)\langle r\rangle_{j,-\alpha}\right] \quad (j = 1, 2, \ldots).$$

For $\epsilon = 0$, there holds the asymptotic expansion (3.9) of $S(n + \mu, \mu, \mu r) \equiv S(n + \mu, \mu, \alpha, \beta, \mu r)$ for n with $n = o(\mu^{1/2})$ $(\mu \to \infty)$, where $W(n, j)$ is defined by (3.1) with

$$a_j = \frac{1}{(j+1)!\beta}\left[\langle r+\beta\rangle_{j+1,-\alpha} - \langle r\rangle_{j+1,-\alpha}\right] \quad j = 0, 1, \ldots.$$

Since the formulas (3.8) and (3.9) with $W(n, j)$ and a_j presented in (3.1) and Theorem 3.1, respectively, are algebraic analytic identities, we may replace r by r/μ in the formulas and obtain the following corollary.

Corollary 3.2 *For $\epsilon \neq 0$, by replacing the quantity r by r/μ, the asymptotic expansion (3.8) is also applicable to $S(n, \mu, r; \epsilon) \equiv S(n, \mu, \alpha, \beta, r; \epsilon)$ for n with $n = o(\mu^{1/2})$ $(\mu \to \infty)$, where $W(n, j)$ is defined by (3.1) with $a_0 = (e^\epsilon - 1)/\beta$ and*

$$a_j = \frac{1}{j!\beta}\left[\left\langle \frac{r}{\mu}+\beta\right\rangle_{j,-\alpha} + (e^\epsilon - 2)\left\langle \frac{r}{\mu}\right\rangle_{j,-\alpha}\right] \quad (j = 1, 2, \ldots).$$

For $\epsilon = 0$, by replacing the quantity r by r/μ, the asymptotic expansion (3.9) is also applicable to $S(n + \mu, \mu, r) \equiv S(n + \mu, \mu, \alpha, \beta, r)$ for n with $n = o(\mu^{1/2})$ $(\mu \to \infty)$, where $W(n, j)$ is defined by (3.1) with

$$a_j = \frac{1}{(j+1)!\beta}\left[\left\langle \frac{r}{\mu}+\beta\right\rangle_{j+1,-\alpha} - \left\langle \frac{r}{\mu}\right\rangle_{j+1,-\alpha}\right] \quad j = 0, 1, \ldots.$$

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