Characterization of (c)-Riordan Arrays, Gegenbauer-Humbert-type polynomial sequences, and (c)-Bell polynomials

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February 7, 2012

Abstract

Here presented are the definitions of (c)-Riordan arrays and (c)-Bell polynomials which are extensions of the classical Riordan arrays and Bell polynomials. The characterization of (c)-Riordan arrays by means of the A- and Z-sequences is given, which corresponds to a horizontal construction of a (c)-Riordan array rather than its definition approach through column generating functions. There exists a one-to-one correspondence between Gegenbauer-Humbert-type polynomial sequences and the set of (c)-Riordan arrays, which generates the sequence characterization of Gegenbauer-Humbert-type polynomial sequences. The sequence characterization is applied to construct readily a (c)-Riordan array. In addition, subgrouping of (c)-Riordan arrays by using the characterizations is discussed. The (c)-Bell polynomials and its identities by means of convolution families are also studied. Finally, the characterization of (c)-Riordan arrays in terms of the convolution families and (c)-Bell polynomials is presented.

Key words Riordan arrays, (c)-Riordan arrays, A-sequence, Z-sequence, (c)-Bell polynomials, (c)-hitting-time subgroup.

Mathematics subject Classification (2000) 05A10 - 05A15 - 05A19

1 Introduction

In the recent literature, special emphasis has been given to the concept of *Riordan arrays*, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [26]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli [29, 30], on subgroups of the Riordan group in Peart and Woan [18] and Shapiro [23], on some characterizations of Riordan matrices in Rogers [19], Merlini et al. [16] and [14] by the authors, and on many interesting related results in Cheon et al. [3, 4], He et al. [12], Nkwanta [17], Shapiro [24, 25], [11], and so forth.

In [10] a generalized Sheffer type polynomials are defined. Based on the isomorphism between the Sheffer type polynomials and Riordan arrays (see for example, [20] and [12]), one may give the definition of the generalized Riordan arrays. More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t;\{c_k\}]\!]$ (where $c=(c_0,c_1,c_2,\ldots)$ satisfies $c_0=1$, $c_k>0$ for all $k=1,2,\ldots$); the *order* of $f(t)\in\mathcal{F}$, $f(t)=\sum_{k=0}^\infty f_k t^k/c_k$, is the minimal number $r\in\mathbb{N}$ such that $f_r\neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s.'s $f(t)$ for which the compositional inverse $\overline{f}(t)$ exists such that $f(\overline{f}(t))=\overline{f}(f(t))=t$.

We call the element $g \in \mathcal{F}$ with the form $g(x) = \sum_{k\geq 0} \frac{x^k}{c_k}$ a generalized power series (GPS) associated with $\{c_n\}$ or, simply, a (c)-GPS, and $\mathcal{F}$ the GPS set associated with $\{c_n\}$.

Definition 1.1 *Let g be a (c)-GPS associated with $\{c_n\}$, where $\{c_k\}_{k\geq 0}$ is a sequence of non-zero constants with $c_0 = 1$. A (c)-Riordan array generated by $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$ with respect to $g(x)$ and $\{c_k\}_{k\geq 0}$ is an infinite complex matrix $[d_{n,k}]_{0\leq k\leq n}$, whose bivariate generating function has the form*

$$F(t,x) = \sum_{n,k} d_{n,k} \frac{t^n}{c_n} x^k = d(t)g(xh(t)). \tag{1.1}$$

Hence, we denote $[d_{n,k}] = (d(t), h(t))$. In particular, if $h'(0) \neq 0$, the corresponding Riordan array is called a proper Riordan array. Otherwise, it is called a non-proper Riordan array.

Furthermore, if $c_k = 1$ $(k = 0, 1, 2, \ldots)$, i.e., the corresponding series $d(t)$ and $h(t)$ are ordinary power series, then expression (1.1) is written as

$$F(t,x) = \sum_{n,k} d_{n,k} t^n x^k = \frac{d(t)}{1 - xh(t)}, \tag{1.2}$$

which deffines the classical Riordan array, called (1)-Riordan array. If $c_k = k!$ $(k = 0, 1, 2, \ldots)$, i.e., the corresponding series $d(t)$ and $h(t)$ are exponential power series, then expression (1.1) is written as

$$F(t,x) = \sum_{n,k} d_{n,k} \frac{t^n}{n!} x^k = d(t)e^{xh(t)}, \tag{1.3}$$

which defines the Sheffer type Riordan array. If $c_0 = 1$ and $c_k = 1/k$ $(k = 1, 2, \ldots)$, i.e., the corresponding series $d(t)$ and $h(t)$ are Dirichlet series, then expression (1.1) is written as

$$F(t,x) = d_{0,0} + \sum_{1\leq k\leq n} d_{n,k} \frac{t^n}{n} x^k = d(t)\left(1 - \ln(1 - xh(t))\right), \tag{1.4}$$

which is called the Dirichlet Riordan series.

Theorem 1.1 *In definition (1.1), the (n,k) entry of (c)-Riordan array $[d_{n,k}]$ is*

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] d(t) \frac{(h(t))^k}{c_k} = [t^n] \frac{c_n}{c_k} d(t)(h(t))^k \tag{1.5}$$

for all $0 \leq k \leq n$ and $d_{n,k} = 0$ otherwise. In particular, for the classical $(1)-$Riordan array, $d_{n,k} = [t^n]d(t)(h(t))^k$ $(o \leq k \leq n)$; for the Sheffer Riordan array, $d_{n,k} = [t^n/n!]d(t)(h(t))^k/k!$ $(0 \leq k \leq n)$; and for the Dirichlet Riordan array, $d_{n,k} = [t^n/n]d(t)(h(t))^k/k$ $(1 \leq k \leq n)$.

Proof: First, we can see (1.5) in Theorem 1.1 satisfies (1.1). By substituting (1.5), the expression of $d_{n,k}$, into the left-hand side of (1.1):

$$\begin{aligned}
& \sum_{n\geq 0} \sum_{k=0}^{n} d_{n,k} \frac{t^n}{c_n} x^k \\
= & \sum_{n\geq 0} \sum_{k\geq 0} \left[\frac{t^n}{c_n}\right] d(t) \frac{(h(t))^k}{c_k} \frac{t^n}{c_n} x^k \\
= & \sum_{n\geq 0} \left[\frac{t^n}{c_n}\right] d(t) \frac{t^n}{c_n} \sum_{k\geq 0} \frac{(xh(t))^k}{c_k} \\
= & d(t)g(xh(t)),
\end{aligned}$$

which implies (1.1). Conversely, from (1.1), one may obtain (1.5). ∎

[33] defined a type of generalized Riordan arrays in terms of a given non-zero sequence $\{c_n\}$ with $c_0 = 1$ using expression (1.5). Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the Riordan array $D = (d_{n,k})_{n,k\in\mathbb{N}}$ having

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] d(t)\frac{h(t)^k}{c_k} = \frac{c_n}{c_k}[t^n]d(t)\,(h(t))^k \tag{1.6}$$

or, in other words, having $\frac{c_n}{c_k}d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k. For the relationship of the $(1)-$Riordan arrays and $(c)-$Riordan arrays, we have the following result.

Theorem 1.2 *A lower triangular array $[d_{n,k}]$ is a (c)-Riordan array if and only if the array $(c_k d_{n,k}/c_n)$ is a (1)-Riordan array. Equivalently, $[d_{n,k}] = (d(t), h(t))$ can be written as*

$$[d_{n,k}] = D[[t^n]d(t)(h(t))^k]_{n\geq k\geq 0}D^{-1}, \tag{1.7}$$

where $D = diag(c_0, c_1, c_2, \ldots)$.

Proof: From (1.6), if $(d_{n,k})$ is a $(c)-$Riordan array, then

$$\frac{c_k}{c_n}d_{n,k} = [t^n]\,d(t)\,(h(t))^k\,,$$

which implies $(c_k d_{n,k}/c_n)$ is a $(1)-$Riordan array. Conversely, if $(c_k d_{n,k}/c_n)$ is a $(1)-$Riordan array $(d(t), h(t))$, then

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] d(t)\frac{(h(t))^k}{c_k}.$$

Thus using the argument in the proof of Theorem 1.1, we have the bivariate GF of lower triangular matrix $(d_{n,k})$

$$\sum_{0\leq k\leq n} d_{n,k}\frac{t^n}{c_n}x^k = d(t)g(xh(t)),$$

which means $(d_{n,k})$ is a $(c)-$Riordan array. ∎

From now on, we denote by $(d(t), h(t))$, $D^{-1}[[t^n]d(t)(h(t))^k]D$, or $[d_{n,k} = (c_n/c_k)[t^n]d(t)(h(t))^k]_{n\geq k\geq 0}$ a (c)-Riordan array generated by $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$ associated with the (c)-GPS $g(t) = \sum_{k\geq 0} t^k/c_k$.

Rogers [19] introduced the concept of the A-*sequence* for the classical Riordan arrays; Merlini et al. [16] introduced the related concept of the Z-*sequence* and showed that these two concepts, together with the element $d_{0,0}$, completely characterize a proper classical Riordan array. In [14], the authors consider the characterization of Riordan arrays by means of the A- and Z-sequences. [14] also shows how the A- and Z-sequences of the product of two Riordan arrays are derived from those of the two factors; similar results are obtained for the inverse. This paper aims to study (c)-Riordan arrays through their A- and Z-sequence characterization, simply referred to as *sequence characterization*. Thus, Theorem 1.2 can be applied to find out the characterization of (c)-Riordan arrays from the characterization of classical Riordan arrays.

The generating functions (GFs) of the Gegenbauer-Humbert-type polynomials defined by (see, for example, [6], Gould [9], and Shiue, Hsu and one of the authors [13]).

$$\Phi(t) = (1 - mxt + yt^m)^{-\lambda} = \sum_{k=0}^{\infty} P_k^{m,y,\lambda}(x)t^k. \tag{1.8}$$

We consider $P_k^{m,y,\lambda}(x)$ as follows

$$\begin{aligned}
&P_k^{2,1,1}(x) = U_k(x),\ \textit{Chebyshev polynomial of the 2nd kind},\\
&P_k^{2,1,1/2}(x) = \psi_k(x),\ \textit{Legendre polynomial},\\
&P_k^{2,-1,1}(x) = P_{k+1}(x),\ \textit{Pell polynomial},\\
&P_k^{2,-1,1}(x/2) = F_{k+1}(x),\ \textit{Fibonacci polynomial},\\
&P_k^{2,2,1}(x/2) = \Phi_{k+1}(x),\ \textit{Fermat polynomial of the 1st kind},
\end{aligned}$$

where $F_{k+1} = F_{k+1}(1)$ is the Fibonacci number. From (1.8) we immediately have

$$\sum_{k=0}^{\infty} P_k^{m,y,\lambda}(x)t^k = \frac{1}{(1-mxt+yt^m)^\lambda} = \frac{1}{(1+yt^m)^\lambda}\left[\frac{1}{(1-mxt)^\lambda}\right]_{t=t/(1+yt^m)}. \tag{1.9}$$

Since

$$\frac{1}{(1-mxt)^\lambda} = \sum_{k\geq 0}\binom{k+\lambda-1}{k}x^k,$$

from Theorem 1.2, we have

Corollary 1.3 *The generating functions of the Gegenbauer-Humbert-type polynomials defined by (1.8) can be represented by (c)-Riordan arrays*

$$D^{-1}\left(\frac{1}{(1+yt^m)^\lambda}, \frac{t}{1+yt^m}\right)D,$$

where the matrices are

$$D = diag\left(1, 1/\lambda, 1/\binom{\lambda+1}{2}, 1/\binom{\lambda+2}{3}, \ldots\right).$$

We will establish a one-to-one correspondence between the set of Gegenbauer-Humbert-type polynomial sequences and the set of (c)-Riordan arrays in next section (see Theorem 2.4).

[15] presented a relationship between the convolution family and the power of series. From the result, one may find the convolution polynomial sequence including the binomial sequence make an important rule in the construction of Riordan arrays (Although the paper itself does not mention the concept of Riordan arrays). Let us give the definition of (c)-convolution polynomial sequence first.

Definition 1.2 *A family of polynomials $\{F_k(x)\}_{k\geq 0}$ forms a convolution family if $F_n(x)$ posses degree $\leq n$ and if the convolution condition*

$$F_n(x+y) = \sum_{k=0}^{n} F_k(x)F_{n-k}(y) \tag{1.10}$$

holds for all x and y and for all $n \geq 0$. We say $\{F_k(x)\}_{k\geq 0}$ forms a convolution family associated with a (c)-sequence $\{c_k\}_{k\geq 0}$ ($c_0 = 1$, $c_k > 0$ for every $k > 0$) if $\{F_k(x)\}_{k\geq 0}$ is a convolution family and $F_k(x) = \tilde{F}_k(x)/c_k$, namely, $\{\tilde{F}_k(x)\}_{k\geq 0}$ satisfies

$$\tilde{F}_n(x+y) = \sum_{k=0}^{n} \frac{c_n}{c_{n-k}c_k}\tilde{F}_k(x)\tilde{F}_{n-k}(y).$$

In particular, if $\{c_k = k!\}_{k\geq 0}$, the corresponding $\{F_k = \tilde{F}_k(x)/k!\}_{k\geq 0}$ is called binomial sequence.

As examples (see e.g., P. 87 in [32]), power x^n, falling factorial $(x)_n$, Abel polynomial $x(x-na)^{n-1}$ ($a \neq 0$), generating functions $\sum_{k=0}^n s(n,k)(x)_k$ and $\sum_{k=0}^n S(n,k)x^k$ form convolution family, where $s(n,k)$ and $S(n,k)$ are respectively the Stirling numbers of the first kind and the second kind.

In [15], the convolution polynomials arise as the coefficients of the power x of a power series via its expansion. Although [15] did not mention it, the expansion actually defines the Bell polynomials. We now extend the result of [15] starting from the following definition of the (c)-Bell polynomials.

Definition 1.3 *The (c)-Bell polynomials associated with (c)-sequence $\{c_k\}_{k\geq 0}$ ($c_0 = 1$, $c_k > 0$ for every $k > 0$) are the polynomials $B^c_{n,k}(x_1, x_2, \ldots)$ in an infinite number of variables x_1, $x_2, \ldots$, defined by*

$$\frac{1}{c_k}\left(\sum_{m\geq 1} x_m \frac{z^m}{c_m}\right)^k = \sum_{n\geq k\geq 0} B^c_{n,k}(x_1, x_2, \ldots)\frac{z^n}{c_n}, \tag{1.11}$$

in which

$$B^c_{n,k}(x_1,x_2,\ldots)=\sum_{\pi(n)}\frac{n!}{k_1!k_2!\cdots}\left(\frac{x_1}{c_1}\right)^{k_1}\left(\frac{x_2}{c_2}\right)^{k_2}\ldots, \tag{1.12}$$

where $\pi(n)$ denotes a partition of n with $k_1+2k_2+\cdots=n$; k_i is the number of parts of size i. In addition, $k_1+k_2+\cdots=k$, the number of parts of partition.

In particular, if $\{c_k=k!\}_{k\geq 0}$, then the (c)-Bell polynomials are the exponential Bell polynomials $B_{n,k}(x_1,x_2,\ldots)$. For $\{c_k=1\}_{k\geq 0}$ and $\{c_k=1/k\}_{k>0}$ with $c_0=1$, the corresponding Bell polynomials are called, respectively, the ordinary and Dirichlet Bell polynomials.

Similar to [6] (see P. 133), $B^c_{n,k}$ can be also defined by the formal double series expansion of $g(x)=\sum_{k\geq 0}x^k/c_k$

$$\begin{aligned}\Phi &\equiv \Phi(z,x):=g\left(x\sum_{m\geq 1}x_m\frac{z^m}{c_m}\right)=\sum_{n,k\geq 0}B^c_{n,k}\frac{z^n}{c_n}x^k\\ &= 1+\sum_{n\geq 1}\frac{z^n}{c_n}\left\{\sum_{k=1}^n x^kB^c_{n,k}(x_1,x_2,\ldots)\right\}.\end{aligned} \tag{1.13}$$

Bell polynomials are a wide class of polynomials related to many well-known numbers such as (see e.g., [6], P. 135) Stirling numbers of the first kind, $|s(n,k)|=B_{n,k}(0!,1!,2!,\ldots)$, and the second kind, $S(n,k)=B_{n,k}(1,1,1\ldots)$, Lah number, $\binom{n-1}{k-1}n!/k!=B_{n,k}(1!,2!,3!,\ldots)$, Idempotent number, $\binom{n}{k}k^{n-k}=B_{n,k}(1,2,3,\ldots)$, etc.

By using (c)-Bell polynomials, we can rewrtite $d_{n,k}$, the entries of the Riordan array $(d(t),h(t))$ shown in (1.5), as

$$d_{n,k}=\sum_{\ell=k}^n\frac{c_n}{c_\ell c_{n-\ell}}\alpha_{n-\ell}B^c_{\ell,k}(h_1,h_2,\ldots), \tag{1.14}$$

where $\{\alpha_m\}_{m\geq 0}$ and $\{h_n\}_{n\geq 1}$ are the set of coefficients of (c)-power series $d(t)$ and $h(t)$, respectively. In fact, from (1.5) and (1.11) we have

$$\begin{aligned}d_{n,k} &= \left[\frac{t^n}{c_n}\right]d(t)\frac{(h(t))^k}{c_k}=\left[\frac{t^n}{c_n}\right]\sum_{m\geq 0}\alpha_m\frac{t^m}{c_m}\sum_{\ell\geq k\geq 0}B^c_{\ell,k}\frac{t^\ell}{c_\ell}\\ &= [t^n]c_n\sum_{n\geq 0}\sum_{\ell=k}^n\frac{\alpha_{n-\ell}}{c_\ell c_{n-\ell}}B^c_{\ell,k}(h_1,h_2,\ldots,h_{n-\ell-k+1})t^n,\end{aligned}$$

which implies (1.14). A special case of $d_{n,k}=k!\sum_{\ell=k}^n\binom{n}{r}\alpha_{n-\ell}B^c_{\ell,k}(h_1,h_2,\ldots)$ for $c_n=n!$ $(n\geq 0)$ was given in [12].

Denote

$$s_n(x):=\sum_{k=0}^n x^kd_{n,k}, \tag{1.15}$$

where $d_{n,k}$ is the (c)-Riordan array. Then (1.1) can be written as

$$F(t,x)=d(t)g(xh(t))=\sum_{n\geq 0}s_n(x)\frac{t^n}{c_n}. \tag{1.16}$$

And we call $s_n(x)$ the Sheffer polynomials for $(d(t),h(t))$ with respective to sequence $\{c_k\}_{k\geq 0}$. When $d(t)=1$, we say the corresponding Sheffer for $(1,h(t))$ is the (c)-associated sequence, and the Sheffer for $(d(t),t)$ the (c)-Appell sequence for $d(t)$. Here, the definition of Sheffer polynomial for $(d(t),h(t))$ is different from those defined in [20, 22], where the Sheffer is for $(1/d(\bar{h}(t)), \bar{h}(t))$ and $\bar{h}(t)$ is the composite inversion of $h(t)$. Those two Sheffer polynomials consist of a pair that was studied in [12]. From (1.14), we immediately have

$$s_n(x) = \sum_{k=0}^{n} x^k \sum_{\ell=k}^{n} \frac{c_n}{c_\ell c_{n-\ell}} \alpha_{n-\ell} B^c_{\ell,k}(h_1, h_2, \ldots). \tag{1.17}$$

We will establish more connections between (c)-Riordan arrays and (c)-Bell polynomials. In [29], the Riordan arrays are used to construct some identities related to the Stirling numbers. We will study the problem in a more deeper way by extending the sum problems to more general case such as Bell's polynomials and Bell's numbers based on the above observation.

[15] presents the following characterization of all convolution family defined in Definition 1.2. Here, we give a modified expression via a different approach in its proof by using the Bell polynomials.

Theorem 1.4 *A family of polynomials $\{F_k(x)\}_{k\geq 0}$ forms a convolution family if and only if there exists a delta series $f(z)$ (i.e., $f \in \mathcal{F}_1$) with $f(z) = \sum_{n\geq 1} f_n z^n/n!$ ($f_1 \neq 0$), such that*

$$F_n(x) = \sum_{k=0}^{n} x^k B_{n,k}\left(\frac{f_1}{1!}, \frac{f_2}{2!}, \ldots, \frac{f_{n-k+1}}{(n-k+1)!}\right). \tag{1.18}$$

Proof: Let $F(z) \in \mathcal{F}_0$ with $F(0) = 1$. [15] proves that polynomials

$$F_n(x) = [z^n]F(z)^x$$

form a convolution family. Conversely, every convolution family can be constructed in this way or is identically zero. Let

$$f(z) = \ln F(z) = \sum_{n\geq 1} f_n \frac{z^n}{n!}.$$

Then $F(z)^x = \exp(xf(z))$. Hence,

$$F_n(x) := [z^n]F(z)^x = \sum_{k=0}^{n} x^k B_{n,k}\left(\frac{f_1}{1!}, \frac{f_2}{2!}, \ldots, \frac{f_{n-k+1}}{(n-k+1)!}\right),$$

which proves the theorem. ∎

We shall show how the (c)-Bell polynomials are used to perform the elements of a (c)-Riordan array. Hence, the identities of (c)-Bell polynomials and those of the entries of the corresponding (c)-Riordan array can be interchanged each other.

It is immediate can be shown from Definition 1.1 with $x = 1$ that the usual row-by-column product of two (c)-Riordan arrays associated with the same sequence $\{c_k > 0\}_{k\geq 0}$ ($c_0 = 1$) is also a (c)-Riordan array associated with the sequence:

$$\begin{aligned}(d(t),\, h(t)) * (f(t),\, g(t)) &= D^{-1}[[t^n]d(t)(h(t))^k]DD^{-1}[[t^n]f(t)(g(t))^k]D \\ &= D^{-1}[[t^n]d(t)f(h(t))(g(h(t))^k]D = (d(t)f(h(t)),\, g(h(t)))\end{aligned} \tag{1.19}$$

because $[[t^n]d(t)(h(t))^k] \cdot f(t)(g(t))^k = d(t)f(h(t))(g(h(t)))^k$. The Riordan array $I = (1,\, t)$ is zero everywhere except that it contains all 1's on the main diagonal; it is easily seen that I acts as an identity for this product, that is, $(1,\, t) * (d(t),\, h(t)) = (d(t),\, h(t)) * (1, t) = (d(t),\, h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(\overline{h}(t))},\, \overline{h}(t)\right) \tag{1.20}$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper (c)-Riordan arrays is a group. And there is a semidirect product decomposition of the group of (c)-Riordan arrays $R \approx \mathcal{A} \rtimes \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are shown below, since $(d(t), h(t)) = \left(\frac{td(t)}{h(t)}, t\right)\left(\frac{h(t)}{t}, h(t)\right)$. Particular subgroups of $\mathcal{R}$ including the mentioned $\mathcal{A}$ and $\mathcal{B}$ are important and have been considered in the literature:

- the set $\mathcal{A}$ of *(c)-Appell arrays*, that is the (c)-Riordan arrays $D = (d(t),\, h(t))$ for which $h(t) = t$; it is an invariant subgroup and is isomorphic to $\mathcal{F}_0$, the group of f.p.s.'s of order 0, with the usual product as group operation;

- the set $\mathcal{L}$ of *(c)-Lagrange arrays*, that is the (c)-Riordan arrays $D = (d(t),\, h(t))$ for which $d(t) = 1$; it is also called the *associated subgroup*; it is isomorphic to $\mathcal{F}_1$, the group of f.p.s.'s of order 1, with composition as group operation;

- the set $\mathcal{B}$ of *(c)-Bell* or *(c)-renewal arrays*, that is the (c)-Riordan arrays $D = (d(t),\, h(t))$ for which $h(t) = td(t)$; it is the set originally considered by Rogers in [19];

- the set $\mathcal{C}$ of *(c)-Checkerboard arrays*, that is, the (c)-Riordan arrays $D = (d(t),\, h(t))$ for which $d(t)$ is an even function and $h(t)$ is an odd function.

- the set $\mathcal{H}$ of *(c)-hitting time arrays*, that is, the (c)-Riordan arrays $D = (d(t), h(t))$ for which $d(t) = \frac{th'(t)}{h(t)}$; it is the subgroup with the usual Riordan product defined by Peart and Woan in [18].

The paper has the following structure; in Section 2 we consider the sequence characterization of (c)-Riordan arrays, proving its main properties and its relation to the previously defined subgroups; in addition, we shall show how the sequence approach characterizes the product of two (c)-Riordan arays and the inversion of a (c)-Riordan array. Section 3 will discuss the convolution family and (c)-Bell polynomials and sums and (c)-Riordan arrays related to those concepts. In Section 4 we show the relationship between the (c)-Bell polynomials and the entries of (c)-Riordan arrays.

2 The sequence characterization of (c)-Riordan arrays

In [19], Rogers states that for every proper Riordan array $D = (d(t),\, h(t))$ there exists a sequence $A = (a_k)_{k\in\mathbb{N}}$ such that for every $n, k \in \mathbb{N}$ we have:

$$[t^{n+1}]d(t)(h(t))^{k+1} = a_0[t^n]d(t)(h(t))^k + a_1[t^n]d(t)(h(t))^{k+1} + a_2[t^n]d(t)(h(t))^{k+2} + \cdots = \sum_{j=0}^{\infty} a_j[t^n]d(t)(h(t))^{k+j} \tag{2.21}$$

where the sum is actually finite since $d_{n,k} = 0,\ \forall k > n$. A proof of this fact was given in Sprugnoli [31], and we can reformulate it according to the modified definition of a (c)-Riordan array used in the present paper.

Theorem 2.1 *An infinite lower triangular array $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ is a (c)-Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ relation*

$$\frac{c_{k+1}}{c_{n+1}} d_{n+1,k+1} = \frac{c_0}{c_n} a_0 d_{n,k} + \frac{c_1}{c_n} a_1 d_{n,k+1} + \frac{c_2}{c_n} a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} \frac{c_{k+j}}{c_n} a_j d_{n,k+j} \tag{2.22}$$

holds. In addition, the generating function $A(t)$ of $A-$ sequence satisfies $tA(h(t)) = h(t)$.

Proof: Using Theorem 1.1 and expression (2.21), we obtain formula (2.22) immediately. From a similar argument of the proof of Theorem 2.1, we have $tA(h(t)) = h(t)$, where $A(t)$ is the generating function of $A-$ sequence. ∎

The sequence $A = (a_k)_{k\in\mathbb{N}}$ is the A-sequence of the Riordan array $D = (d(t),\, h(t))$ and it only depends on $h(t)$. In fact, as we have shown during the proof of the theorem, we have:

$$h(t) = tA(h(t)) \qquad \text{or} \qquad A(y) = \left[\frac{h(t)}{t} \;\middle|\; y = h(t)\right] = \left[\frac{y}{t} \;\middle|\; y = h(t)\right] \tag{2.23}$$

and this uniquely determines A when $h(t)$ is given and vice versa, $h(t)$ is uniquely determined when A is given.

Although the two functions $d(t)$ and $A(t)$ completely characterize a proper Riordan array, we are mainly interested in another type of characterization. Let us consider the following result:

Theorem 2.2 *Let $(d_{n,k})_{n,k\in\mathbb{N}}$ be any infinite, lower triangular array with $d_{n,n} \neq 0$, $\forall n \in \mathbb{N}$ (in particular, let it be a proper GRA); then a unique sequence $Z = (z_k)_{k\in\mathbb{N}}$ exists such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, i.e.:*

$$\frac{1}{c_{n+1}} d_{n+1,0} = \frac{1}{c_n} z_0 d_{n,0} + \frac{c_1}{c_n} z_1 d_{n,1} + \frac{c_2}{c_n} z_2 d_{n,2} + \cdots = \frac{1}{c_n} \sum_{j=0}^{\infty} c_j z_j d_{n,j}. \tag{2.24}$$

Proof: Let $z_0 = d_{1,0}/(c_1 d_{0,0})$. Now we can uniquely determine the value of z_1 by expressing $d_{2,0}$ in terms of the elements in row 1, i.e.:

$$\frac{1}{c_2} d_{2,0} = \frac{1}{c_1} \left(z_0 d_{1,0} + c_1 z_1 d_{1,1} \right) \qquad \text{or} \qquad z_1 = \frac{c_1^2 d_{0,0} d_{2,0} - c_2 d_{1,0}^2}{c_1^2 c_2 d_{0,0} d_{1,1}}.$$

Similarly, we can determine z_2 by expressing $d_{3,0}$ in terms of the elements in row 2 with the substitution of the obtained values of z_0 and z_1. By using (2.24) and the same processing succursively, we determine the elements of the sequence Z one-by-one uniquely. ∎

The sequence Z is called the *Z-sequence* of the (c)-Riordan array; it characterizes column 0, except for the element $d_{0,0}$. Therefore, we can say that the triple $(d_{0,0}, A(t), Z(t))$ completely characterizes a proper Riordan array. Usually, we denote $d_{0,0}$ as δ.

To see how the Z-sequence is obtained by starting with the usual definition of a Riordan array, let us prove the following result.

Theorem 2.3 *Let $(d(t), h(t))$ be a proper Riordan array and let $Z(t)$ be the generating function of the corresponding Z-sequence. We have:*

$$d(t) = \frac{\delta}{1 - tZ(h(t))} \qquad \text{or} \qquad Z(y) = \left[\frac{d(t) - \delta}{t d(t)} \;\Big|\; t = \overline{h}(y) \right]. \tag{2.25}$$

Proof: By the preceding theorem, the Z-sequence exists and is unique. Therefore, equation (2.24) is valid for every $n \in \mathbb{N}$, and we can pass to its generating function form. Since $d(t)h(t)^k$ is the generating function for column k, we have:

$$\frac{d(t) - d_{0,0}}{t} = z_0 d(t) + z_1 d(t) h(t) + z_2 d(t) h(t)^2 + \cdots = d(t)(z_0 + z_1 h(t) + z_2 h(t)^2 + \cdots) = d(t) Z(h(t)).$$

By solving this equation in $d(t)$, we obtain the first relation of (2.25). The second relation is an immediate consequence, when we set $y = h(t)$. ∎

Example 2.1 *Riordan arrays were introduced as a generalization of the Pascal triangle P. Actually, we have*

$$P = \left(\frac{1}{1-t}, \frac{t}{1-t} \right);$$

as can be easily checked $P_{n,k} = \binom{n}{k}$, the A-sequence is $(1, 1, 0, 0, \ldots)$ and the Z-sequence is $(1, 0, 0, 0, \ldots)$. See, e.g., [29].

From Corollary 1.3, we have the following result.

Theorem 2.4 *There exists a one-to-one correspondence between the set of Gegenbauer-Humbert-type polynomial sequences and the set of (c)-Riordan arrays*

$$D^{-1} \left(\frac{1}{(1+yt^m)^\lambda}, \frac{t}{1+yt^m} \right) D,$$

where the matrices

$$D = diag\left(1, 1/\lambda, 1/\binom{\lambda+1}{2}, 1/\binom{\lambda+2}{3}, \ldots \right).$$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	−1	0	1				
3	0	−2	0	1			
4	1	0	−3	0	1		
5	0	3	0	−4	0	1	
6	−1	0	6	0	−5	0	1

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	1	0	1				
3	0	2	0	1			
4	1	0	3	0	1		
5	0	3	0	4	0	1	
6	1	0	6	0	5	0	1

Table 1: The Chebyshev triangle U and the Pell triangle P

The generating functions of corresponding $A-$ and $Z-$ sequences are

$$A(s) = \frac{s}{t}, \quad s = \frac{t}{1+yt^m}$$

and

$$Z(s) = \frac{1-(1+yt^m)^\lambda}{t}, \quad t = \bar{h}(s),$$

where $\bar{h}$ is the compositional inverse of $h(t) = \frac{t}{1+yt^m}$. If $m=2$, then $t = \bar{h}(s) = \frac{1-\sqrt{1-4ys^2}}{2ys}$.

Proof: The proof of the one-to-one correspondence is straightforward. The generating functions $A(t)$ and $Z(t)$ of the $A-$ and $Z-$ sequences can be constructed using theorems 2.1 and 2.3, respectively. For the case $m=2$, another inverse function of $h(t)$, $\frac{1+\sqrt{1-4ys^2}}{2ys}$, cannot be considered because we need $h(0) = \bar{h}(0) = 0$. ∎

Example 2.2 *For the Chebyshev polynomials of the 2nd kind, $U_k(x)$, we have $U_k(x) = P_k^{2,1,1}(x)$. From (1.8), by noting $m=2$ and $y=\lambda=1$, we may find the corresponding c sequence is $c=(1,1,1,\ldots)$ and the (c)-Riordan array is $\left(\frac{1}{1+t^2}, \frac{t}{1+t^2}\right)$ which entries are shown in table 1.*

It is easy to see the A-sequence is $(1,0,-1,0,-1,0,-2,0,-5,\ldots)$ and the Z-sequence is $(0,-1,0,-1,0,-2,0,-5,\ldots)$, or equivalently,

$$A(s) = \frac{2s^2}{1-\sqrt{1-4s^2}}, \quad \textit{and} \quad Z(s) = \frac{\sqrt{1-4s^2}-1}{2s}.$$

Similarly, Pell polynomial sequence $\{P_k^{2,-1,1}(x)\}_{k\geq 0}$ has a triangle representation, the Pell triangle

$$P = \left(\frac{1}{1-t^2}, \frac{t}{1-t^2}\right),$$

which possesses $A-$ sequence $(1,0,1,0,-1,0,2,0,-5,\ldots)$ and $Z-$ sequence $(0,1,0,-1,0,2,0,-5,\ldots)$. And the generating functions of $A-$ sequence and $Z-$ sequence are

$$A(s) = \frac{2s^2}{\sqrt{1+4s^2}-1}, \quad \textit{and} \quad Z(s) = \frac{\sqrt{1+4s^2}-1}{2s},$$

respectively.

Example 2.3 *For the Legendre polynomials, $P_k^{2,1,1/2}(x) = \psi_k(x)$, noting $m=2$, $y=1$, $\lambda=1/2$, and*

$$\binom{1/2+k-1}{k} = \frac{(1/2+k-1)_k}{k!} = \frac{(1/2+k-1)(1/2+k-2)\cdots(1/2)}{k!}$$

for $k=1,2,\ldots$, we obtain the corresponding c sequence

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	$-\frac{1}{2}$	0	1				
3	0	$-\frac{3}{2}$	0	1			
4	$\frac{3}{8}$	0	$-\frac{5}{2}$	0	1		
5	0	$\frac{15}{8}$	0	$-\frac{7}{2}$	0	1	
6	$-\frac{5}{16}$	0	$\frac{35}{8}$	0	$-\frac{9}{2}$	0	1

Table 2: The triangle $(1/\sqrt{1+t^2}, t/(1+t^2))$ used to constructed the Legendre triangle

$$c = (1, 2, \frac{8}{3}, \frac{16}{5}, \ldots)$$

and

$$(d(t), h(t)) = \left(\frac{1}{\sqrt{1+t^2}}, \frac{t}{1+t^2}\right),$$

which generates the Legendre triangle $L = D^{-1}(d(t), h(t))D$, *where* $D = diag(1, 2, 8/3, 16/5, \ldots)$ *and* $(d(t), h(t))$ *is shown in Table 2. The A− and Z− sequences are* $(1, 0, -1, 0, -1, 0, -2, 0, -5, \ldots)$ *and* $(0, -\frac{1}{2}, 0, -\frac{3}{8}, 0, -\frac{11}{16}, 0, -\frac{211}{128}, \ldots)$, *which have the generating functions*

$$A(s) = \frac{2s^2}{1-\sqrt{1-4s^2}}, \quad \text{and} \quad Z(s) = \frac{1-\sqrt{1+t^2}}{t}, \quad \text{where } t = \frac{1-\sqrt{1-4s^2}}{2s},$$

respectively.

Example 2.4 *Clearly, in the Pascal triangle we have* $Z_P = (1, 0, 0, 0, \ldots)$ *and so* $Z_P(t) = 1$. *Formula (2.25) gives immediately* $d_P(t) = 1/(1-t)$. *The Z-sequence for the Fibonacci triangle is a bit more complicated. If we set* $y = h_F(t) = (1-\sqrt{1-4t})/2$, *we can invert the function* $h_F(t)$ *and find* $t = y - y^2$. *We now substitute this expression in:*

$$\frac{d_F(t)-\delta_F}{td_F(t)} = \left(\frac{1}{1-t-t^2} - 1\right) \cdot \frac{1-t-t^2}{t} = 1+t$$

and find $Z_F(t) = 1 + t - t^2$; *the Z-sequence is* $Z_F = (1, 1, -1, 0, 0, \ldots)$.

Obviously, we can characterize the main subgroups of $\mathcal{R}$ by means of their A- and/or Z-sequences as the same as Theorems 2.4, 2.5, and 6.3 in [14]. Here, we list them as follows.

Theorem 2.5 *Let us consider a Riordan array* $(d(t), h(t))$. *(1)* $(d(t), h(t))$ *belongs to the (c)-Appell subgroup* $\mathcal{A}$ *if and only if its A-sequence satisfies* $A(t) = 1$. *Besides, in that case, we also have:*

$$Z(t) = \frac{d(t)-\delta}{td(t)}.$$

(2) $(d(t), h(t))$ *belongs to the (c)-Lagrange subgroup* $\mathcal{L}$ *if and only if its Z-sequence satisfies* $Z(t) = 0$ *and* $\delta = 1$. *(3)* $(d(t), h(t))$ *belongs to the (c)-checkerboard subgroup* $\mathcal{C}$ *if and only if the generating functions* $A(t)$ *of its A-sequence is even and* $Z(t)$ *of its Z-sequence is odd.*

For the Bell subgroup of renewal arrays we have a more elaborate result:

Theorem 2.6 *Let* $(d(t), h(t))$ *be a (c)-Riordan array in which* $d(0) = h'(0) \neq 0$. *Then* $d(t) = h(t)/t$ *if and only if:* $A(y) = \delta + yZ(y)$.

The reader can check this result for the Pascal triangle, which is a renewal array.

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	0	1						
3	0	0	0	1					
4	-1	0	0	0	1				
5	0	-2	0	0	0	1			
6	-1	0	-3	0	0	0	1		
7	0	-2	0	-4	0	0	0	1	
8	0	0	-3	0	-5	0	0	0	1

$n\backslash k$	0	1	2	3	4	5	6		
0	1								
1	0	1							
2	0	0	1						
3	0	0	0	1					
4	-1	0	0	0	1				
5	0	-2	0	0	0	1			
6	1	0	-3	0	0	0	1		
7	0	2	0	-4	0	0	0	1	
8	0	0	3	0	-5	0	0	0	1

Table 3: The triangles $Q = P * U$ and $G = U * P$

Theorem 2.7 *A monic Riordan array $D = (d(t), h(t))$ belongs to the (c)-hitting-time subgroup if and only if $Z(t) = A'(t)$, where $A(t)$ and $Z(t)$ are the generating functions of its A- and Z-sequences.*

According to Formula (1.19), the product of two (c)-GRAs, $(d_1(t), h_1(t))$ and $(d_2(t), h_2(t))$, associated with sequence $\{c_n \neq 0\}$ and $c_0 = 1$ is a (c)-GRA associated with $\{c_n\}$. The A- and Z-sequences of the product depend on the analogous sequences of the two factors are exactly equal to those given in [14] for the classical Riordan Arrays. In order to describe the results, let us consider two proper (c)-GRAs $(d_1(t),\ h_1(t))$ and $(d_2(t),\ h_2(t))$ and their product:

$$(d_1(t)d_2(h_1(t)),\ h_2(h_1(t)))$$

so that:

$$d_3(t) = d_1(t)d_2(h_1(t)) \qquad \text{and} \qquad h_3(t) = h_2(h_1(t)).$$

We can now give the formula for the A-sequence and the Z-sequence of the product of two GRAs (see [14]):

Theorem 2.8 *The A-sequence and the Z-sequence, $A_3(t)$ and $Z_3(t)$ are given by:*

$$A_3(t) = A_2(t)A_1\left(\frac{t}{A_2(t)}\right).$$

and

$$Z_3(t) = \left(1 - \frac{t}{A_2(t)}Z_2(t)\right)Z_1\left(\frac{t}{A_2(t)}\right) + A_1\left(\frac{t}{A_2(t)}\right)Z_2(t).$$

Since the proofs are the same as [14], we omit them.

Example 2.5 *Let us perform the products $Q = P * U$ and $G = U * P$. The $A-$ sequence and $Z-$ sequence of Q are $A = (1, 0, 0, 0, -1, 0, -1, \ldots)$ and $Z = (0, 0, 0, -1, 0, -1, 0, \ldots)$, respectively. The $A-$ sequence and $Z-$ sequence of G are $A = (1, 0, 0, 0, -1, 0, 1, \ldots)$ and $Z = (0, 0, 0, -1, 0, 1, 0, \ldots)$, respectively.*

Formula (1.20) allows us to find the inverse of any (c)-GRA given by the pair $(d(t),\ h(t))$, provided we are able to find the compositional inverse $\overline{h}(t)$ of $h(t)$.

The sequence characterization of the inverse of a (c)-GRA $(d(t),\ h(t)) = D^{-1}[[t^n]d(t)(h(t))^k]D$, where $D = diag(c_0, c_1, c_2, \ldots)$, can be found using a pattern similar to the one used in [14]. In fact,

$$\begin{aligned}(d(t), h(t))^{-1} &= \left(D^{-1}[[t^n]d(t)(h(t))^k]D\right)^{-1} = D^{-1}[[t^n]d(t)(h(t))^k]^{-1}D \\ &= D^{-1}\left[[t^n]\frac{1}{d(\bar{h}(t))}(\bar{h}(t))^k\right]D = \left(\frac{1}{d(\bar{h}(t))}, \bar{h}(t)\right).\end{aligned}$$

Let $A(t)$ and $Z(t)$ be the generating functions of the A- and Z-sequences of $(d(t),\ h(t))$ and let us denote by $d^*(t)$, $h^*(t)$, $A^*(t)$ and $Z^*(t)$ the corresponding functions for the inverse $(d(t),\ h(t))^{-1}$. We immediately observe that, by Formula (1.20), $h^*(t) = \overline{h}(t)$. Now we have:

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	1	0	1				
3	0	2	0	1			
4	2	0	3	0	1		
5	0	5	0	4	0	1	
6	5	0	9	0	5	0	1

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	$\frac{1}{2}$	0	1				
3	0	$\frac{3}{2}$	0	1			
4	$\frac{7}{8}$	0	$\frac{5}{2}$	0	1		
5	0	$\frac{27}{8}$	0	$\frac{7}{2}$	0	1	
6	$\frac{33}{16}$	0	$\frac{55}{8}$	0	$\frac{9}{2}$	0	1

Table 4: The inverse triangles U^{-1} and $(1/\sqrt{1+t^2}, t/(1+t^2))^{-1}$

Theorem 2.9 *The A-sequence and the Z-sequence of the inverse (c)-GRA* $(d(t),\, h(t))^{-1}$ *are:*

$$A^*(y) = \left[\frac{1}{A(t)} \;\Big|\; y = \frac{t}{A(t)}\right] = \left[\frac{y}{t} \;\Big|\; y = \frac{t}{A(t)}\right]$$

and

$$Z^*(y) = \left[\frac{\delta - d(y)}{\delta t} \;\Big|\; y = \overline{h}(t)\left(= \frac{t}{A(t)}\right)\right] = \left[\frac{-yZ(t)}{t(1-yZ(t))} \;\Big|\; y = \frac{t}{A(t)}\right].$$

Example 2.6 *The initial parts of the inverses of (c)-GRAs* U^{-1} *and* $(1/\sqrt{1+t^2}, t/(1+t^2))^{-1}$ *are shown in Table 4. The inverse of Legendre triangle L is* $D^{-1}(1/\sqrt{1+t^2}, t/(1+t^2))^{-1}D$*, where* $D = diag(1, 2, 8/3, 16/5, \ldots)$*. The A− sequence and Z− sequence of* U^{-1} *are* $A = (1, 0, 1, 0, 0, \ldots)$ *and* $Z = (0, 1, 0, 0, 0, 0, \ldots)$*, respectively. The A− sequence and Z− sequence of* $(1/\sqrt{1+t^2}, t/(1+t^2))^{-1}$ *are* $A = (1, 0, 1, 0, 0, 0, \ldots)$ *and* $Z = (0, \frac{1}{2}, 0, \frac{1}{8}, 0, 0, -\frac{1}{16}, 0, \ldots)$*, respectively. It is easy to see both generating functions of A− sequences of* U^{-1} *and* $(1/\sqrt{1+t^2}, t/(1+t^2))^{-1}$ *are* $A(t) = 1 + t^2$*.*

For the Pascal triangle, the 0-th element of every row is 1, and every other element is computed by the A-sequence (1, 1, 0, 0, ...) corresponding to the basic recurrence of binomial coefficients. In the Pascal triangle the Z-sequence is particularly simple, [14] pointed out that this is not always the case. Sometimes it is more favorable another characterization of Riordan arrays:

Theorem 2.10 *A Riordan array* $D = (d(t),\, h(t))$ *is completely characterized by the function* $d(t)$ *and by its A-sequence.*

Proof: By Formula 2.23, the A-sequence determines the function $h(t)$. ∎

Using the above result and Theorem 3.1 we have the following corollary.

Corollary 2.11 *A subset of* $\mathcal{R}$ *characterized by* $A(t) = a_0 + a_1 t$ $(a_0, a_1 \in \mathbb{R})$ *is a subgroup.*

[14] also pointed out that an important point relative to the construction of a Riordan array using A-sequence, is our ability in finding the compositional inverse of a delta series, that is a formal power series $h(t) \in \mathcal{F}_1$. We may set $y = h(t)$ and solved this equation in $t = t(y)$, but this is not always feasible. In the simple case that $h(t)$ is a polynomial of third degree, technical problems can arise, and if the degree is greater than 4, the solution (in terms of radicals) can become at all impossible. In these cases, we can use the Lagrange Inversion Formula (LIF) to find out the coefficients of $\overline{h}(t)$. In fact, the LIF gives:

$$\overline{h}_n = \frac{1}{n}[t^{n-1}]\left(\frac{t}{h(t)}\right)^n$$

and we can compute the coefficients of the A-sequence (or vice versa):

Theorem 2.12 *Let* $D = (d(t),\, h(t))$ *be a (c)-Riordan array; then the coefficients of its A-sequence are given by:*

$$a_n = \frac{1}{n}[t^{n-1}]\frac{h_*'(t)}{h_*(t)^n} \qquad \text{and conversely} \qquad h_n = \frac{1}{n}[t^{n-1}]A(t)^n,$$

where $h_*(t) = h(t)/t$*.*

From this theorem, it follows $h_1 = a_0$, $h_2 = a_0 a_1$, $h_3 = a_0^2 a_2 + a_0 a_1^2$, and so on.

3 The convolution family and (c)-Bell polynomials

Theorem 3.1 *Let $\{F_n(x)\}_n$ be a convolution family associated with a (c)-sequence $\{c_k\}_{k\geq 0}$ ($c_0=1$, $c_k>0$ for every $k>0$) defined on I, $\mathbb{N}\subset I\subset\mathbb{R}$, with $F_0(x)\neq 0$. Then*

$$B^c_{n,k}\left(\frac{c_1}{c_0}F_0(x),\frac{c_2}{c_1}F_1(x),\ldots\right)=\frac{c_n}{c_kc_{n-k}}F_{n-k}(kx). \tag{3.26}$$

Proof: Let $\Phi(x,t)$ be the generating function of $\{F_n(x)\}_n$: $\Phi(x,z)=\sum_{n\geq 0}F_n(x)z^n/c_n$. Since $\{F_n(x)\}_n$ is a convolution family associated with $\{c_n\}_n$, we have

$$\Phi(x+y,z)=\Phi(x,z)\Phi(y,z),\qquad \Phi(kx,z)=(\Phi(x,z))^k\ (k\in\mathbb{N}_0).$$

In fact, using (1.10) yields

$$\begin{aligned}\Phi(x,z)\Phi(y,z)&=\sum_{n\geq 0}\sum_{k\geq 0}F_n(x)F_k(y)\frac{z^{n+k}}{c_nc_k}\\ &=\sum_{n\geq 0}\sum_{k=0}^{n}F_{n-k}(x)F_k(y)\frac{z^n}{c_{n-k}c_k}=\sum_{n\geq 0}F_n(x+y)\frac{z^n}{c_n}.\end{aligned}$$

Similarly, we have $\Phi(kx,z)=(\Phi(x,z))^k$ for $k\geq 0$. Thus,

$$\begin{aligned}\frac{1}{c_k}\left(z\Phi(x,z)\right)^k&=\frac{1}{c_k}z^k\Phi(kx,z)=\frac{1}{c_k}\sum_{n\geq 0}F_n(kx)\frac{z^{n+k}}{c_n}\\ &=\sum_{n\geq k}\frac{c_n}{c_kc_{n-k}}F_{n-k}(kx)\frac{z^n}{c_n}.\end{aligned} \tag{3.27}$$

On the other hand,

$$\begin{aligned}\frac{1}{c_k}\left(z\Phi(x,z)\right)^k&=\frac{1}{c_k}\left(\sum_{n\geq 0}F_n(x)\frac{z^{n+1}}{c_n}\right)^k\\ &=\frac{1}{c_k}\left(\sum_{n\geq 1}\frac{c_n}{c_{n-1}}F_{n-1}(x)\frac{z^n}{c_n}\right)^k\\ &=\sum_{n\geq k\geq 0}B^c_{n,k}\left(\frac{c_1}{c_0}F_0(x),\frac{c_2}{c_1}F_1(x),\ldots,\frac{c_{n-k+2}}{c_{n-k+1}}F_{n-k+1}(x),\right)\frac{z^n}{c_n}.\end{aligned} \tag{3.28}$$

Comparing the rightmost sides of (3.27) and (3.28), we immediately obtain (3.26). ∎

If $c_k=k!$ and $x=1$, we recover the result on the exponential Bell polynomials shown in Theorem 6 of [1]:

$$B_{n,k}(F_0(1),2F_1(1),3F_2(1),\ldots)=\binom{n}{k}F_{n-k}(k).$$

Example 3.1 *For binomial sequence $\{x^n\}_n$, (3.26) gives*

$$B^c_{n,k}\left(\frac{c_1}{c_0},\frac{c_2}{c_1}x,\frac{c_3}{c_2}x^2,\ldots\right)=\frac{c_n}{c_kc_{n-k}}(kx)^{n-k},$$

which includes the well-known identity

$$B_{n,k}(1,2x,3x^2,\ldots)=\binom{n}{k}(kx)^{n-k}$$

when $c_k=k!$.

Example 3.2 *Let $S(n,k)$ denote the Stirling number of the second kind, and denote*

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

Since $\{B_n(x)\}$ is a binomial sequence, from (3.26) we obtain the identity

$$B^c_{n,k}\left(\frac{c_1}{c_0}, \frac{c_2}{c_1}B_1(x), \ldots\right) = \frac{c_n}{c_k c_{n-k}} B_{n-k}(kx) = \frac{c_n}{c_k c_{n-k}} \sum_{j=0}^{n-k} S(n-k,j)(kx)^j,$$

where $B_0(x) \equiv 1$ is used. If $c_k = k!$, we get an extension of Corollary 7 of [1]:

$$B_{n,k}(1, 2B_1(x), 3B_2(x), \ldots) = \binom{n}{k} \sum_{j=0}^{n-k} S(n-k,j)(kx)^j.$$

Similarly, if $c_k = 1$ $(k \geq 0)$, $\left\{F_n(x) = \binom{x}{n}\right\}$ and $\left\{G_n(x,t) = \frac{x}{x+tn}\binom{x+tn}{n}\right\}$ $(t \geq 0)$ are two (c)-convolution families because of

$$F_n(x+y) = \sum_{k=0}^{n} F_k(x)F_{n-k}(y) \quad G_n(x+y) = \sum_{k=0}^{n} G_k(x)G_{n-k}(y).$$

Thus, from (3.26) we have

$$B^c_{n,k}\left(1, \binom{x}{1}, \binom{x}{2}, \ldots\right) = \binom{kx}{n-k}$$

and

$$B^c_{n,k}\left(1, \frac{x}{x+t}\binom{x+t}{1}, \frac{x}{x+2t}\binom{x+2t}{2}, \ldots\right) = \frac{x}{x+(n-k)t}\binom{x+(n-k)t}{n-k}.$$

Remark 3.1 The $G_n(x,t)$ shown above are the Hagen-Rothe type polynomials that figure into many of one author's papers starting with two papers [7] and [8] in 1956 and 1957 in Amer. Math. Monthly dealing with the generalized Vandermonde convoltions. Some writers have dubbed those polynomials as Gould polynomials.

Remark 3.2 We may use suitable (c)-Riordan arrays to obtain the well-known Gould's identity shown in (11) of [7]. From (1.1) with $x = 1$, one may obtain

$$\sum_{k=0}^{n} d_{n,k}\frac{g_k}{c_k} = [t^n]d(t)g(h(t)) \qquad (3.29)$$

because

$$[t^n]\, d(t)g(h(t)) = [t^n]\, d(t)\sum_{k\geq 0} g_k \frac{(h(t))^k}{c_k}$$
$$= \sum_{k\geq 0} g_k \frac{1}{c_k}[t^n]d(t)(h(t))^k.$$

Denote $f(t) := g(h(t))$. Then

$$g_k = \left[\frac{t^k}{c_k}\right] g(t) = c_k[t^k]f(y)\big|_{y=\overline{h}(t)},$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$, i.e., $y = \overline{h}(t)$ iff $t = h(y)$. Using Lagrange inverse formula (see, for instance, Theorem 5.1 in [34]) and noting $y = ty/h(y)$, we may calculate g_k by

$$g_k = \frac{c_k}{k}[y^{k-1}]\left\{f'(y)\left(\frac{y}{h(y)}\right)^k\right\} \tag{3.30}$$

Let $(d(t), h(t)) = (1+at)^{\alpha+n\beta}, t(1+at)^{-\beta})$, and let $f(t) = t^b(1+at)^\gamma$, where α, β, γ, $n \in \mathbb{N}_0$, and $b \in \mathbb{N}_0$ satisfying $b \leq n$. Then, (3.30) yields

$$g_k = c_k \frac{\gamma+\beta b}{\gamma+\beta k}\binom{\gamma+\beta k}{k-b}a^{n-b}.$$

Since

$$d_{n,k} = [t^n](1+at)^{\alpha+n\beta}\left(t(1+at)^{-\beta}\right)^k = \binom{\alpha+(n-k)\beta}{n-k}$$

and

$$[t^n]d(t)f(t) = [t^n](1+at)^{\alpha+n\beta}t^b(1+at)^\gamma = \binom{\alpha+\gamma+n\beta}{n-b},$$

from (3.29) we have b-th order Gould's identity

$$\sum_{k=b}^{n}\frac{\gamma+\beta b}{\gamma+\beta k}\binom{\gamma+\beta k}{k-b}\binom{\alpha+(n-k)\beta}{n-k} = \binom{\alpha+\gamma+n\beta}{n-b}, \quad 0 \leq b \leq n, \tag{3.31}$$

which includes Gould's identity (11) in [7] as the case of $b = 0$. By using the concept of "natural range of summation" shown in [28], we may write (3.31) as

$$\sum_{k=0}^{n}\frac{\gamma+\beta b}{\gamma+\beta k}\binom{\gamma+\beta k}{k-b}\binom{\alpha+(n-k)\beta}{n-k} = \binom{\alpha+\gamma+n\beta}{n-b} \tag{3.32}$$

for all $b \in \mathbb{N}_0$. In fact, for $0 \leq b \leq n$, the natural range of the summation on the left-hand side of (3.31) is $b \leq k \leq n$; for $b > n$, (3.31) is trivial because its both sides are zero.

Remark 3.3 From (3.29), we can find another type identities. Let $(d(t),\ h(t)) = (d_{n,k})_{n\geq k\geq 0}$ be a (c)-generalized Riordan array with respect to a (c)-sequence $\{c_k\}_{k\geq 0}$ with $c_0 = 1$ and $c_k \neq 0$ for all $k > 0$, where $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$, and let $f(t) = \sum_{n\geq 0} f_n t^n/c_n$. Then, there holds identity

$$\sum_{k=1}^{n} d_{n,k}\frac{f_k}{c_k} = \sum_{j=1}^{n}[t^j]f(h(t))[t^{n-j}]d(t). \tag{3.33}$$

Furthermore, denote by $h^*(t)$ is the compositional inverse of $h(t)$, then (3.33) can be written as

$$\sum_{k=1}^{n} d_{n,k}\frac{f_k}{c_k} = \sum_{j=1}^{n}\frac{1}{j}[u^{j-1}]f'(u)\left(\frac{u}{h^*(u)}\right)^j[t^{n-j}]d(t). \tag{3.34}$$

Indeed, substituting $g_k = f_k$ and $g(h(t)) = f(h(t))$ into (3.29), we can write it as

$$\begin{aligned} & \sum_{k=0}^{n} d_{n,k}\frac{f_k}{c_k} = [t^n]d(t)f(h(t)) \\ = & \sum_{j=1}^{n}[t^j]f(h(t))[t^{n-j}]d(t) \\ = & f_0 d_{n,0} + \sum_{j=1}^{n}[t^j]f(h(t))[t^{n-j}]d(t), \end{aligned}$$

which implies (3.33). Noting the compositional inverse of $h(t)$, which exists due to $h(t) \in \mathcal{F}_1$, satisfies $u = t(u/h^*(u))$, we may use Lagrange inverse formula to write

$$[t^j]f(h(t)) = \frac{1}{j}[u^{j-1}]f'(u)\left(\frac{u}{h^*(u)}\right)^j, \quad j \geq 1.$$

Thus (3.34) is followed from (3.33).

From the proof of (3.33), it is easy to see that the initial terms of both sides of (3.33) can be extended to $k=0$ and $\ell=0$, respectively, namely,

$$\sum_{k=0}^{n} d_{n,k}\frac{f_k}{c_k} = \sum_{j=0}^{n}[t^j]f(h(t))[t^{n-j}]d(t).$$

As an example, we consider $(d(t),\ h(t)) = (1/(1-t),\, t/(1-t))$. From (3.34) we obtain the identity

$$\sum_{k=1}^{n}\binom{n}{k}\left[t^k\right] f(t) = \sum_{j=1}^{n}\frac{1}{j}[u^{j-1}]f'(u)\,(1+u)^j\,,$$

which yields identity

$$\sum_{k=1}^{n}\frac{1}{k!}\binom{n}{k} = \sum_{j=1}^{n}\sum_{\ell=0}^{j-1}\frac{1}{j\ell!}\binom{j}{j-1-\ell}$$

when $f(t)=e^t$ and the identity

$$2^n - 1 = \sum_{j=1}^{n}\sum_{\ell=0}^{j-1}\frac{\ell+1}{j}\binom{j}{j-1-\ell}$$

when $f(t) = 1/(1-t)$. In the following example, we consider parameter $0<q<1$ and denote

$$c_n = n!_q := \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}. \tag{3.35}$$

[21] defines

$$\epsilon_a(t) = \sum_{k\geq 0}\frac{a^k}{c_k}t^k$$

with a non-zero constant a, and evaluates

$$\frac{1}{\epsilon_a(t)} = \sum_{k\geq 0} q^{\binom{k}{2}}\frac{(-at)^k}{c_k}.$$

Hence, we have the q-analogue (c)-Riordan array

$$\left(\frac{1}{\epsilon_a(t)}, \frac{t}{1-t}\right) = \left(\sum_{\ell=k}^{n}\binom{\ell-1}{k-1}\frac{a^{n-\ell}}{c_{n-\ell}}\right)_{n\geq k\geq 0},$$

where we use the formula

$$[t^n]f(t)g(t) = \sum_{\ell=0}^{n}[t^\ell]f(t)[u^{n-\ell}]g(y)$$

to evaluate

$$d_{n,k} = [t^n]\epsilon_a(t)\left(\frac{t}{1-t}\right)^k.$$

Therefore we obtain $q-$identities

$$\sum_{k=1}^{n}\frac{1}{k!}\sum_{\ell=k}^{n}\binom{\ell-1}{k-1}\frac{a^{n-\ell}}{c_{n-\ell}}=\sum_{j=1}^{n}\frac{a^{n-j}}{c_{n-j}}\sum_{\ell=0}^{j-1}\frac{1}{j\ell!}\binom{j}{j-1-\ell}$$

and

$$\sum_{k=1}^{n}\sum_{\ell=k}^{n}\binom{\ell-1}{k-1}\frac{a^{n-\ell}}{c_{n-\ell}}=\sum_{j=1}^{n}\frac{a^{n-j}}{c_{n-j}}\sum_{\ell=0}^{j-1}\frac{\ell+1}{j}\binom{j}{j-1-\ell}$$

from (3.34) with $f(t)=e^t$ and $f(t)=1/(1-t)$, respectively, where c_k are defined by (3.35) and note $\binom{0}{k}=\delta_{k,0}$.

4 (c)-Riordan arrays and (c)-Bell polynomials

First, we discuss the composition of formal (c)-power series associated with a sequence $\{c_k\}_{k\geq 0}$, where $c_0=1$ and $c_k>0$ for every $k\geq 1$. Similar to Faá di Bruno formula (see, for example, Theorem A in [6]), we have the following result.

Theorem 4.1 *Let f and g be two formal (c)-power series:*

$$f:=\sum_{k\geq 0}f_k\frac{t^k}{c_k},\quad g:=\sum_{m\geq 1}g_m\frac{t^m}{c_m}, \tag{4.36}$$

and let h be the formal (c)-power series of the composition of g and f:

$$h:=\sum_{n\geq 0}h_n\frac{t^n}{c_n}=f\circ g=f[g]. \tag{4.37}$$

Then the coefficients h_n are given by the following expression:

$$h_0=f_0,\qquad h_n=\sum_{n\geq k\geq 1}f_kB^c_{n,k}(g_1,g_2,\ldots,g_{n-k+1}). \tag{4.38}$$

Proof: First, h_n are linear combinations of the f_k, namely,

$$h_n=\sum_{k=1}^{n}A_{n,k}f_k, \tag{4.39}$$

and $A_{n,k}$ only dependent on g_ℓ ($\ell\geq 1$) associated with the sequence $\{c_k\}$. Secondly, we determine $A_{n,k}$ by choosing for $f(t)$ the special formal series $\tilde{f}(t)=e^{at}$, where a is a parameter. Then, the expansion coefficients of $\tilde{f}(t)$ are $\tilde{f}_k=(\partial^k\tilde{f}/\partial t^k)\Big|_{t=0}=a^k$. Hence, from (4.39)

$$\begin{aligned}\tilde{h}:=\tilde{f}\circ g &= \sum_{n\geq 0}\tilde{h}_n\frac{t^n}{c_n}=1+\sum_{n\geq 1}\left\{\frac{t^n}{c_n}\sum_{k=1}^{n}A_{n,k}\tilde{f}_k\right\}\\ &= 1+\sum_{n\geq k\geq 1}A_{n,k}\frac{t^n}{c_n}a^k. \end{aligned}\tag{4.40}$$

On the other hand, from (1.13), we obtain

$$\begin{aligned}\tilde{h}=\tilde{f}\circ g &= exp(ag)=exp\left(a\sum_{m\geq 1}g_m\frac{t^m}{c_m}\right)\\ &= 1+\sum_{n\geq k\geq 1}B^c_{n,k}(g_1,g_2,\ldots)\frac{t^n}{c_n}a^k. \end{aligned}\tag{4.41}$$

Comparing the rightmost terms of (4.40) and (4.41) yields $A_{n,k} = B^c_{n,k}$, which implies (4.38). ∎

From Theorem 4.1, we immediately have the following corollary.

Corollary 4.2 *Let f be a formal (c)-power series in $\mathcal{F}_1$ and $g = \bar{f}$, the compositional inverse of f, and let $B^c_{n,k}(f_1, f_2, \ldots)$ and $B^c_{n,k}(g_1, g_2, \ldots)$ be the (c)-Bell polynomials generated by $(f(t))^m/c_m$ and $(g(t))^m/c_m$, i.e.,*

$$\left[\frac{t^n}{c_n}\right]\frac{1}{c_m}(f(t))^m = B^c_{n,k}(f_1, f_2, \ldots), \quad \left[\frac{t^n}{c_n}\right]\frac{1}{c_m}(g(t))^m = B^c_{n,k}(g_1, g_2, \ldots)$$

for all $n \geq k \geq 0$. Then

$$\begin{aligned}
&\sum_{n\geq k\geq m} B^c_{n,k}(f_1, f_2, \ldots, f_{n-k+1})B^c_{k,m}(g_1, g_2, \ldots, g_{k-m+1}) \\
=&\sum_{n\geq k\geq m} B^c_{n,k}(g_1, g_2, \ldots, g_{n-k+1})B^c_{k,m}(f_1, f_2, \ldots, f_{k-m+1}) = \delta_{n,m},
\end{aligned} \tag{4.42}$$

where $\delta_{n,m}$ is the Knocker symbol, i.e., $\delta_{n,m} = 1$ when $n = m$ and zero otherwise.

Proof: It is obvious that

$$\begin{aligned}
&\sum_{n\geq k\geq m} B^c_{n,k}(f_1, f_2, \ldots, f_{n-k+1})B^c_{k,m}(g_1, g_2, \ldots, g_{k-m+1}) = \left[\frac{t^n}{c_n}\right]\left[\frac{1}{c_m}(f(y))^m\Big| y = g(t)\right] \\
=&\left[\frac{t^n}{c_n}\right]\left(\frac{t^m}{c_m}\right) = \delta_{n,m}.
\end{aligned}$$

∎

Example 4.1 *As an example, we consider the ordinary generating functions of the Stirling numbers of the first kind, $\left[\begin{array}{c} n \\ m \end{array}\right]$ (signless), and the second kind, $\left\{\begin{array}{c} n \\ m \end{array}\right\}$:*

$$\left[\frac{t^n}{n!}\right]\frac{1}{m!}\left(\log\frac{1}{1-t}\right)^m = \left[\begin{array}{c} n \\ m \end{array}\right], \quad \left[\frac{t^n}{n!}\right]\frac{1}{m!}\left(e^{-t} - 1\right)^m = (-1)^n\left\{\begin{array}{c} n \\ m \end{array}\right\}.$$

Thus, Corollary 4.2 gives

$$\sum_{n\geq k\geq m}(-1)^{n-k}\left[\begin{array}{c} n \\ k \end{array}\right]\left\{\begin{array}{c} k \\ m \end{array}\right\} = \delta_{n,m}.$$

We now use Bell polynomials to characterize (c)-Riordan arrays. In [20], the Sheffer identity is used to characterize Sheffer polynomials $s_n(x)$ for $(d(t), h(t))$ (see (1.15). Since there exists an isomorphism between the Riordan group and Sheffer group (see [12]), the Sheffer identity can also be used to characterize Riordan arrays. Here, the Riordan group is the collection of all proper Riordan arrays with operation $(d_1(t), h_1(t)) * (d_2(t), h_2(t)) = (d_1(t)d_2(h_1(t)), h_2(h_1(t))) \equiv (d_3(t), h_3(t))$, and the Sheffer group is the collection of all Sheffer polynomials for proper Riordan arrays $(d(t), h(t))$ with operation $s^{(1)}_n(x) * s^{(2)}_n(x) = s^{(3)}_n(x)$, where $s^{(i)}_n(x)$ are associated with $(d_i(t), h_i(t))$, $i = 1, 2, 3$, respectively. Recall the Sheffer identity is presented as

$$s_n(x + y) = \sum_{k=0}^{n}\binom{n}{k}p_k(y)s_{n-k}(x), \tag{4.43}$$

where $s_n(x)$ is the Sheffer for $(d(t), h(t))$, and $p_k(y)$ is associated to $h(t)$. Hence, we can establish the following result.

Theorem 4.3 *$\{s_n(x)\}_{n\geq 0}$ is a Sheffer sequence generated by $(d(t),h(t))$, $h(t)=\sum_{n\geq 1}h_nt^n/n!$, for some $d(t)\in\mathcal{F}_0$ with $d(t)=\sum_{n\geq 0}\alpha_nt^n/n!$ if and only if*

$$s_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(x)\sum_{j=0}^{k}y^jB_{k,j}(h_1,h_2,h_{k-j+1}), \tag{4.44}$$

or equivalently,

$$s_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}x^i\sum_{\ell=i}^{n-k}\binom{n-k}{\ell}\alpha_{n-k-\ell}B_{\ell,i}(h_1,h_2,\ldots)\sum_{j=0}^{k}y^jB_{k,j}(h_1,h_2,h_{k-j+1}) \tag{4.45}$$

for every x and y.

Proof: Noting that $p_k(y)$ is associated with $h(t)$, we can write $p_k(y)$ using (1.17) with $\alpha_0=1$ and $\alpha_n=0$ $(n\geq 1)$ as follows:

$$p_k(y)=\sum_{j=0}^{k}y^jB_{k,j}(h_1,h_2,h_{k-j+1}).$$

Substituting the above expression into the Sheffer identity (4.43), we immediately have (4.44). (4.45) can be obtained from (4.44) by substituting (1.17) with the transform $k\mapsto i$ and then $n\mapsto n-k$. ∎

After transferring $x\mapsto x+y$ in (1.17), we substitute the resulting expression of $s_n(x+y)$ into (4.45) and obtain the following corollary.

Corollary 4.4 *Let $d(t)\in\mathcal{F}_0$ and $h(t)\in\mathcal{F}_1$ with $d(t)=\sum_{n\geq 0}\alpha_nt^n/n!$ and $h(t)=\sum_{n\geq 1}h_nt^n/n!$. $(d(t),h(t))$ is a proper Riordan array if and only if*

$$\begin{aligned}&\sum_{k=0}^{n}(x+y)^k\sum_{\ell=k}^{n}\frac{c_n}{c_\ell c_{n-\ell}}\alpha_{n-\ell}B^c_{\ell,k}(h_1,h_2,\ldots)\\ =\ &\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}x^i\sum_{\ell=i}^{n-k}\binom{n-k}{\ell}\alpha_{n-k-\ell}B^c_{\ell,i}(h_1,h_2,\ldots,h_{\ell-i+1})\sum_{j=0}^{k}y^jB_{k,j}(h_1,h_2,\ldots,h_{k-j+1})\end{aligned}$$

for every x and y.

Since any Riordan array $(d(t),h(t))$ can be split as

$$(d(t),h(t))=(d(t),t)*(1,h(t)),$$

we may characterize the Riordan array by characterizing its Appell factor $(d(t),t)$ and its associate factor $(1,h(t))$. From [15] (see also the proof of Theorem 1.4), we have the following result.

Theorem 4.5 *Let $h(t)\in\mathcal{F}_1$ with $h(t)=\sum_{n\geq 1}h_nt^n/n!$. $(1,h(t))$ is an associate Riordan array if and only if $F_n(x)=[t^n]exp(xh(t))$ is a convolution family, or equivalently,*

$$\sum_{k=0}^{n}(x+y)^kB_{n,k}(h_1,h_2,h_{k-j+1})=\sum_{k=0}^{n}\binom{n}{k}\sum_{\ell=0}^{n-k}x^\ell B_{n-k,\ell}(h_1,h_2,h_{n-k-\ell+1})\sum_{j=0}^{k}y^jB_{k,j}(h_1,h_2,h_{k-j+1}).$$

We now extend the above results to the (c)-Riordan arays. (c)-Sheffer sequences are defined by (1.16), namely,

$$F(t,x)=d(t)g(xh(t))=\sum_{n\geq 0}s_n(x)\frac{t^n}{c_n}.$$

Then, we present the (c)-Sheffer identity as follows. First, the formal power series $f(t) \in \mathcal{F}$, $f(t) = \sum_{k\geq 0} \frac{a_k}{c_k} t^k$, defines a linear functional on the algebra of polynomials in the single variable x over the field $\mathbb{C}$ by setting

$$\langle f(t)|x^n\rangle = a_n.$$

Secondly, from (1.16), we immediately have

$$\langle d(t)(h(t))^k|s_n(x)\rangle = c_n\delta_{n,k} \tag{4.46}$$

for all $n, k \geq 0$, which implies the expansion formula:

$$g(t) = \sum_{k\geq 0} \frac{\langle g(t)|s_k(x)\rangle}{c_k} d(t)(h(t))^k \tag{4.47}$$

for every $g \in \mathcal{F}$. In fact, the expansion formula can be found as follows.

$$\begin{aligned} &\langle \sum_{k\geq 0} \frac{\langle g(t)|s_k(x)\rangle}{c_k} d(t)(h(t))^k \Big| s_n(x)\rangle = \sum_{k\geq 0} \frac{\langle g(t)|s_k(x)\rangle}{c_k} \langle d(t)(h(t))^k|s_n(x)\rangle \\ = &\sum_{k\geq 0} \frac{\langle g(t)|s_k(x)\rangle}{c_k} c_n\delta_{n,k} = \langle g(t)|s_n(x)\rangle \end{aligned}$$

for all $s_n(x)$ $(n = 0, 1, 2, \ldots)$.

Thirdly, from the expansion formula and an obvious identity $\langle g(yt)|p(x)\rangle = p(y)$ for any $g \in \mathcal{F}$, we have

$$g(yt) = \sum_{k\geq 0} \frac{\langle g(t)|s_k(x)\rangle}{c_k} d(t)(h(t))^k = \sum_{k\geq 0} \frac{s_k(y)}{c_k} d(t)(h(t))^k. \tag{4.48}$$

We now establish the following (c)- Sheffer identity..

Theorem 4.6 *$\{s_n(x)\}_{n\geq 0}$ is a (c)-Sheffer sequence generated by $(d(t), h(t))$ in (1.16) associated with the given (c)-GPS $g(t)$ from if and only if*

$$g(yt)s_n(x) = \sum_{k=0}^{n} \frac{c_n}{c_k c_{n-k}} p_k(y)s_{n-k}(x) \tag{4.49}$$

for all y, where $p_k(x)$ is the (c)-Sheffer sequence generated by $(1, h(t))$ with respective to $g(t)$.

Proof: We may use a similar argument in the proof of the classical Sheffer identity in [20]. Suppose $s_n(x)$ and $p_n(x)$ are the (c)-Sheffer sequences for, respectively, $(d(t), h(t)$ and $(1, h(t))$ with respect to $g(t) = \sum_{n\geq 0} t^n/c_n$. By the expansion formula, we obtain

$$g(yt) = \sum_{k\geq 0} \frac{\langle g(yt)|p_k(x)\rangle}{c_k} (h(t))^k = \sum_{k\geq 0} \frac{p_k(y)}{c_k} (h(t))^k.$$

Applying both sides of this equation to $s_n(x)$ and using

$$h(t)s_n(x) = \frac{c_n}{c_{n-1}} s_{n-1}(x),$$

yields

$$g(yt)s_n(x) = \sum_{k\geq 0} \frac{p_k(y)}{c_k} (h(t))^k s_n(x) = \sum_{k\geq 0} \frac{c_n}{c_{n-k}c_k} p_k(y)s_{n-k}(x). \tag{4.50}$$

Conversely, suppose sequence $s_n(x)$ satisfies (4.49). We define a linear operator $Ts_n(x) = p_n(x)$, where $p_n(x)$ is the Sheffer sequence associated with $(1, h(t))$. Thus,

$$
\begin{aligned}
g(yt)Ts_n(x) &= g(yt)p_n(x) = \sum_{k\geq 0} \frac{c_n}{c_{n-k}c_k} p_k(y)s_{n-k}(x) \\
&= T\sum_{k\geq 0} \frac{c_n}{c_{n-k}c_k} p_k(y)s_{n-k}(x) = Tg(yt)s_n(x),
\end{aligned}
$$

which implies $g(yt)T = Tg(yt)$. Therefore, T has the form $g \in \mathcal{F}$ completing the proof of the theorem. ∎

If $g(yt) = e^{yt}$, i.e., $c_n = n!$, then Theorem 4.6 is reduced to the case of the classical Sheffer sequences. From Theorems 4.3 and 4.6, we obtain the following corollary.

Corollary 4.7 *$\{s_n(x)\}_{n\geq 0}$ is a (c)-Sheffer sequence generated by $(d(t), h(t))$, $h(t) = \sum_{n\geq 1} h_n t^n/c_n$, for some $d(t) \in \mathcal{F}_0$ with $d(t) = \sum_{n\geq 0} \alpha_n t^n/c_n$ if and only if*

$$
g(yt)s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(x) \sum_{j=0}^{k} y^j B^c_{k,j}(h_1, h_2, h_{k-j+1}), \tag{4.51}
$$

or equivalently,

$$
g(yt)s_n(x) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} x^i \sum_{\ell=i}^{n-k} \binom{n-k}{\ell} \alpha_{n-k-\ell} B^c_{\ell,i}(h_1, h_2, \ldots) \sum_{j=0}^{k} y^j B^c_{k,j}(h_1, h_2, h_{k-j+1}) \tag{4.52}
$$

for every x and y.

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