Characterizations of Orthogonal Generalized Gegenbauer-Humbert Polynomials and Orthogonal Sheffer-type Polynomials

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Dedicated to Professor Leetsch C. Hsu on the Occasion of his 90th Birthday

Abstract

We present characterizations of the orthogonal generalized Gegen-bauer-Humbert polynomial sequences and the orthogonal Sheffer-type polynomial sequences. Using a new polynomial sequence transformation technique presented in [12], we give a method to evaluate the measures and their supports of some orthogonal generalized Gegenbauer-Humbert polynomial sequences.

AMS Subject Classification: 41A80, 65B10, 33C45, 33D45, 39A70, 42C05.

Key Words and Phrases: generalized Gegenbauer-Humbert polynomial sequence, Sheffer-type polynomial sequence, Chebyshev polynomial, Legendre polynomial, Morgan-Voyc polynomial, Fermat polynomial, Dickson polynomial of the second kind, and Laguerre polynomial, measures, supports.

1 Introduction

A system of polynomials $\{p_n(x), n \in \mathbb{N}\}$, where $p_n(x)$ is a polynomial of exact degree n and $\mathbb{N} = \{0, 1, 2, \ldots\}$ or $\{0, 1, 2, \ldots, N\}$ for a finite nonnegative integer N, is an orthogonal system of polynomials with respect to some real positive measure μ on X, if $\{p_n(x)\}$ is a set linearly independent in $L_2(X, \mu)$ and satisfies the orthogonality relation

$$\langle p_i, p_j \rangle_\mu := \int_S p_i(x)p_j(x)d\mu(x) = d_i^2\delta_{ij}, \quad i, j \in \mathbb{N}, \tag{1}$$

where S is the support of the measure μ and d_i are nonzero constants. If these constants $d_i = 1$, we say the system is orthonormal.

The measure μ usually has a density $\mu'(x) = w(x)$ or is a discrete measure with weights $w(i)$ at the points x_i. The relation (1) then becomes

$$\int_S p_i(x)p_j(x)w(x)dx = d_i^2\delta_{ij}, \quad i, j \in \mathbb{N}, \tag{2}$$

in the former case and

$$\sum_{n=0}^{M} p_i(x_n)p_j(x_n)w_n = d_i^2\delta_{ij}, \quad i, j \in \mathbb{N}, \tag{3}$$

in the latter case where it is possible that $M = \infty$.

In this paper, we shall present a characterization of the orthogonal generalized Gegenbauer-Humbert polynomial sequences and give a method to find the density functions and their supports for a class of orthogonal generalized Gegenbauer-Humbert polynomial sequences. We shall also give a characterization of the orthogonal Sheffer-type polynomial sequences. We now start from a general result on orthogonal polynomial sequences.

It is well-known that all orthogonal polynomials $\{p_n(x)\}$ on the real line satisfy a recurrence relation of order 2 (see, for examples, [1], [2], [3], [4])

$$-xp_n(x) = b_np_{n+1}(x) + \gamma_np_n(x) + c_np_{n-1}(x), \quad n \geq 1, \tag{4}$$

where $b_n, c_n \neq 0$ and $c_n/b_{n-1} > 0$. Note that if for all $n \in \mathbb{N}$, $p_n(0) = 1$, we have $\gamma_n = -(b_n + c_n)$ and the polynomials $p_n(x)$ can be defined by the recurrence relation

$$-xp_n(x) = b_np_{n+1}(x) - (b_n + c_n)p_n(x) + c_np_{n-1}(x), \quad n \geq 1 \tag{5}$$

together with $p_{-1}(x) = 0$ and $p_0(x) = 1$. Favard proved a converse result (see, for example, [4]).

Theorem 1.1 *(Favard's Theorem) Let A_n, B_n, and C_n be arbitrary sequences of real numbers and let $\{p_n(x)\}$ be defined by the recurrence relation of order 2*

$$p_{n+1}(x) = (A_nx + B_n)p_n(x) - C_np_{n-1}(x), \quad n \geq 0, \tag{6}$$

together with $p_0(x) = c \neq 0$ *and* $p_{-1}(x) = 0$. *Then* $\{p_n(x)\}$ *is a sequence of orthogonal polynomials if and only if* $A_n \neq 0$, $C_n \neq 0$, *and* $C_n A_n A_{n-1} > 0$ *for all* $n \geq 1$.

For more references of the orthogonal polynomial sequences, readers may find from a recently published very nice survey, [5], by Chihara.

In this paper, we will discuss the characterization of the orthogonal generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$, which are defined by the expansion (see, for example, [6], Gould [7], and Shiue, Hsu and the author [8])

$$\Phi(t) \equiv (C - 2xt + yt^2)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x)t^n, \tag{7}$$

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (7), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [8])

$$\begin{aligned}
&P_n^{1,1,1}(x) = U_n(x),\ \textit{Chebyshev polynomial of the second kind},\\
&P_n^{1/2,1,1}(x) = \psi_n(x),\ \textit{Legendre polynomial},\\
&P_n^{1,-1,1}(x) = P_{n+1}(x),\ \textit{Pell polynomial},\\
&P_n^{1,-1,1}\left(\frac{x}{2}\right) = F_{n+1}(x),\ \textit{Fibonacci polynomial},\\
&P_n^{1,1,1}\left(\frac{x}{2}+1\right) = B_n(x),\ Morgan-Voyc\ polynomial\ ([9]\ by\ Koshy),\\
&P_n^{1,2,1}\left(\frac{x}{2}\right) = \Phi_{n+1}(x),\ Fermat\ polynomial\ of\ the\ first\ kind,\\
&P_n^{1,2a,2}(x) = D_n(x,a),\ \textit{Dickson polynomial of the second kind},\\
&a \neq 0\ (see,\ for\ example,\ [10]\ by\ Lidl,\ Mullen,\ and\ Turnwald),
\end{aligned}$$

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called Gegenbauer polynomials (see [6]). More results on the Gegenbauer-type polynomials can be found in Hsu[11] and Shiue and the author [12], etc. It is interesting that for each generalized Gegenbauer-Humbert polynomial sequence there exists a non-generalized Gegenbauer-Humbert polynomial sequence, for instance, corresponding to the Chebyshev polynomials of the second kind, Pell polynomials, Fibonacci polynomials, Fermat polynomials of the first kind, and the Dickson polynomials of the second kind, we have the Chebyshev polynomials of the first kind, Pell-Lucas polynomials (see [13] by Horadam and Mahon), Lucas polynomials, Fermat polynomials of the second kind (see [14] by Horadam), and the Dickson polynomials of the first kind, respectively.

The class of the generalized Gegenbauer-Humbert polynomial sequences satisfy (see [12])

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x) \tag{8}$$

for all $n \geq 2$ with initial conditions

$$\begin{aligned} P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda}, \\ P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1}, \end{aligned}$$

[12] also obtained the explicit expression of $\{P_n^{\lambda,y,C}(x)\}$ as follows.

Theorem 1.2 *([12]) Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion (7) can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \tag{9}$$

One may write (8) into the form

$$xP_n^{\lambda,y,C}(x) = \frac{C(n+1)}{2(\lambda n)}P_{n+1}^{\lambda,y,C}(x) + \frac{y(2\lambda+n-1)}{2(\lambda+n)}P_{n-1}^{(\lambda,y,C}(x). \tag{10}$$

In [2], Dombrowski and Nevai presented properties of the measures associated with orthogonal polynomial sequences $\{P_n(x) = \gamma_n x^n + \cdots\}_{n\geq 0}$ $(\gamma > 0)$ defined by the following recurrence relation of order 2:

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \tag{11}$$

$n = 0, 1, \ldots$, where $P_{-1}(x) = 0$, $P_0(x) = \gamma_0$, $a_0 = 0$, $a_n = \gamma_{n-1}/\gamma_n$ and $b_n = \int_{-\infty}^{\infty} xP_n^2(x)d\mu(x)$. Comparing (10) and (11), we immediately learn that the polynomial sequences generated by the above recurrence relation and having generating function shown in (7) must be $\{P_n^{1,C,C}(x)\}_{n\geq 0}$, $C \neq 0$.

In this paper, we shall discuss the characterization of the orthogonal Sheffer-type polynomial sequences, which are polynomial sequences possessing a different type generating functions. Sheffer-type polynomial sequences have applications to variable subjects including Lévy processes, financial mathematics, wavelet analysis, mathematical physics, etc. We now present the definition of Sheffer-type polynomial sequences.

Definition 1.3 *Let $A(t)$ and $g(t)$ be any given formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $A(0) = 1$, $g(0) = 0$ and $g'(0) \neq 0$. Then the polynomials $p_n(x)$ $(n = 0, 1, 2, \cdots)$ defined by the generating function (GF)*

$$A(t)e^{xg(t)} = \sum_{n\geq 0} p_n(x)t^n \tag{12}$$

are called Sheffer-type polynomials with $p_0(x) = 1$.

Sheffer-type polynomials include a lot of famous polynomials as the special cases such as the Bernoulli polynomials, Euler polynomials, Laguerre polynomials, etc. Here, we present a short list of the Sheffer-type polynomials in terms of different choices of $(A(t), g(t))$.

$$\begin{aligned}
&For\ (t/(e^t-1), t),\ p_n(x) = \frac{1}{n!}B_n(x),\ Bernoulli\ polynomials,\\
&For\ (2/(e^t+1), t),\ p_n(x) = \frac{1}{n!}E_n(x),\ Euler\ polynomials,\\
&For\ (e^t, log(1+t)),\ p_n(x) = (PC)_n(x),\ Poisson-Charlier\ polynomials,\\
&For\ (e^{-\alpha t}(\alpha \neq 0), log(1+t)),\ p_n(x) = \hat{C}_n^{(\alpha)}(x),\ Charlier\ polynomials\\
&For\ (1, log(1+t)/(1-t)),\ p_n(x) = (ML)_n(x)\ Mittag-Leffler\ polynomials\\
&For\ ((1-t)^{-1}, log(1+t)/(1-t)),\ p_n(x) = (Pi)_n(x),\ Pidduck\ polynomials\\
&For\ ((1-t)^{(-p)}, t/(t-1))(p>0),\ p_n(x) = L_n^{(p-1)}(x),\ Laguerre\ polynomials
\end{aligned}$$

$$\begin{aligned}
&For\ (e^{\lambda t}(\lambda \neq 0), 1-e^t),\ p_n(x) = (Tos)_n^{(\lambda)}(x),\ Toscano\ polynomials\\
&For\ (1, e^t-1),\ p_n(x) = \tau_n(x),\ Touchard\ polynomials\\
&For\ (1/(1+t), t/(t-1)),\ p_n(x) = A_n(x),\ Angelescu\ polynomials\\
&For\ ((1-t)/(1+t)^2, t/(t-1)),\ p_n(x) = (De)_n(x)\ Denisyuk\ polynomials\\
&For\ ((1-t)^{-p}, e^t-1)(p>0),\ p_n(x) = T_n^{(p)}(x),\ Weighted-Touchard\ polynomials
\end{aligned}$$

The set of all Sheffer-type polynomial sequences $\{p_n(x) = [t^n]A(t)e^{xg(t)}\}$ with an operation, "umbral composition" (*cf.* [15] and [16]), forms a group called the Sheffer group. Some properties and characterizations of Sheffer group are shown in [17]. In addition, a higher dimensional extension of the Sheffer-type polynomial sequences are discussed in [18].

In Sections 2 and 3, we shall give characterizations of the orthogonal generalized Gegenbauer-Humbert polynomial sequences and the orthogonal Sheffer-type polynomial sequences, respectively. In Section 4, we shall present a method to find the densities of the measures $\mu(x)$ and their supports S shown in (1) for generalized Gegenbauer-Humbert polynomial sequences $\{P_n^{1,y,C}(x)\}$ using a technique of representing a polynomial sequence $\{p_n(x)\}$ generated by a linear recurrence relation of order two in terms of one or two terms of a orthogonal generalized Gegenbauer-Humbert polynomial sequence.

2 A characterization of the orthogonal generalized Gegenbauer-Humbert polynomials

First, we consider the characterization of the orthogonal generalized Gegenbauer-Humbert polynomials defined by (8). From Favard's Theorem, one may obtain the following result.

Theorem 2.1 *A generalized Gegenbauer-Humbert polynomial sequence defined by (8) is an orthogonal polynomial sequence if and only if $yC > 0$.*

Proof. Writing the recurrence relation (8) into the standard form in Theorem 1.1, we have

$$C_n = y\frac{2\lambda + n - 1}{C(n+1)} \text{ and } A_n = 2\frac{\lambda + n}{C(n+1)}.$$

Thus from Theorem 1.1, $\{P_n^{\lambda,y,C}(x)\}$ is an orthogonal polynomial sequence if and only if

$$C_n A_n A_{n-1} = 4y\frac{(\lambda+n)(\lambda+n-1)(2\lambda+n-1)}{C^3 n(n+1)^2} > 0$$

for all $n \geq 1$. Noting $\lambda > 0$ and $n \geq 1$, we immediately learn that the above inequality is equivalently $yC > 0$, which completes the proof.

■

Example 1 Using Theorem 2.1, we may identify the Chebyshev polynomial sequence of the second kind $\{P_n^{1,1,1}(x) = U_n(x)\}$ and the Legendre polynomial sequence $\{P_n^{1/2,1,1}(x) = \psi_n(x)\}$ are orthogonal, while Pell polynomial sequence and Fibonacci polynomial sequence are not orthogonal. Morgan-Voyc polynomial sequence $\{B_n(2(x-1)) = P_n^{1,1,1}(x)\}$ (and $\{B_n(x)\}$) and the sequence of the Fermat polynomials of the first kind, $\{\Phi_n(2x) = P_{n-1}^{1,2,1}(x)\}$ (and $\Phi_n(x)\}$), are orthogonal polynomial sequences. Dickson polynomials of the second kind are orthogonal when $a > 0$ and non-orthogonal when $a < 0$. We will evaluate the measures and their supports for Morgan-Voyc polynomials, Fermat polynomials, and Dickson polynomials of the second kind in Section 4.

We need the following lemma to find out the recurrence structure of an orthogonal generalized Gegenbauer-Humbert polynomial sequence.

Lemma 2.2 *If $\{p_n(x)\}$ is an orthogonal polynomial sequence, then there exist sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, and $\{C_n\}_{n\geq 1}$ so that*

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \tag{13}$$

where

$$\begin{aligned} A_n &= \frac{k_{n+1}}{k_n}, \; C_n = \frac{A_n h_n}{A_{n-1}h_{n-1}} = \frac{k_{n+1}k_{n-1}h_n}{k_n^2 h_{n-1}}, \; \textit{and} \\ B_n &= -\frac{A_n}{h_n}\int_S x p_n(x)^2 d\mu(x) = -\frac{k_{n+1}}{k_n h_n}\int_S x p_n(x)^2 d\mu(x), \end{aligned}$$

k_n *is the leading coefficient of* $p_n(x)$*, and*

$$h_n = \int_S p_n(x)^2 d\mu(x)$$

is a structural constant.

Proof. The proof can be found in [4] and [3]. However, for the sake of convenience, we present a brief proof as follows.

We first determine A_n so that $p_{n+1}(x) - A_n x p_n(x) \in \pi_n$, a collection of all polynomials of degree $\leq n$. Hence,

$$p_{n+1}(x) - A_n x p_n(x) = \sum_{j=0}^{n} c_j p_j(x).$$

Using the orthogonality of $\langle p_{n+1}(x), p_j(x)\rangle_\mu = 0$ and $\langle p_n(x), x p_j(x)\rangle_\mu = 0$ for all $j = 0, 1, \ldots, n-2$, it is readily seen that $c_j = 0$ for all $j = 0, 1, \ldots, n-2$. Therefore, (13) follows and the expression of A_n is a consequence of (13). To obtain the expression of C_n, we take inner product of (13) with $p_{n-1}(x)$ and consider

$$\int_S p_{n+1}(x)p_{n-1}(x)d\mu(x) = 0 = A_n \int_S x p_n(x) p_{n-1}(x) d\mu(x) - C_n h_{n-1},$$

in which the integral of the right-hand member can be written as

$$\int_S p_n(x)(k_{n-1}x^n + \textit{lower powers}) d\mu(x) = \frac{k_{n-1}}{k_n} h_n = \frac{h_n}{A_{n-1}}.$$

Thus the relation

$$A_n \frac{h_n}{A_{n-1}} - C_n h_{n-1} = 0$$

yields the expression of C_n. Taking the inner product with $p_n(x)$ on the both sides of (13) yields

$$0 = A_n \int_S x p_n(x) p_n(x) d\mu(x) + B_n h_n,$$

which implies the expression of B_n.

$\blacksquare$

From Lemma 2.2, one may obtain

Theorem 2.3 *If the generalized Gegenbauer-Humbert polynomial sequence $\{P_n^{\lambda,y,C}(x)\}$ defined by (8) is an orthogonal polynomial sequence, then*

$$\frac{y}{C} = \frac{nh_n(\lambda+n)}{h_{n-1}(\lambda+n-1)(2\lambda+n-1)} \tag{14}$$

for all $n \geq 1$, where $h_n = \int_S (P_n^{\lambda,y,C}(x))^2 d\mu(x)$. In addition, every element of the sequence $\{P_n^{\lambda,y,C}(x)\}$ satisfies

$$\int_S x P_n^{\lambda,y,C}(x)^2 d\mu(x) = 0. \tag{15}$$

Proof. From the definition (8) of $\{P_n^{\lambda,y,C}(x)\}$ and the expression of C_n in Lemma 2.2, we have

$$y\frac{2\lambda+n-1}{C(n+1)} = 2\frac{h_n(\lambda+n)}{C(n+1)} \Big/ 2\frac{h_{n-1}(\lambda+n-1)}{Cn},$$

which implies (14). Comparing (8) and the standard recurrence relation (13), we know $B_n = 0$ for all $n \geq 0$, which is equivalent to (15).

$\blacksquare$

Remark 1 From (14) one immediately have

$$h_n = \frac{y(\lambda+n-1)(2\lambda+n-1)}{nC(\lambda+n)} h_{n-1},$$

which implies

$$h_n = \left(\frac{y}{C}\right)^n \frac{(\lambda+n-1)^{\underline{n}}(2\lambda+n-1)^{\underline{n}}}{n!(\lambda+n)^{\underline{n}}} h_0,$$

where the falling factorial notation $x^{\underline{r}}$ (sometimes also denoted $(x)_r$) is defined by $x^{\underline{r}} = x(x-1)^{\underline{r-1}} (r \geq 1)$ with $x^{\underline{0}} = 1$. Using the above equations and equation (15), we may evaluate the measures and their supports.

Example 2 For the orthogonal sequence of the Chebyshev polynomials of the second order $\{P_n^{1,1,1}(x) = U_n(x)\}$, we have $y/C = 1$ that implies $h_n = h_1 = \pi/2$ and

$$\int_{-1}^{1} x\sqrt{1-x^2}(U_n(x))^2 dx = 0$$

for all $n \geq 0$. The above equation is obviously true by observing that $U_{2n-1}(x)$ are odd and $U_{2n}(x)$ are even.

For the sequence of the Legendre polynomials $\{P_n^{1/2,1,1}(x) = \psi_n(x)\}$, we have

$$\frac{h_n}{h_{n-1}} = \frac{n-1/2}{n+1/2},$$

which implies $h_n = 2/(2n+1)$, and

$$\int_{-1}^{1} x(\psi_n(x))^2 dx = 0$$

for all $n \geq 0$. The last formula holds obviously because $\psi_{2n+1}(x)$ are odd and $\psi_{2n}(x)$ are even.

Example 3 We know both $U_n(x)$ and $\psi_n(x)$ are special cases of Gegenbauer polynomials $\{P_n^{\lambda,1,1}(x)\}$ $(\lambda > 0)$. From Theorem 2.1, we know $\{P_n^{\lambda,1,1}(x)\}$ $(\lambda > 0)$ is orthogonal. Using Theorem 2.3, we obtain

$$\frac{h_n}{h_{n-1}} = \frac{(\lambda+n-1)(2\lambda+n-1)}{n(\lambda+n)},$$

which implies

$$h_n = \frac{\pi\Gamma(2\lambda+n)}{2^{2\lambda-1}n!(\lambda+n)\left(\Gamma(\lambda)\right)^2},$$

where $\Gamma(x)$ is the gamma function. In addition, we have

$$\int_{-1}^{1} x(1-x^2)^{\lambda-1/2}\left(P_n^{\lambda,1,1}(x)\right)^2 dx = 0.$$

3 A characterization of the orthogonal Sheffer-type polynomial sequences

Meixner determined all sets of monic orthogonal Sheffer-type polynomials in his historic paper [19]. Here, a polynomial is said to be monic if the coefficient of its highest order term is 1. We now use a modified Meixner's approach to give a characterization of all orthogonal Sheffer-type polynomials. Denote $D = d/dx$ and $f = g^{-1}$, the composition inverse of g. Expansion (12) suggests

$$f(D)p_m(x) = mp_{m-1}(x) \tag{16}$$

because of

$$\begin{aligned} f(D)A(t)e^{xg(t)} &= A(t)e^{xg(t)}f(g(t)) = tA(t)e^{xg(t)} \\ &= \sum_{n\geq 0} p_n(x)\frac{t^{n+1}}{n!} = \sum_{n\geq 0} np_{n-1}(x)\frac{t^n}{n!}, \end{aligned}$$

where we have used $p_{-1}(x) = 0$.

Theorem 3.1 *Let $A(t)$ and $g(t)$ be defined as Definition 1.3. Then the polynomial sequence $\{p_n(x)\}$ defined by (12) is orthogonal if and on if it satisfies*

$$p_{n+1}(x) = (A_0x + B_0 + n\lambda)p_n(x) - n(C_1 + (n-1)\gamma)p_{n-1}(x), \tag{17}$$

where $A_0 \neq 0, B_0, C_1, \lambda, and \gamma$ are constant, and $C_1, \gamma > 0$. Furthermore, $g(t)$ and $A(t)$ satisfy

$$g'(t) = \frac{A_0}{1-\lambda t+\gamma t^2}, \ and \ \frac{A'(t)}{A(t)} = \frac{B_0 - C_1 t}{1-\lambda t+\gamma t^2}. \tag{18}$$

Proof. All orthogonal polynomial sequences including orthogonal Sheffer-type polynomial sequences, $\{p_n(x)\}$, satisfy the recurrence relation (13) shown in Lemma 2.2:

$$p_{n+1}(x) = (A_nx + B_n)p_n(x) - C_np_{n-1}(x). \tag{19}$$

We now apply $f(D)$ defined by (16) on the both sides of the relation and note that $f(0) = 0$ and $f'(0) \neq 0$ implies $f(D)x = f'(D)$. Thus,

$$\begin{aligned} (n+1)p_n(x) &= f(D)p_{n+1}(x) = f(D)\left[(A_nx + B_n)p_n(x) - C_np_{n-1}(x)\right] \\ &= A_nf'(D)p_n(x) + n(A_nx + B_n)p_{n-1}(x) - (n-1)C_np_{n-2}(x), \end{aligned} \tag{20}$$

where we need $C_nA_nA_{n-1} > 0$, which is a necessary and sufficient condition of the orthogonality of $\{p_n(x)\}$ presented in (19) (See Lemma 2.2). On the other hand, multiplying n to the both sides of relation (13) for $p_n(x)$ yields

$$np_n(x) = n(A_{n-1}x + B_{n-1})p_{n-1}(x) - nC_{n-1}p_{n-2}(x). \tag{21}$$

Subtracting (21) from (20), we obtain

$$\begin{aligned}(1-A_nf'(D))p_n(x) &= n[(A_n-A_{n-1})x+(B_n-B_{n-1})]p_{n-1}(x)\\ &\quad -n(n-1)\left(\frac{C_n}{n}-\frac{C_{n-1}}{n-1}\right)p_{n-2}(x). \qquad (22)\end{aligned}$$

Applying $f(D)$ on the leftmost and rightmost sides of (22) yields

$$\begin{aligned}&n(1-A_nf'(D))p_{n-1}(x)\\ =\ &n(A_n-A_{n-1})f'(D)p_{n-1}(x)\\ &+n(n-1)[(A_n-A_{n-1})x+(B_n-B_{n-1})]p_{n-2}(x)\\ &-n(n-1)(n-2)\left(\frac{C_n}{n}-\frac{C_{n-1}}{n-1}\right)p_{n-3}(x).\end{aligned}$$

By transferring n to $n+1$, the above equation implies

$$\begin{aligned}(1+(A_n-2A_{n+1})f'(D))p_n(x) &= n[(A_{n+1}-A_n)x+(B_{n+1}-B_n)]p_{n-1}(x)\\ &\quad -(n)(n-1)\left(\frac{C_{n+1}}{n+1}-\frac{C_n}{n}\right)p_{n-2}(x).\end{aligned} \qquad (23)$$

From (22) and (23) we have identity

$$\begin{aligned}&-(1-A_nf'(D))p_n(x)+n[(A_n-A_{n-1})x+(B_n-B_{n-1})]p_{n-1}(x)\\ &-n(n-1)\left(\frac{C_n}{n}-\frac{C_{n-1}}{n-1}\right)p_{n-2}(x)\\ =\ &-(1+(A_n-2A_{n+1})f'(D))p_n(x)+n[(A_{n+1}-A_n)x+(B_{n+1}-B_n)]p_{n-1}(x)\\ &-(n)(n-1)\left(\frac{C_{n+1}}{n+1}-\frac{C_n}{n}\right)p_{n-2}(x). \qquad (24)\end{aligned}$$

Comparing the nth degree terms on the both sides of (24) yields

$$\begin{aligned}&-(1-A_nf'(D))p_n(x)+n(A_n-A_{n-1})xp_{n-1}(x)\\ =\ &-(1+(A_n-2A_{n+1})f'(D))p_n(x)+n(A_{n+1}-A_n)xp_{n-1}(x).\end{aligned} \qquad (25)$$

In (25) the constant terms on the both sides are equal, which implies

$$-(1 - A_n f'(D))p_n(x) = -(1 + (A_n - 2A_{n+1})f'(D))p_n(x),$$

or equivalently, $A_n = A_{n+1}$ for every $n \geq 0$. Hence, (25) holds if and only if

$$A_n = A_0, \tag{26}$$

a nonzero constant for every $n \geq 0$. Comparing the terms of degree $n-1$ and $n-2$ on the both sides of (24), we have the results

$$B_{n+1} - B_n = \lambda$$

and

$$\frac{C_{n+1}}{n+1} - \frac{C_n}{n} = \gamma$$

for every $n \geq 0$, where λ and γ are constants. Hence,

$$B_n = B_0 + n\lambda \textit{ and } C_n = n(C_1 + (n-1)\gamma) \tag{27}$$

for all $n \geq 1$, where $C_1, \gamma > 0$ because of the request $C_n A_n A_{n-1} = C_n A_0^2 > 0$ for all $n \geq 1$ (see Theorem 2.1). Substituting all of the established relationship of the sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, and $\{C_n\}_{n\geq 1}$ into (19) and (22), we obtain, respectively,

$$p_{n+1}(x) = (A_0 x + B_0 + n\lambda)p_n(x) - n(C_1 + (n-1)\gamma)p_{n-1}(x), \tag{28}$$

where $A_0 \neq 0$ and $C_1, \gamma > 0$, and

$$(1 - A_0 f'(D))p_n(x) = \lambda f(D)p_n(x) - \gamma f^2(D)p_n(x). \tag{29}$$

From (29), we further have

$$f'(y) = \frac{1}{A_0}(1 - \lambda f(y) + \gamma f^2(y)),$$

which implies

$$g'(t) = \frac{A_0}{1 - \lambda t + \gamma t^2}$$

by using the inverse function theorem.

From (28), we have

$$p_{n+1}(0) = (B_0 + n\lambda)p_n(0) - n(C_1 + (n-1)\gamma)p_{n-1}(0). \tag{30}$$

Noting $A(t) = \sum_{n\geq 0} p_n(0)\frac{t^n}{n!}$, (30) implies

$$\frac{A'(t)}{A(t)} = \frac{B_0 - C_1 t}{1 - \lambda t + \gamma t^2}$$

because

$$\begin{aligned}
A(t)(B_0 - C_1 t) &= \sum_{n\geq 0}(B_0 p_n(0) - nC_1 p_{n-1}(0))\frac{t^n}{n!} \\
&= \sum_{n\geq 0}(p_{n+1}(0) - n\lambda p_n(0) + n(n-1)\gamma p_{n-1}(0))\frac{t^n}{n!} \\
&= (1 - \lambda t + \gamma t^2)\sum_{n\geq 0} p_{n+1}(0)\frac{t^n}{n!} \\
&= A'(t)(1 - \lambda t + \gamma t^2),
\end{aligned}$$

which completes the proof of the theorem.

■

Let the zeros of the denominator of $g'(t)$ shown in (18) be α and β. Then one may solve $g(t)$ and $A(t)$ from (18) as follows.

Corollary 3.2 *Let $A(t)$ and $g(t)$ be defined as Definition 1.3. Then the polynomial sequence $\{p_n(x)\}$ defined by (12) is orthogonal if and on if*

$$g(t) = \begin{cases} \frac{A_0}{\alpha-\beta}\ln\left(\frac{1-\beta t}{1-\alpha t}\right), & if\ \alpha \neq \beta, \\ \frac{A_0 t}{1-\alpha t}, & if\ \alpha = \beta. \end{cases}$$

and

$$\ln f(t) = \begin{cases} \frac{C_1-\alpha B_0}{\alpha(\alpha-\beta)}\ln(1-\alpha t) - \frac{C_1-\beta B_0}{\beta(\alpha-\beta)}\ln(1-\beta t), & if\ 0 \neq \alpha \neq \beta \neq 0, \\ -\frac{C_1}{\alpha^2}\ln(1-\alpha t) - \frac{C_1-\alpha B_0}{\alpha}\frac{t}{1-\alpha t}, & if\ \alpha = \beta \neq 0, \\ \frac{C_1-\alpha B_0}{\alpha^2}\ln(1-\alpha t) + \frac{C_1}{\alpha}t, & if\ \alpha \neq \beta = 0, \\ -\frac{C_1}{2}t^2 + B_0 t, & if\ \alpha = \beta = 0, \end{cases}$$

Example 4 As an example, we set $A_0 = -1$, $B_0 = C_1 = 1$, and $\alpha = \beta = 1$ in Corollary 3.2 and obtain

$$g(t) = \frac{-t}{1-t} \quad and \quad A(t) = \frac{1}{1-t}.$$

Thus, from Theorem 3.1, the Laguerre polynomial sequence, $\{L_n(x)\}$, generated by (12) in Definition 1.3 with $(A(t), g(t)) = (1/(1-t), -t/(1-t))$ is an orthogonal polynomial sequence. Furthermore, from the expansion of $(1-t)^2$, we can read $\lambda = 2$ and $\gamma = 1$, which implies the following recurrence relation for $\{L_n(x)\}$:

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2 L_{n-1}(x)$$

with the initial conditions $L_{-1}(x) = 0$ and $L_0(x) = 1$. Thus, $L_1(x) = 1-x$, $L_2(x) = 2-4x+x^2$, $L_3(x) = 6-18x+9x^2-x^3$, etc. Using Lemma 2.2, one may check the assumption of $B_0 = C_1 = 1$ is satisfied for $\{L_n(x)\}$. Since

$$h_0 = \int_S L_0^2(x)d\mu(x) = \int_0^\infty e^{-x}dx = 1$$

and

$$h_1 = \int_S L_1^2(x)d\mu(x) = \int_0^\infty (1-x)^2 e^{-x}dx = 1,$$

we have

$$B_0 = -\frac{A_0}{h_0}\int_S xL_0^2(x)d\mu(x) = \int_0^\infty e^{-x}dx = 1$$

and

$$C_1 = \frac{A_1 h_1}{A_0 h_0} = \frac{h_1}{h_0} = 1.$$

4 Evaluate the measures and their supports of orthogonal sequences $\{P_n^{1,y,C}(x)\}$

In this section, we will present a method to find the densities of measures $\mu(x)$ and their supports S (see (1)) of orthogonal generalized Gegenbauer-Humbert polynomial sequences, $\{P_n^{1,y,C}(x)\}$ $(Cy > 0)$, using a technique of transferring a polynomial sequence defined by a recurrence relation of order two to an orthogonal Gegenbauer-Humbert polynomial sequence. This transfer technique can also give an orthogonal representation of non-orthogonal polynomials satisfying recurrence relation of order 2 in terms of only one or two terms of an orthogonal polynomial sequence. Thus, many useful approximation properties for orthogonal polynomials (for instance, Gaussian quadratures) can be transfered to some non-orthogonal polynomials.

Many number and polynomial sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders.

A polynomial sequence $\{a_n(x)\}$ is called sequence of order 2 if it satisfies the linear recurrence relation of order 2:

$$a_n(x) = p(x)a_{n-1} + q(x)a_{n-2}(x), \quad n \geq 2, \tag{31}$$

for some coefficient $p(x) \not\equiv 0$ and $q(x) \not\equiv 0$ and initial conditions $a_0(x)$ and $a_1(x)$. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (See Comtet [6], Hsu [20], Strang [21], Wilf [22], etc.) [12] presented a new method to construct an explicit formula of $\{a_n(x)\}$ generated by (31). For the sake of reader's convenience, we cite this result as follows (see also Miller and Takloo-Bighash [23] with different approaches).

Proposition 4.1 *Let $\{a_n(x)\}$ be a sequence of order 2 satisfying the linear recurrence relation (31). Then*

$$a_n(x) = \begin{cases} \left(\frac{a_1(x)-\beta(x)a_0(x)}{\alpha(x)-\beta(x)}\right)\alpha^n(x) - \left(\frac{a_1(x)-\alpha(x)a_0(x)}{\alpha(x)-\beta(x)}\right)\beta^n(x), & if\ \alpha(x) \neq \beta(x); \\ na_1(x)\alpha^{n-1}(x) - (n-1)a_0(x)\alpha^n(x), & if\ \alpha(x) = \beta(x), \end{cases} \tag{32}$$

where $\alpha(x)$ and $\beta(x)$ are roots of $t^2 - p(x)t - q(x) = 0$, namely,

$$\alpha(x) = \frac{1}{2}(p(x) + \sqrt{p^2(x) + 4q(x)}), \beta(x) = \frac{1}{2}(p(x) - \sqrt{p^2(x) + 4q(x)}). \tag{33}$$

We now give a transfer formula between different generalized Gegenbauer-Humbert polynomial sequences. This technique can be used to transfer any polynomials defined by recurrence relations of order 2 to a generalized Gegenbauer-Humbert polynomials.

Theorem 4.2 *If $\{a_n(x) = P_n^{1,C',y'}(x)\}$, a generalized Gegenbauer-Humbert polynomial sequence with parameters C' and y', which is defined by (7) with coefficient polynomials $p(x) = 2x/C'$ and $q(x) = -y'/C'$ and initial conditions $a_0(x) = 1/C'$ and $a_1(x) = 2x/(C')^2$, then we have the following transfer formula from $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ to $\{P_n^{1,y',C'}(x)\}_{n\geq 0}$:*

$$P_n^{1,y',C'}(x) = \frac{C^{n+2}}{C'}\left(\pm\sqrt{\frac{y'}{yCC'}}\right)^n P_n^{1,y,C}\left(\pm\frac{x\sqrt{yC}}{\sqrt{y'C'}}\right). \tag{34}$$

In particular, every polynomials sequence $\{P_n^{1,y',C'}(x)\}$ defined by (7) can be transfered to the Chebyshev polynomial sequence of the second kind by using the formula

$$P_n^{1,y',C'}(x) = \frac{1}{C'}\left(\pm\sqrt{\frac{y'}{C'}}\right)^n U_n\left(\pm\frac{x}{\sqrt{y'C'}}\right). \tag{35}$$

Proof. We first modify the explicit formula of the polynomial sequences defined by linear recurrence relation (32) of order 2. If $\alpha(x)\neq\beta(x)$, the first formula in (32) can be written as

$$a_n(x) = \frac{a_1(x)((\alpha(x))^n-(\beta(x))^n) - a_0(x)\alpha(x)\beta(x)((\alpha(x))^{n-1}-(\beta(x))^{n-1})}{\alpha(x)-\beta(x)}.$$

Noting that $-\alpha(x)\beta(x) = \alpha(x)(\alpha(x)-p(x)) = \beta(x)(\beta(x)-p(x))$, we may further write the above expression of $a_n(x)$ as

$$\begin{aligned}
& a_n(x) \\
= \; & \frac{1}{\alpha(x)-\beta(x)}\left[a_1(x)((\alpha(x))^n-(\beta(x))^n) + a_0(x)\alpha(x)(\alpha(x)-p(x))\right. \\
& \left.\times(\alpha(x))^{n-1} - a_0(x)\beta(x)(\beta(x)-p(x))(\beta(x))^{n-1}\right] \\
= \; & \frac{a_0(x)((\alpha(x))^{n+1}-(\beta(x))^{n+1}) + (a_1(x)-a_0(x)p(x))((\alpha(x))^n-(\beta(x))^n)}{\alpha(x)-\beta(x)}.
\end{aligned} \tag{36}$$

Denote $r(x) = x+\sqrt{x^2-Cy}$ and $s(x) = x-\sqrt{x^2-Cy}$. To find a transfer formula between expressions (9) and (36), we set

$$\alpha(x) := \frac{r(x)}{k(x)} \quad \text{and} \quad \beta(x) := \frac{s(x)}{k(x)} \tag{37}$$

for a nonzero real or complex valued function $k(x)$, which are two roots of $t^2-p(x)t-q(x)=0$. Thus, adding and multiplying two equations of (37) side by side, we obtain

$$\begin{aligned}
\alpha(x)+\beta(x) &= p(x) = \frac{2x}{k(x)} \\
\alpha(x)\beta(x) &= -q(x) = \frac{yC}{(k(x))^2}.
\end{aligned}$$

The above system implies

$$k(x) = \pm\sqrt{\frac{Cy}{-q(x)}},$$

and at

$$x=\frac{p(x)k(x)}{2}=\pm\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}},$$

$r(x)$ and $s(x)$ give expressions of $\alpha(x)$ and $\beta(x)$ as

$$\alpha(x)=\frac{r\left(\pm\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right)}{\pm\sqrt{\frac{yC}{-q(x)}}},\ \text{and}\ \beta(x)=\frac{s\left(\pm\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right)}{\pm\sqrt{\frac{yC}{-q(x)}}}. \tag{38}$$

It is clear that $\alpha(x)$ and $\beta(x)$ satisfy $\alpha(x)+\beta(x)=p(x)$ and $\alpha(x)\beta(x)=-q(x)$.

We first consider the case of $k(x)=\sqrt{-yC/q(x)}$. Substituting the corresponding (38) with positive sign into (36), we have

$$\begin{aligned}
&a_n(x)\\
=\ &\frac{a_0(x)(r^{n+1}(x)-s^{n+1}(x))+k(x)(a_1(x)-a_0(x)p(x))(r^n(x)-s^n(x))}{k^n(x)(r(x)-s(x))}\\
=\ &a_0(x)C^{n+2}\left(\sqrt{\frac{-q(x)}{yC}}\right)^n P_n^{1,y,C}\left(\frac{k(x)p(x)}{2}\right)\\
&+(a_1(x)-a_0(x)p(x))C^{n+1}\left(\sqrt{\frac{-q(x)}{yC}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{k(x)p(x)}{2}\right)\\
=\ &a_0(x)C^{n+2}\left(\sqrt{\frac{-q(x)}{yC}}\right)^n P_n^{1,y,C}\left(\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right)\\
&+(a_1(x)-a_0(x)p(x))C^{n+1}\left(\sqrt{\frac{-q(X)}{yC}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right).
\end{aligned}\tag{39}$$

Similarly, for $k(x)=-\sqrt{-yC/q(x)}$, we have

$$\begin{aligned}
a_n(x)\ =\ &a_0(x)C^{n+2}\left(-\sqrt{\frac{-q(x)}{yC}}\right)^n P_n^{1,y,C}\left(-\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right)\\
&+(a_1(x)-a_0(x)p(x))C^{n+1}\left(-\sqrt{\frac{-q(x)}{yC}}\right)^{n-1} P_{n-1}^{1,y,C}\left(-\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right).
\end{aligned}\tag{40}$$

Therefore, $a_n(x)$ defined by (31) can be presented as

$$\begin{aligned}a_n(x) = a_0(x)C^{n+2}\left(\pm\sqrt{\frac{-q(x)}{yC}}\right)^n P_n^{1,y,C}\left(\pm\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right)\\ +(a_1(x)-a_0(x)p(x))C^{n+1}\left(\pm\sqrt{\frac{-q(X)}{yC}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\pm\frac{p(x)}{2}\sqrt{\frac{yC}{-q(x)}}\right),\end{aligned} \tag{41}$$

where $\{P_n^{1,y,c}\}$ is the sequence of any generalized Gegenbauer-Humbert polynomials with $\lambda = 1$. In particular, $a_n(x)$ can be expressed in terms of $\{P_n^{1,1,1} = U_n\}$, the sequence of Chebyshev polynomials of the second kind:

$$\begin{aligned}a_n(x) &= a_0(x)\left(\pm\sqrt{-q(x)}\right)^n U_n\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right)\\ &\quad +(a_1(x)-a_0(x)p(x))\left(\pm\sqrt{-q(x)}\right)^{n-1} U_{n-1}\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right),\end{aligned} \tag{42}$$

which is a special case of (41) for $(y, C) = (1, 1)$.

If $a_n(x) = P_n^{1,y'C'}(x)$ defined by (7) with coefficient polynomials $p(x) = 2x/C'$ and $q(x) = -y'/C'$ and initial conditions $a_0(x) = 1/C'$ and $a_1(x) = 2x/(C')^2$, then $a_1(x) - a_0(x)p(x) = 0$ and (41) and (42) are reduced to (34) and (35), respectively.

■

From Theorem 4.2, we immediately have transfer formulas

$$\begin{aligned}P_{n+1}(x) &= (\pm i)^n U_n(\mp xi),\\ F_{n+1}(x) &= (\pm i)^n U_n\left(\mp\frac{xi}{2}\right),\\ B_n(x) &= (\pm 1)^n U_n\left(\pm\left(\frac{x}{2}+1\right)\right),\\ \Phi_{n+1}(x) &= \left(\pm\sqrt{2}\right)^n U_n\left(\pm\frac{x}{2\sqrt{2}}\right),\\ D_n(x,a) &= \frac{1}{2}\left(\pm\sqrt{a}\right)^n U_n\left(\pm\frac{x}{2\sqrt{a}}\right).\end{aligned}$$

Remark 2 It is obvious that when both y and C are integers, the corresponding generalized Gegenbauer-Humbert polynomials have integer coefficients. Formulas (34) can be used to transfer between the generalized Gegenbauer-Humbert polynomials with integer coefficients and the generalized Gegenbauer-Humbert polynomials with non-integer coefficients. For instance, the last transfer formula shown above presents the Dickson polynomial of the second kind with real coefficients in terms of the Chebyshev polynomials of the second kind.

If $yC > 0$, from Theorem 2.1 we know that $\{P_n^{1,y,C}(x)\}$ is an orthogonal polynomial sequence. Let $w(x)$ and $S = [a, b]$ be the density function and its support interval of $\{P_n^{1,y,C}(x)\}$. We now use Theorem 4.2 to find the density function and its support interval of $\{P_n^{1,y,C}(g(x))\}$, where $g(x)$ is a one-to-one and differentiable function.

Theorem 4.3 *Let $\{P_n^{1,y,C}(x)\}$ be a polynomial sequence defined by (7), and let $g(x)$ be a one-to-one and differential function. Then sequence $\{P_n^{1,y,C}(g(x))\}$ is an orthogonal polynomial sequence associated with the density function*

$$w(x) = g'(x)\sqrt{1-(g(x))^2/(yC)}$$

with support interval between $g^{-1}(-\sqrt{yC})$ and $g^{-1}(\sqrt{yC})$, where $g^{-1}(x)$ is the composition inverse of $g(x)$, i.e., $(g^{-1}\circ g)(x) = (g\circ g^{-1})(x) = x$. Furthermore,

$$\int_{g^{-1}(-\sqrt{yC})}^{g^{-1}(\sqrt{yC})} P_n^{1,y,C}(x)P_m^{1,y,C}(x)g'(x)\sqrt{1-\frac{(g(x))^2}{yC}}dx = \frac{\pi\sqrt{yC}}{2C^2}\left(\frac{y}{C}\right)^n \delta_{n,m}, \tag{43}$$

where $\delta_{n,m}$ is the Kronecker symbol.

In particular, if $g(x) = x$, then $\{P_n^{1,y,C}(x)\}$ is an orthogonal polynomial sequence with respect to density function $\sqrt{1-x^2/(yC)}$ over support interval $\left[-\sqrt{yC}, \sqrt{yC}\right]$, and $\{P_n^{1,y,C}(x)\}$ satisfies (43) when $g(x) = g^{-1}(x) = x$.

Proof. Let us consider inner product $\langle P_n^{1,y,C}(x), P_m^{1,y,C}(x)\rangle_{\sqrt{1-x^2/(yC)}}$ over $[-\sqrt{yC}, \sqrt{yC}]$, in which the transfer formula (35) will be applied:

$$
\begin{aligned}
&\int_{-\sqrt{yC}}^{\sqrt{yC}} P^{1,y,C}(x)P_m^{1,y,C}(x)\sqrt{1-\frac{x^2}{yC}}dx\\
=&\int_{-\sqrt{yC}}^{\sqrt{yC}}\frac{1}{C^2}\left(\pm\sqrt{\frac{y}{C}}\right)^{n+m}U_n\left(\pm\frac{x}{\sqrt{yC}}\right)U_m\left(\pm\frac{x}{\sqrt{yC}}\right)\sqrt{1-\frac{x^2}{yC}}dx\\
=&\frac{1}{C^2}\left(\pm\sqrt{\frac{y}{C}}\right)^{n+m}\int_{-1}^{1}U_n(x)U_m(x)\sqrt{1-x^2}\sqrt{yC}dx\\
=&\frac{\sqrt{yC}}{C^2}\left(\frac{y}{C}\right)^n\frac{\pi}{2}\delta_{n,m},
\end{aligned}
$$

where the rightmost integral yields $(\pi/2)\delta_{n,m}$ due to the orthogonality of $\{U_n(x)\}$ (see, for examples, [24] by Mason and Handscomb and [25] by Rivlin).

Hence, using a transformation we obtain

$$
\begin{aligned}
&\int_{g^{-1}(-\sqrt{yC})}^{g^{-1}(\sqrt{yC})} P^{1,y,C}(x)P_m^{1,y,C}(x)g'(x)\sqrt{1-\frac{(g(x))^2}{yC}}dx\\
=&\int_{-\sqrt{yC}}^{\sqrt{yC}} P^{1,y,C}(x)P_m^{1,y,C}(x)\sqrt{1-\frac{x^2}{yC}}dx\\
=&\frac{\sqrt{yC}}{C^2}\left(\frac{y}{C}\right)^n\frac{\pi}{2}\delta_{n,m}.
\end{aligned}
$$

■

Corollary 4.4 *Let $\{P_n^{1,C,C}(x)\}$, $C\neq 0$, be a polynomial sequence defined by (7) with $\lambda=1$, and let $g(x)$ be a one-to-one and differential function. Then sequence $\{P_n^{1,C,C}(g(x))\}$ is an orthogonal polynomial sequence satisfying recurrence relation (10) associated with the density function*

$$w(x)=\frac{g'(x)}{|C|}\sqrt{C^2-(g(x))^2}$$

with support interval between $g^{-1}(-|C|)$ and $g^{-1}(|C|)$, where $g^{-1}(x)$ is the composition inverse of $g(x)$, i.e., $(g^{-1}\circ g)(x)=(g\circ g^{-1})(x)=x$. Furthermore,

$$\int_{g^{-1}(-|C|)}^{g^{-1}(|C|)} P_n^{1,C,C}(x)P_m^{1,C,C}(x)\frac{g'(x)}{|C|}\sqrt{C^2-(g(x))^2}dx=\frac{\pi|C|}{2C^2}\delta_{n,m}, \tag{44}$$

where $\delta_{n,m}$ is the Kronecker symbol.

In particular, if $g(x) = x$, then $\{P_n^{1,C,C}(x)\}$ is an orthogonal polynomial sequence with respect to density function $\sqrt{1-x^2/C^2}$ over support interval $[-|C|, |C|]$, and $\{P_n^{1,C,C}(x)\}$ satisfies (44) when $g(x) = g^{-1}(x) = x$.

Example 5 From Theorem 4.3, Morgan-Voyc polynomial sequence $\{B_n(x) = P_n^{1,1,1}\left(\frac{x}{2}+1\right)\}$ is orthogonal with respect to the density function $w(x) = \sqrt{-4x-x^2}/4$ with support $[-4, 0]$. The sequence of Fermat polynomials of the first kind, $\{\Phi_n(x) = P_{n-1}^{1,2,1}(x/2)\}$, is orthogonal with respect to the density function $w(x) = \sqrt{8-x^2}/(4\sqrt{2})$ with support $[-2\sqrt{2}, 2\sqrt{2}]$. Dickson polynomials $\{D_n(x,a) = P_n^{1,2a,2}(x)\}$ of the second kind are orthogonal when $a > 0$ with respect to the density function $w(x) = \sqrt{4a-x^2}/(2\sqrt{a})$ over the support interval $[-2\sqrt{a}, 2\sqrt{a}]$. In addition, we have

$$\int_{-4}^{0} B_n(x)B_m(x)\frac{\sqrt{-4x-x^2}}{4}dx = \frac{\pi}{2}\delta_{n,m},$$

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \Phi_n(x)\Phi_m(x)\frac{\sqrt{8-x^2}}{4\sqrt{2}}dx = \pi 2^{n-(1/2)}\delta_{n,m},$$

$$\int_{-2\sqrt{a}}^{2\sqrt{a}} D_n(x,a)D_m(x,a)\frac{\sqrt{4a-x^2}}{2\sqrt{a}}dx = \frac{\pi}{4}a^{n+1/2}\delta_{n,m}.$$

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