Matrix Characterizations of Riordan Arrays

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Abstract

Here we discuss two matrix characterizations of Riordan arrays, P-matrix characterization and A-matrix characterization. P-matrix is an extension of the Stieltjes matrix defined in [25] and the production matrix defined in [7]. By modifying the marked succession rule introduced in [18], a combinatorial interpretation of the P-matrix is given. The P-matrix characterizations of some subgroups of Riordan group are presented, which are used to find some algebraic structures of the subgroups. We also give the P-matrix characterizations of the inverse of a Riordan array and the product of two Riordan arrays. A-matrix characterization is defined in [17], and it is proved to be a useful tool for a Riordan array, while, on the other side, the A-sequence characterization is very complex sometimes. By using the fundamental theorem of Riordan arrays, a method of construction of A-matrix characterizations from Riordan arrays is given. The converse process is also discussed. Several examples and applications of two matrix characterizations are presented.

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1 Introduction

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [26]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Luzón et al. [16] and Sprugnoli [27, 28], on subgroups of the Riordan group in Jean-Louis and Nkwanta [15], Peart and Woan [20], and Shapiro [23], on some characterizations of Riordan matrices in Rogers [22], Merlini et al. [17], and He and Sprugnoli [13], and on many interesting related results in Cheon et al. [3, 4], Gould et al. [9], He [10, 11], He et al. [12], Nkwanta [19], Shapiro [24, 25], Wang et al. [30], and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that

$f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $f^{-1}(t)$ exists such that $f(f^{-1}(t)) = f^{-1}(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t),\, h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k.

It immediately knows that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t)) * (d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{2}$$

The Riordan array $I = (1,\, t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it can be easily proved that I acts as an identity for this product, that is, $(1,\, t) * (d(t),\, h(t)) = (d(t),\, h(t)) * (1, t) = (d(t),\, h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(h^*(t))},\, h^*(t)\right) \tag{3}$$

where $h^*(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays forms a group.

Several subgroups of $\mathcal{R}$ are important and have been considered in the literature:

- the *Appell subgroup* is the set $\mathcal{A}$ of the Riordan arrays $D = (d(t),\, t)$; it is an invariant subgroup and is isomorphic to the group of f.p.s.'s of order 0, with the usual product as group operation;
- the *associated subgroup* is the set $\mathcal{L}$ of the Riordan arrays $D = (1,\, h(t))$; it is isomorphic with the group of f.p.s.'s of order 1, with composition as group operation;
- the *Bell subgroup* or *renewal subgroup* is the set $\mathcal{B}$ of the Riordan arrays $D = (d(t),\, td(t))$; the set is originally considered by Rogers in [22];
- the *checkerboard subgroup* is the set $\mathcal{C}$ of the Riordan arrays $D = (d(t),\, h(t))$ for which $d(t)$ is an even function and $h(t)$ is an odd function;
- the *hitting-time subgroup* is the set $\mathcal{H}$ of the Riordan arrays $D = (d(t),\, h(t))$ for which $d(t) = th'(t)/h(t)$; the subgroup is originally defined in [20].
- the *derivative subgroup* is the set $\mathcal{D}$ of the Riordan arrays $D = (h'(t),\, h(t))$.

It is clear that there exists a semidirect product decomposition for the Riordan group $\mathcal{R}$: $\mathcal{R} \simeq \mathcal{A} \rtimes \mathcal{B}$, since $(d(t), h(t)) = \left(\frac{td(t)}{h(t)}, t\right)\left(\frac{h(t)}{t}, h(t)\right)$.

From [22], an infinite lower triangular array $[d_{n,k}]_{n,k\in\mathbb{N}} = (d(t), h(t))$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ there holds

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n}, \tag{4}$$

which is shown in [13] to be equivalent to

$$h(t) = tA(h(t)). \tag{5}$$

Here, $A(t)$ is the generating function of the A-sequence. In [13, 17] it is also shown that a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ exists such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{6}$$

or equivalently,

$$d(t) = \frac{d_{0,0}}{1 - tZ(h(t))}. \tag{7}$$

We may write (7) and (5) as

$$\frac{d(t) - d_{0,0}}{t} = Z(h(t)), \quad \frac{d(t)h^n(t)}{t} = t^{n-1}A(h(t)). \tag{8}$$

Denote the upper shift matrix by U, i.e.,

$$U = (\delta_{i+1,j})_{i,j\geq 0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left(Z(t), A(t), tA(t), t^2A(t), \ldots\right), \tag{9}$$

where the rightmost expression is the presentation of P by using its column generating functions. Let M be a matrix. Then UM is the matrix obtained from M by removing the first row of M and elevating the remaining rows of M up one row, i.e., premultiplying a matrix M by an upper shift matrix U results in the entries of M being shifted upward by one position. Thus, (8) can be written in a matrix form by using upper shift matrix U:

$$U(d(t), h(t)) = (d(t), h(t))P \tag{10}$$

because its left-hand side and right-hand side are the same. In fact, due to (8) and the fundamental theorem of Riordan arrays (see [24]) we can write

$$LHS = U(d(t), h(t)) = \left(\frac{d(t) - d_{0,0}}{t}, \frac{d(t)h(t)}{t}, \frac{d(t)h^2(t)}{t}, \frac{d(t)h^3(t)}{t}, \cdots\right)$$

and

$$\begin{aligned} & RHS = (d(t), h(t))P \\ = \; & \left(d(t)Z(h(t)), d(t)A(h(t)), d(t)h(t)A(h(t)), d(t)h^2(t)A(h(t)), \cdots\right), \end{aligned}$$

where the Riordan array $(d(t), h(t))$ can be written as the form $(d(t), d(t)h(t), d(t)h^2(t), \ldots)$ in terms of the generating functions of the columns of $(d(t), h(t))$. Thus, two sides of (10) are equal. Conversely, if $R = (d_{n,k})_{n,k\geq 0}$ is a lower triangle matrix satisfying $UR = RP$, where P is as in (9), then we have

$$\begin{aligned} d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n} \quad for\ n \geq 0, \\ d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_{n-k} d_{n,n}, \quad for\ n \geq k \geq 0, \end{aligned}$$

which implies that R is a Riordan array because of the sequence characterization of Riordan arrays shown in (6) and (4). Hence, we call $P = (p_{n,k})_{n,k\geq 0}$ the *characterization matrix* and have the following matrix characterization of Riordan arrays. If $p_{n,k} \in \mathbb{N}_0$, then P is also named production matrix in [7].

Proposition 1.1 *Let $d(t) \in \mathcal{F}_0$, $h(t) \in \mathcal{F}_1$, and $U = (\delta_{i+1,j})_{i,j\geq 0}$. Then a lower triangle matrix R is a Riordan array if and only if*

$$UR = RP, \tag{11}$$

where P is defined by (9).

If there exists a solution of $MS_M = UM$, then its solution S_M is called a Stieltjes matrix of M (see, for example, [25]). If $M = R$, a proper Riordan array, then it is a lower triangular matrix with nonzero entries on the main diagonal. Thus, its Stieltjes matrix S_R always exists and it is unique. In other words, Equation (11) has a unique solution $P = R^{-1}UR$ when R is a proper Riordan array. Therefore, the characterization matrix P defined by (11) is a Stieltjes matrix. An application of the Stieltjes matrix in the LDL^T decomposition of Hankel matrix is presented in [21].

If the Stieltjes matrix $P = (p_{n,k})_{n,k\geq 0}$ has non-negative integer entries, i.e., $p_{n,k} \in \mathbb{N}_0$, then P is also called a production matrix in [7]. Hence, Proposition 1.1 is reduced to Proposition 3.2 of [7]. More precisely, denoting the so-called *ECO matrix* induced by P by $A_P := (u^T, u^T P, u^T P^2, \ldots)^T \equiv (d_{n,k})_{n,k\geq 0}$, [7] shows

$$d_{n+1,k} = \sum_{j\geq 0} d_{n,j} p_{j,k} = d_{n,0} p_{0,k} + d_{n,1} p_{1,k} + d_{n,2} p_{2,k} + \cdots \tag{12}$$

for $n \geq 0$, and $d_{0,k} = \delta_{0,k}$, the Kronecker delta. Thus, Proposition 3.2 of [7] presents that A_P is a Riordan array if and only if P can be written as in (9). One may see that Proposition 1.1 is an extension of Proposition 3.2 of [7] for the case of an arbitrary Stieltjes matrix P with the form (9). In other words, in Proposition 1.1, we do not need the entries of P to be non-negative.

The ECO method is a constructive method to produce all the objects of a given class, according to the growth of a certain parameter (in terms of the size) of the objects. A complete description of the ECO method and its applications for the enumeration of several classes of combinatorial objects is given in [1]. The roots of the ECO method can be traced back to the paper [5], where the authors study Baxter permutations: for the first time, a combinatorial construction is presented which can be described by means of a generating tree, as it usually happens for every ECO construction. If an ECO construction is sufficiently regular, then it is often possible to describe it using a succession rule, whose definition is first

introduced by Julian West in [31]. Some algebraic properties of succession rules have been determined in [8]. Succession rules play an important role in the general framework of the ECO method: in practice, they are the tool to translate the recursive construction given by the ECO operator, onto a system of functional equation satisfied by the generating function of the studied class. Intimately related to the concept of succession rule is the notion of a generating tree, which is the most common way of representing a succession rule.

A *succession rule* is a formal system consisting of an axiom (a), $a \in \mathbb{N}_+$, and a set of productions

$$\{(k_j) \to (e_1(k_j))(e_2(k_j)) \cdots (e_j(k_j)) : j \in \mathbb{N}\},$$

where $e_i : \mathbb{N}_+ \to \mathbb{N}_+$, for deriving the successors $(e_1(k))(e_2(k)) \cdots (e_k(k))$ of any given label (k), $k \in \mathbb{N}$. Thus, we may write a succession rule, denoted by Ω, as

$$\Omega : \begin{cases} (a) \\ (k) \to (e_1(k))(e_2(k)) \cdots (e_k(k)). \end{cases} \tag{13}$$

$(a), (k), (k)$, and $(e_i(k))$ $(a, k, e_i(k) \in \mathbb{N}_+)$ are called the labels of Ω. The succession rule can be presented as a *generating rooted tree* called a *generating tree*, whose vertices are the labels of Ω, where (a) is the label of the root, and each node labeled (k) has k sons labeled by $(e_1(k)), \ldots, (e_k(k))$, respectively. Denote by f_n the number of nodes at level n in the generating tree determine by Ω. We call $\{f_n\}_{n\geq 0}$ the sequence $\{f_n\}_{n\geq 0}$ associated with Ω. In [6] the authors represent the succession rule Ω by an infinite matrix $P = (p_{i,j})_{i,j\geq 0}$, called the *production matrix* of the succession rule Ω, where $p_{i,j}$ is the number of labels ℓ_j produced by label ℓ_i, and $\{(\ell_k)\}_{k\geq 0}$ is the label set of Ω with the axiom label ℓ_0. Simply speaking, if Ω containing root (a) satisfies succession rule $(a) \to (a) \ldots (a)(a+1) \ldots (a+1) \ldots (a+m) \ldots (a+m)$, which we shorten to $(a)^{n_1}(a+1)^{n_2} \ldots (a+m)^{n_m}$, then the first row of P is $(n_1, n_2, \cdots, n_m, 0, \cdots)$, where n_j $(j = 1, 2, \ldots, m)$ may be zero, say $n_k = 0$, if $(a+k)$ is not a label. Similarly, we may write other rows of P row by row.

In [7] the following result is proved, which can be viewed as a particular case of Proposition 1.1 when A- and Z- sequences are non-negative sequences.

Proposition 1.2 *[7] Let P be an infinite production matrix and let A_P be the ECO matrix induced by P defined before. Then A_P is a Riordan matrix if and only if P is of the form*

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14}$$

where $z_j, a_j \geq 0$ for $j = 0, 1, \ldots$.

Considering succession rule (13) as a *family succession rule* from the kth generation, the node labeled (k), to the $k+1$st generation, we allow a negative exponent for some of $(e_i(k))$ $(i = 1, 2, \ldots, k)$ to distinct the sexual of the $k+1$st generation from a kth generation member. More precisely, if $(e_i(k))$ contains a negative exponent -1, then $(e_i(k))$ is from a kth generation member that has an opposite sexual, otherwise it keeps the same sexual of the kth member. Here, the assumption of two elements have the different sexual might be replaced by that they have opposite properties, for instance, one is in a set and another

is in the complement of the set. Actually, the generating tree of this modified succession rule possesses all properties of the marked labeled generating tree introduced in [18] and later studied in [2]. In this sense, our modification is not new. However, our modification introduces the negative exponent, which can generalize production matrix P to the matrix with negative entries so that any integer matrix defined in (9) can be viewed as a general production matrix associated with a modified succession rule. Hence, an ECO succession rule can be applied in finding a combinatorial interpretation to a Riordan array via its characterization matrix. More precisely, for any integer matrix P defined by (9), we define succession rule presented as

$$\Omega_P : \left\{(a+k)^{\pm 1} \to (a)^{\pm z_k}(a+1)^{\pm a_k}(a+2)^{\pm a_{k-1}}\cdots(a+k+1)^{\pm a_0}, \quad k \geq 0, \right. \tag{15}$$

where label number $a \neq 0$, $(a)^{z_k} = (a)^{z_k}$ if $z_k \geq 0$ and $(1/a)^{-z_k}$ if $z_k < 0$, and $(a+\ell)^{a_j}$ are defined similarly. Again, we understand that $(a)^n$ is the short form of $(a)\cdots(a)$, the list of n produced labels (a). From the definition, we have $(a)^m(a)^n = (a)^{m+n}$ for $m, n \in \mathbb{Z}$. And $(a+k)^{-1}$ produces $(a)^{-z_k}(a+1)^{-a_k}(a+2)^{-a_{k-1}}\cdots(a+k+1)^{-a_0}$ for $k \geq 0$. As we defined above, the element with negative exponent -1, or equivalently, the element $(1/a)$, in the $k+1$st generation of the succession rule (15) is considered as an opposite sexual family member from the previous generation member $(a+k)$ or $(a+k)^{-1}$.

As an example, let $z_0 = 1$, $z_1 = -1$, $a_0 = 1$, $z_k = 0$ for $k \geq 2$, and $a_k = 0$ for $k \geq 1$. Then,

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which has a combinatorial interpretation shown as the following succession rule

$$\Omega_P : \begin{cases} (a) \to (a)(a+1) \\ (a+1) \to (a)^{-1}(a+2) \\ (a+k) \to (a+k+1), \quad k \geq 2. \end{cases}$$

Or equivalently,

$$\begin{array}{cccccccccc} & & & & & (a) & & & & \\ & & (a) & & & & & (a+1) & & \\ & (a) & & (a+1) & & & (a)^{-1} & & (a+2) & \\ (a) & (a+1) & (a)^{-1} & & (a+2) & (a)^{-1} & & (a+1)^{-1} & & (a+3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \end{array}$$

From Proposition 1.1, the corresponding Riordan array is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 1 & 0 & \cdots \\ -1 & -1 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, from (5) and (7), we obtain $R = (d(t), h(t)) = (1/(1-t+t^2), t)$. The row sum sequence has generating function

$$\left(\frac{1}{1-t+t^2}, t\right)\begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix} = \frac{1}{(1-t)(1-t+t^2)},$$

which generates the sequence $\{a_{n+2}\}_{n\geq 0}$. It can be seen that $\{a_n\}_{n\geq 0}$ is the sequence A021823. The P-matrix with negative entries is very useful in studying recursive polynomial sequences such as the Gegenbauer-Humbert-Type polynomial sequences discussed in [9].

In the next section, we shall present some properties of P- matrix characterizations of Riordan arrays including the P-matrix characterizations of some subgroups of the Riordan group, the inverse of a Riordan array, the product of two Riordan arrays, and the extended Riordan arrays. Here, an extended Riordan array is an extension of a proper Riordan array with integer indices, i.e., $(d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{Z}}$. The P-matrix characterization is also used to construct a one-to-one correspondence between the associate subgroup and the Bell subgroup. Section 3 discusses A-matrix characterization, which is defined in [17], by means of which a Riordan array can be easily studied while the A-sequence characterization is very complex sometimes, although the A-matrix of a Riordan array may not be unique. By using the fundamental theorem of Riordan arrays, a method of construction of A-matrix characterizations from given Riordan arrays and its converse process are given. Several examples and applications of two matrix characterizations are given in Section 3.

2 Properties of P-matrix characterization

First, we give the P-matrix characterizations of some important subgroups of the Riordan group.

Proposition 2.1 *Denote by P_M the characterization matrix of the set M. Then, the characterization matrix P of the Appell subgroup is of the form*

$$P_A = \begin{bmatrix} z_0 & 1 & 0 & 0 & 0 & \cdots \\ z_1 & 0 & 1 & 0 & 0 & \cdots \\ z_2 & 0 & 0 & 1 & 0 & \cdots \\ z_3 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{16}$$

where the GF $Z(t)$ of the sequence $\{z_n\}_{n\geq 0}$ satisfies (7) with $h(t) = t$.

The characterization matrix P of the associated subgroup is of the form

$$P_L = \begin{bmatrix} 1 & a_0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & a_0 & 0 & 0 & \cdots \\ 0 & a_2 & a_1 & a_0 & 0 & \cdots \\ 0 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{17}$$

where the GF $A(t)$ of the sequence $\{a_n\}_{n\geq 0}$ satisfies (5) and the GF $Z(t)$ of the sequence $\{z_n\}_{n\geq 0}$ is $Z(t)=1$.

The characterization matrix P_B of the Bell subgroup is of the form

$$P_B = \begin{bmatrix} z_0 & d(0) & 0 & 0 & 0 & \cdots \\ z_1 & z_0 & d(0) & 0 & 0 & \cdots \\ z_2 & z_1 & z_0 & d_(0) & 0 & \cdots \\ z_3 & z_2 & z_1 & z_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{18}$$

where the GF $Z(t)$ of the sequence $\{z_n\}_{n\geq 0}$ satisfies (7) with $h(t)=td(t)$.

The characterization matrix P_C of the checkerboard subgroup is of the form

$$P_C = \begin{bmatrix} 0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & 0 & a_0 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & a_0 & 0 & \cdots \\ z_3 & 0 & a_2 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{19}$$

where the GFs $Z(t)$ and $A(t)$ of the sequences $\{z_n\}_{n\geq 0}$ and $\{a_n\}_{n\geq 0}$ are odd and even functions, respectively.

The characterization matrix P_H of the hitting-time subgroup is of the form

$$P_H = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & \cdots \\ 2a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ 3a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ 4a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{20}$$

where the GFs $Z(t)$ and $A(t)$ of the sequences $\{z_n\}_{n\geq 0}$ and $\{a_n\}_{n\geq 0}$, respectively, satisfy $Z(t)=A'(t)$.

It is obvious that $U^T=(\delta_{i+1,j})^T_{i,j\in N_0}=(\delta_{i,j+1})_{i,j\in N_0}$, which is the lower shift matrix. Thus, $UU^T=I$ and $U^TU=I-diag(1,0,0,\ldots)$, because the shift arrays U and U^T are infinite. However, for an $n\times n$ shift matrix U_n, we have $U_nU_n^T=I-diag(0,\ldots,0,1)_n$ and $U_n^TU_n=I-diag(1,0,\ldots,0)_n$. Hence, U^T is the right inverse of U, and by abuse of notation - if not otherwise specified - we will represent the right inverse U^T simply by U^-1.

Definition 2.2 *Let $\mathcal{R}$ be the Riordan group, and let $T:\mathcal{R}\mapsto\mathcal{R}$ be the linear operator defined by*

$$TR = URU^T \tag{21}$$

for every $R\in\mathcal{R}$.

It can be seen that the operator T is well-defined because, for any $R=(d(t),h(t))\in\mathcal{R}$, there holds

$$\begin{aligned} & TR = URU^T \\ = \; & (\delta_{i+1,j})_{i,j\in\mathbb{N}_0}(d(t),d(t)h(t),d(t)h^2(t),\ldots)(\delta_{i,j+1})_{i,j\in\mathbb{N}_0} \\ = \; & \left(\frac{d(t)h(t)}{t},\frac{d(t)h^2(t)}{t},\frac{d(t)h^3(t)}{t},\ldots\right)=\left(\frac{d(t)h(t)}{t},h(t)\right)\in\mathcal{R}. \end{aligned} \tag{22}$$

From Proposition 2.1, we have the following results concerning the algebraic structure of the subgroups of $\mathcal{R}$ and their P-matrices via operator T.

Proposition 2.3 *Let T be the operator defined in Definition 2.2. Then*

$$T(d(t), h(t)) = \left(\frac{d(t)h(t)}{t}, h(t)\right). \tag{23}$$

In addition, T preserves $\mathcal{A}$, i.e., the operator identity of T is the Appell subgroup. More precisely, for any $R_A \in \mathcal{A}$, there holds

$$TR_A = R_A. \tag{24}$$

Furthermore, for any $R \in \mathcal{R}$, if its A-sequence $\{a_n\} \in \mathcal{F}_0$, then there holds

$$TR = R(A(t), t), \tag{25}$$

where $A(t)$ is the generating function of $\{a_n\}$ and $(A(t), t)$ is a Riordan array in $\mathcal{A}$.

In particular, for any $R_B \in \mathcal{B}$, there holds

$$TR_B = R_B(d(0) + tZ(t), t), \tag{26}$$

where $(d(0) + tZ(t), t)$ is a Riordan array in $\mathcal{A}$, $d(0) \neq 0$.

For any $R_C \in \mathcal{C}$, there holds

$$\bar{P}_C T R_B = R_B \tilde{P}_C, \tag{27}$$

where $\bar{P}_C$ is the P-matrix characterization of R_C^{-1}, $\tilde{P}_C$ is the P-matrix characterization of a Riordan array in $\mathcal{C}$, the generating function of $\{z_n\}$ is $Z(t) = (A(t) - a_0)/t$, and the generating function of the A-sequence $\{a_n\}$ is $A(t)$.

Proof. Equation (23) directly follows from (22), and (24) is derived from (22). (24) can also be proved by using the P- matrices, P_A, the P-matrix characterizing an $R_A \in \mathcal{A}$, because

$$R_A = R_A P_A U^T = U R_A U^T.$$

From $UR = RP$ and the fact $PU^T = (A(t), t)$ (see Proposition 2.1), where $(A(t), t) := (A(t), tA(t), t^2A(t), \ldots)$ because $A(t)$ is the generating function of A-sequence $\{a_n\} \in \mathcal{F}_0$, we have

$$TR = URU^T = RPU^T = R(A(t), t).$$

From Proposition 2.1, we also have

$$P_B U^T = (d(0) + tZ(t), t(d(0) + tZ(t)), t^2(d(0) + tZ(t), \ldots) = (d(0) + tZ(t), t),$$

a Riordan array because $d(0) \neq 0$. Thus,

$$TR_B = UR_BU^T = R_B P_B U^T = R_B(d(0) + tZ(t), t).$$

Finally, from Proposition 2.1, we derive

$$UP_CU^T = \left(\frac{A(t) - a_0}{t}, A(t), tA(t), \ldots\right) =: \tilde{P}_C,$$

where $\tilde{P}_C$ is the P-matrix characterization of a Riordan array in $\mathcal{C}$, the generating function of $\{z_n\}$ is $Z(t) = (A(t) - a_0)/t$, and the generating function of the A-sequence $\{a_n\}$ is $A(t)$. Thus,

$$UR_C^{-1}TR_C = UR_C^{-1}UR_CU^T = UP_CU^T = \tilde{P}_C.$$

Noting that $UR_C^{-1} = R_C^{-1}\bar{P}_C$, we obtain (27).

■

In the following we will say two infinite arrays A and B are conjugated with respect to U if they satisfy $A = UBU^{-1} = UBU^T$ or $B = UAU^{-1} = UAU^T$.

Proposition 2.4 *Let $\mathcal{L}$ and $\mathcal{B}$ be the associate subgroup and the Bell subgroup of the Riordan group respectively. Then, there exists a bijection between $\mathcal{L}$ and B, $T : \mathcal{L} \mapsto \mathcal{B}$, defined by $T(1, h(t)) := U(1, h(t))U^T$, and*

$$T(1, h(t)) = \left(\frac{h(t)}{t}, h(t)\right), \tag{28}$$

i.e., $R_L := (1, h(t))$ and $R_B := (h(t)/t, h(t))$ are conjugated with respect to U. In addition, $\mathcal{L}$ and $\mathcal{B}$ are isomorphic. Furthermore, if $R_L = (1, h(t))$ and $R_B = (h(t)/t, h(t))$ are characterized by P_L and P_B, respectively, then, $TP_L = P_B$.

Proof. Consider $T(1, h(t)) := U(1, h(t))U^T$ for any $(1, h(t)) \in L$. Equation (28) may be proved directly from (22). Thus, T maps an element of $\mathcal{L}$ onto a unique element of $\mathcal{B}$. Conversely, for any $R_B = (d(t), td(t)) \in \mathcal{B}$, there exists one and only one $R_L = (1, td(t)) \in \mathcal{L}$ such that $R_B = (d(t), td(t)) = T(1, td(t)) := U(1, td(t))U^T$. Hence, L and B are bijective correspondence, and R_B and R_L are conjugated with respect to U. Moreover, T is also a homomorphism. Indeed, for any two elements $(1, h_1(t))$ and $(1, h_2(t))$ in $\mathcal{L}$, we have

$$\begin{aligned} T\left((1, h_1(t))(1, h_2(t))\right) &= T\left(1, h_2(h_1(t))\right) = \left(\frac{h_2(h_1(t))}{t}, h_2(h_1(t))\right) \\ = \left(\frac{h_1(t)}{t}, h_1(t)\right)&\left(\frac{h_2(t)}{t}, h_2(t)\right) = \left(T(1, h_1(t))\right)\left(T(1, h_2(t))\right). \end{aligned}$$

Thus $\mathcal{L}$ and $\mathcal{B}$ are isomorphic via the linear operator T defined in Definition 2.2.

Denote by P_L and P_B the P-matrix characterizations of $R_L = (1, h(t))$ and $R_B = (h(t)/t, h(t))$, respectively. Then, from Proposition 1.1

$$P_B = (Z(t), A(t), tA(t), \ldots) = (Z(t), d(0) + tZ(t), t(d(0) + tZ(t), \ldots),$$

where $d(0) = (h(t)/t)|_{t=0}$ and $A(t) = d(0) + tZ(t)$, which comes out from the characterization of Bell Riordan arrays (see Theorem 2.5 of [13]). Since $A(t) = d(0) + tZ(t)$ we have $a_0 = A(0) = d(0)$. On the other hand,

$$UP_LU^T = \left(\frac{A(t) - a_0}{t}, A(t), tA(t), \ldots\right) = (Z(t), A(t), tA(t), \ldots),$$

which implies $UP_LU^T = P_B$.

■

Proposition 2.5 *Let $\mathcal{H}$ and $\mathcal{D}$ be the hitting-time subgroup and derivative subgroup of Riordan array group $\mathcal{R}$, respectively, and let T be the linear operator defined in Definition 2.2. Then, there exists a bijection between $\mathcal{H}$ and $\mathcal{D}$, $T : \mathcal{H} \mapsto \mathcal{D}$ such that for any $(th'(t)/h(t), h(t)) \in \mathcal{H}$ there holds*

$$T\left(\frac{th'(t)}{h(t)}, h(t)\right) = (h'(t), h(t)), \tag{29}$$

i.e., $R_H := (th'(t)/h(t), h(t))$ and $R_D := (h'(t), h(t))$ are conjugated with respect to U. In addition, $\mathcal{H}$ and $\mathcal{D}$ are isomorphic.

Proof. Let $R_H(th'(t)/h(t), h(t))$ be an element of $\mathcal{H}$. Then, (22) yields

$$T(R_H) = \left(\frac{th'(t)}{h(t)}\frac{h(t)}{t}, h(t)\right) = (h'(t), h(t)) \in \mathcal{D}.$$

Conversely, for any $R_D = (h'(t), h(t)) \in \mathcal{D}$, there exists a unique $R_H = (th'(t)/h(t), h(t)) \in \mathcal{H}$ such that $TR_H = R_D$. $T : \mathcal{H} \mapsto \mathcal{D}$ is also an isomorphism because of its homomorphic property: for any two elements $(th_1'(t)/h_1(t), h_1(t))$ and $(th_2'(t)/h_2(t), h_2(t))$ in $\mathcal{H}$, there holds

$$\begin{aligned} & T\left(\left(\frac{th_1'(t)}{h_1(t)}, h_1(t)\right)\left(\frac{th_2'(t)}{h_2(t)}, h_2(t)\right)\right) = T\left(\frac{th_1'(t)}{h_1(t)}\frac{h_1(t)h_2'(h_1(t))}{h_2(h_1(t))}, h_2(h_1(t))\right) \\ = \; & (h_1'(t)h_2'(h_1(t)), h_2(h_1(t))) = (h_1'(t), h_1(t))(h_2'(t), h_2(t)) \\ = \; & \left(T\left(\frac{th_1'(t)}{h_1(t)}, h_1(t)\right)\right)\left(T\left(\frac{th_2'(t)}{h_2(t)}, h_2(t)\right)\right). \end{aligned}$$

Thus, $R_H := (th'(t)/h(t), h(t))$ and $R_D := (h'(t), h(t))$ are conjugated with respect to U.

■

We point out that the existence of the isomorphism between L and B was proved in [15] by using an approach different from ours. In fact, our approach provides a constructive proof of the isomorphism between two subgroups by using the linear operator T defined by (21). Since [15] has shown the isomorphism between $\mathcal{D}$ and $\mathcal{L}$, from Propositions 2.4 and 2.5 and the equivalence of isomorphisms, we know that subspaces $\mathcal{H}$, $\mathcal{D}$, $\mathcal{L}$, and $\mathcal{B}$ are isomorphic.

Proposition 2.4 can be used to find P_L-matrix characterization from P_B-matrix characterization and vice versa. For instance, let us consider the Riordan array $(1, t/(1-bt))$. We find its P-matrix is

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & b & 1 & 0 & 0 & \cdots \\ 0 & b^2 & 2b & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then the P_B-matrix for Bell type Riordan array $(1/(1-bt), t/(1-bt))$ is

$$P_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ b & 1 & 0 & 0 & 0 & \cdots \\ b^2 & 2b & 1 & 0 & 0 & \cdots \\ b^3 & 3b^2 & 3b & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Equivalently, the A-sequence generating function and Z-sequence generating function for $(1, t/(1-bt))$ are $A(t) = 1 + bt$ and $Z(t) = 0$, and for $(1/(1-bt), t/(1-bt))$ are $A(t) = 1+bt = 1+tZ(t)$, which are exactly the sequence characterizations of the associated subgroup and Bell subgroup shown in Theorems 2.4 and 2.5, respectively, in [13].

Denote the inverse of $R = (d(t), h(t))$ by $R^{-1} = (f(t), g(t))$. Then from [13], there holds

$$f(t) = \frac{1}{d(\bar{h}(t))}, \quad g(t) = \bar{h}(t), \tag{30}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. From (11) of Proposition 1.1, there holds

$$U = RPR^{-1}. \tag{31}$$

Denote by $\bar{P}$ the characterization matrix of R^{-1}, from (31) we have

$$R^{-1}\bar{P}R = U = RPR^{-1}.$$

Thus, we obtain the following corollary of Proposition 1.1.

Corollary 2.6 *Let $R = (d(t), h(t))$ be a Riordan array with characterization matrix P, and $R^{-1} = (f(t), g(t))$ be the inverse of R with characterization matrix $\bar{P}$. Then, there holds*

$$\bar{P} = R^2 P \left(R^{-1}\right)^2, \tag{32}$$

or equivalently, $\bar{P}R^2 = R^2P$. In particular, if R is an involution (i.e., $R^2 = I$), then $\bar{P} = P$. Note, in general $\bar{P} \neq P^{-1}$.

Proof. The proof of the corollary is obvious. We only need to give the following counterexample to show $\bar{P} \neq P$. Consider Pascal triangle $R = (1/(1-t), t/(1-t))$. Its characterization matrix is $P = (Z(t), A(t)) = (1, 1+t)$, i.e.,

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{33}$$

It is easy to prove that the inverse of R is $R^{-1} = (1/(1+t), t/(1+t))$, whose characterization matrix is $\bar{P} = (Z^*(t), A^*(t)) = (-1, 1-t)$. Therefore,

$$\bar{P} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{34}$$

which implies $\bar{P} \neq P^{-1}$.

■

Proposition 2.7 *Let $R = (d(t), h(t))$ be a Riordan array, and let P be the characterization matrix of R. Then there holds*

$$P = R^{-1}UR, \tag{35}$$

where $R^{-1} = (f(t), g(t))$ is the inverse Riordan array of R, or equivalently, the generating functions $Z(t)$ and $A(t)$ of the first and second columns of P satisfy

$$Z(t) = f(t)\frac{d(g(t)) - 1}{g(t)}, \quad A(t) = tf(t)\frac{d(g(t))}{g(t)}. \tag{36}$$

Proof. It is clear that (35) is another form of (11). Using the fundamental formula of Riordan arrays, we obtain

$$\begin{aligned} & R^{-1}UR \\ = \; & \left(f(t), f(t)g(t), f(t)(g(t))^2, \ldots\right)\left(\frac{d(t)-1}{t}, \frac{d(t)h(t)}{t}, \frac{d(t)(h(t))^2}{t}, \ldots\right) \\ = \; & \left(f(t)\frac{d(g(t))-1}{g(t)}, f(t)\frac{d(g(t))h(g(t))}{g(t)}, f(t)\frac{d(g(t))(h(g(t))^2}{g(t)}, \ldots\right) \\ = \; & \left(f(t)\frac{d(g(t))-1}{g(t)}, f(t)\frac{td(g(t))}{g(t)}, f(t)\frac{t^2d(g(t))}{g(t)}, \ldots\right), \end{aligned}$$

where $h(g(t)) = t$ is used in the last step. Then from the definition of P we have

$$P = \left(Z(t), A(t), tA(t), \ldots\right).$$

Comparing the expressions of $R^{-1}UR$ and P yields (36).

■

Example 2.1 The Pascal triangle $R = (1/(1-t), t/(1-t))$ has the characterization matrix P shown in (33) and the inverse $R^{-1} = (1/(1+t), t/(1+t))$. It can be seen $R^{-1}UR = P$. In addition, it is easy to check that

$$R^2 = \left(\frac{1}{1-2t}, \frac{t}{1-2t}\right), \quad R^{-2} = \left(1+2t, \frac{t}{1+2t}\right),$$

and $\bar{P}R^2 = R^2P$, i.e.,

$$\bar{P}\left(\frac{1}{1-2t}, \frac{t}{1-2t}\right) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & \cdots \\ 4 & 8 & 5 & 1 & 0 & \cdots \\ 8 & 20 & 18 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left(\frac{1}{1-2t}, \frac{t}{1-2t}\right) P.$$

If g is an element of a group G, then the smallest positive integer n such that $g^n = e$, the identity of the group, if it exists, is called the *order* of g. If there is no such integer, then g is said to have *infinite order*. It is well-known (see [24]) that if we restrict all entries of a Riordan array to be integers, then any element of finite order in the Riordan group must have order 1 or 2, and each element of order 2 generates a subgroup of order 2. It is easy to see that the linear operator T defined before maps a Riordan involution to an involution. Hence, we have

Corollary 2.8 *The Riordan array $(d(t), h(t))$ is an involution if and only if $d(t)$ and $h(t)$ satisfy*

$$\bar{h}(t) = h(t), \quad d(t)d(h(t)) = 1,$$

where $^-\bar{h}(t)$ is the compositional inverse of $h(t)$. In addition, $\{T^n(d(t), h(t))\}_{n\geq 0}$ forms a sequence of Riordan involutions.

Denote the characterization matrix of $R = R_1R_2$ by $P_{1,2}$, where R_1 and R_2 are Riordan arrays, and let P_1 and P_2 be the characterization matrices of R_1 and R_2. Thus, we have

Proposition 2.9 *Let P_1, P_2, $P_{1,2}$, and $P_{2,1}$ be the characterization matrices of the Riordan arrays, R_1, R_2, R_1R_2, and R_2R_1, respectively. There hold*

$$P_{1,2} = R_2^{-1}P_1R_2, \quad P_{2,1} = R_1^{-1}P_2R_1. \tag{37}$$

Proof. From Proposition 1.1, we immediately have $(R_1R_2)P_{1,2}(R_1R_2)^{-1} = R_1P_1R_1^{-1}$ and $(R_2R_1)P_{2,1}(R_2R_1)^{-1} = R_2P_2R_2^{-1}$, which clearly imply (37).

■

3 A-matrix characterization of Riordan arrays

The following theorem established in [17] shows that we can characterize a Riordan array by means of an A-matrix, rather than a simple A-sequence with generating function $A(t)$.

Theorem 3.1 *[17] A lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}_0}$ is a Riordan array if and only if there exists another array $(\alpha_{i,j})_{i,j\in\mathbb{N}_0}$ with $\alpha_{0,0} \neq 0$ and a sequence $(\rho_j)_{j\in\mathbb{N}_0}$ such that*

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}d_{n-i,k+j} + \sum_{j\geq 0}\rho_j d_{n+1,k+j+2}. \tag{38}$$

In [17], also given is the following implicit formula to deduce a Riordan array $(d(t), h(t))$ from its associated A-matrix $A = (\alpha_{i,j})_{i,j\in\mathbb{N}_0}$ and the sequence $(\rho_j)_{j\in\mathbb{N}_0}$. It is well-known that the A-sequence and the function $h(t)$ can each be expressed one in terms of the other (see (5)). So after finding the function $h(t)$, we can also find the generating function $A(t)$ of the A-sequence.

$$\frac{h(t)}{t} = \sum_{i\geq 0} t^i \Phi^{[i]}(h(t)) + \frac{h(t)^2}{t}\Psi(h(t)), \tag{39}$$

$$A(t) = \sum_{i\geq 0} t^i A(t)^{-i}\Phi^{[i]}(t) + tA(t)\Psi(t), \tag{40}$$

where $\Phi^{[i]}(t)$ and $\Psi(t)$ are the (ordinary) generating functions of ith row of the A-matrix and the sequence $(\rho_j)_{j\in\mathbb{N}_0}$, i.e.,

$$\Phi^{[i]}(t) = \sum_{j\geq 0} \alpha_{i,j} t^j, \quad i = 1, 2, \ldots,$$

$$\Psi(t) = \sum_{j\geq 0} \rho_j t^j.$$

Formula (40) is easily obtained by transforming t to $\bar{h}(t)$, the compositional inverse of $h(t)$, in formula (39) and then substituting $\bar{h}(t) = t/A(t)$ into the resulting formula. To prove formula (39), we only need to apply (38) to yield

$$\begin{aligned}
[t^{n+1}]d(t)h(t)^{k+1} = d_{n+1,k+1} &= \sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} d_{n-i,k+j} + \sum_{j\geq 0} \rho_j d_{n+1,k+j+2} \\
= \quad & [t^n]d(t)\left[\sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} t^i h(t)^{k+j} + t^{-1}\sum_{j\geq 0} \rho_j h(t)^{k+j+2}\right] \\
= \quad & [t^n]d(t)h(t)^k\left[\sum_{i\geq 0} t^i \Phi^{[i]}(h(t)) + \frac{h(t)^2}{t}\Psi(h(t))\right],
\end{aligned}$$

which implies formula (39).

As mentioned before, in a Riordan array $(d(t), h(t))$, $h(t)$ is related to $A(t)$, the generating function of the A-sequence, while $d(t)$ is related to $Z(t)$, the generating function of the Z-sequence (see (7)). Therefore, [17] presented the following formula to compute the column 0 of Riordan arrays:

$$d_{n+1,0} = \sum_{i\geq 0}\sum_{j\geq 0} \beta_{i,j} d_{n-i,j} + \sum_{j\geq 0} \eta_j d_{n+1,j+2}, \tag{41}$$

which yields generating function $d(t)$ of column 0 as ([17])

$$d(t) = \frac{d_{0,0}}{1 - \sum_{i\geq 0} t^{i+1}\bar{\Phi}^{[i]}(h(t)) - h(t)\bar{\Psi}(h(t))}, \tag{42}$$

where

$$\bar{\Phi}^{[i]}(h(t)) = \sum_{j\geq 0} \beta_{i,j} t^j, \ i = 0, 1, \ldots, \ \textit{and} \ \bar{\Psi}(h(t) = \sum_{j\geq 0} \eta_j t^j.$$

The proof of (42) is similar to the proof of formula (39). We now apply the relation between $d(t)$ and $Z(t)$, the generating function of the Z-sequence, shown in (7), to derive the expression of $Z(t)$ in terms of $\bar{\Phi}$ and $\bar{\Psi}$. From (7), we easily have

$$Z(t) = \frac{1}{\bar{h}(t)}\left(1 - \frac{d_{0,0}}{d(\bar{h}(t))}\right),$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = 1$. Substituting $t = \bar{h}(t))$ into (42) yields

$$d(\bar{h}(t)) = \frac{d_{0,0}}{1 - \sum_{i\geq 0}(\bar{h}(t))^{i+1}\bar{\Phi}^{[i]}(t) - t\bar{\Psi}(t)}.$$

Thus combining the last two expressions of $Z(t)$ and $d(\bar{h}(t))$, we obtain

$$Z(t) = \sum_{i\geq 0}(\bar{h}(t))^{i}\bar{\Phi}^{[i]}(t) - t\bar{\Psi}(t). \tag{43}$$

Actually, formula (40) and formula (43) express a relationship between P-matrix characterization and A-matrix characterization.

In this section, we consider the converse problem, namely, finding an A-matrix from a given Riordan array $(d(t), h(t))$. Denote the bivariate generating function of $(d(t), h(t))$ by $A(x, y)$, i.e.,

$$A(x, y) := \sum_{n\geq 0}\sum_{k\geq 0} d_{n,k}x^n y^k. \tag{44}$$

We have the following result.

Proposition 3.2 *Let $A(x, y)$ be the bivariate generating function of the Riordan array $(d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{N}_0}$, which is characterized by the A-matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}_0}$ and the sequence $(\rho_j)_{j\in\mathbb{N}_0}$. Then, there holds*

$$A(x, y) = (d(x), h(x))\frac{1}{1 - yx} \tag{45}$$

and

$$\begin{aligned} A(x, y) = {} & \frac{1}{1 - \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}x^{i+1}y^{1-j} - \sum_{j\geq 0}\rho_j y^{-j-1}} \\ & \times\left(1 - \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}\left(\sum_{n\geq 0}\sum_{k=0}^{j-2} d_{n,k}x^{n+i+1}y^{k-j+1} - \sum_{n=-i}^{-1}\sum_{k\geq j-1} d_{n,k}x^{n+i+1}y^{k-j+1}\right)\right. \\ & \left. - \sum_{j\geq 0}\rho_j\sum_{n\geq 0}\sum_{k=0}^{j} d_{n,k}x^n y^{k-j-1}\right), \end{aligned} \tag{46}$$

where $d_{n,k} = [t^n]d(t)h(t)^k$, $n, k \in \mathbb{Z}$, in which $d(t)h(t)^k$ is a formal Laurent series (see [16] and the end of Section 2).

Proof. From (44) and the fundamental theorem of Riordan arrays, we have

$$A(x,y)=\sum_{n\geq 0}\sum_{k\geq 0}d_{n,k}x^ny^k=\frac{d(x)}{1-yh(x)}=(d(x),h(x))\frac{1}{1-yx}.$$

By using the A-matrix characterization, we have

$$\begin{aligned} A(x,y) &= \sum_{n\geq 0}\sum_{k\geq 0}d_{n,k}x^ny^k \\ &= 1+\sum_{n\geq 1}\sum_{k\geq 0}\left(\sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}d_{n-i-1,k+j-1}x^ny^k\right)+\sum_{n\geq 0}\sum_{k\geq 0}\left(\sum_{j\geq 0}\rho_jd_{n,k+j+1}\right)x^ny^k, \end{aligned}$$

which implies (46).

■

Proposition 3.2 shows that in the expansion

$$(d(x),h(x))\frac{1}{1-yx}=\frac{1-f(x,y)}{1-g(x,y)},$$

where $g(x,y)\neq 1$, and both $f(x,y)$ and $g(x,y)$ have no constant terms, the coefficients of powers $x^{i+1}y^{1-j}$ and y^{-j-1} of $g(x,y)$ give the entry $\alpha_{i,j}$ of A-matrix and the element ρ_j of the ρ-sequence, respectively. More precisely, we have

Corollary 3.3 *Let $(d(t),h(t))$ be a Riordan array, and let $(d(x),h(x))/(1-yx)=(1-f(x,y))/(1-g(x,y))$, where $g(x,y)\neq 1$ and both $f(x,y)$ and $g(x,y)$ have no non-zero constant terms. Then $(d(t),h(t))$ is characterized by an A-matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}_0}$ and a ρ-sequence $\{\rho_j\}_{j\in\mathbb{N}_0}$, where*

$$\alpha_{i,j}=\left[x^iy^j\right]\frac{y}{x}g\left(x,\frac{1}{y}\right)\ \textit{and}\ \rho_j=\left[y^j\right]\frac{1}{y}g\left(x,\frac{1}{y}\right).$$

Proof. Using Proposition 3.2 and noting

$$\begin{aligned} \left[x^{i+1}y^{1-j}\right]g(x,y) &= \left[x^iy^j\right]\frac{y}{x}g\left(x,\frac{1}{y}\right)\ \textit{and} \\ \left[y^{-j-1}\right]g(x,y) &= \left[y^j\right]\frac{1}{y}g\left(x,\frac{1}{y}\right), \end{aligned}$$

we immediately obtain the results.

■

Example 3.1 Considering the Fibonacci matrix $(d(t),h(t))=(1,t+t^2)$, we have

$$(1,x+x^2)\frac{1}{1-yx}=\frac{1}{1-xy-x^2y}.$$

Then $g(x,y) = xy + x^2y$ and $g(x,1/y) = x/y + x^2/y$. Thus, the A-matrix of $(1, t+t^2)$ is $(\alpha_{i,j})_{i,j\in\mathbb{N}_0}$, where

$$\alpha_{i,j} = \left[x^i y^j\right] \frac{y}{x}\left(\frac{x}{y} + \frac{x^2}{y}\right) = \left[x^i y^j\right](1+x), \quad i,j \in \mathbb{N}_0,$$

which implies

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the $\rho_j = 0$ for all $j = 0, 1, \ldots$. Thus the entry $F_{n,k}$ of $(1, t+t^2)$ satisfies

$$F_{n+1,k+1} = F_{n,k} + F_{n-1,k}.$$

However, the A-sequence of $(1, t+t^2)$ is very complicated, which can be seen from the expansion of its generating function

$$A(t) = \frac{1+\sqrt{1+4t}}{2} = 1 + t - t^2 + 2t^3 - 5t^4 + 14t^5 - 42t^6 + 132t^7 - 429t^8 + \cdots.$$

For some given A-matrices and ρ-sequences, Proposition 3.2 can also be used to construct the corresponding Riordan arrays. In particular, when the ρ-sequence is the zero sequence, the application of Proposition 3.2 is simpler than using formulae (39) and (42). The following example clearly explains the process. If ρ- sequence is the zero sequence, $\rho_j = 0$ ($j = 0, 1, \ldots$), then the A-matrix characterization is defined by

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j=0}^{1} \alpha_{i,j} d_{n-i,k+j},$$

or equivalently,

$$A = \begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & 0 & 0 & 0 & \cdots \\ \alpha_{1,0} & \alpha_{1,1} & 0 & 0 & 0 & \cdots \\ \alpha_{2,0} & \alpha_{2,1} & 0 & 0 & 0 & \cdots \\ \alpha_{3,0} & \alpha_{3,1} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, from Proposition 3.2 the bivariate generating function of the corresponding Riordan array $(d(t), h(t))$ is

$$\begin{aligned} A(x,y) &= \frac{1}{1 - \sum_{i\geq 0}\sum_{j=0}^{1} \alpha_{i,j} x^{i+1} y^{1-j}} \\ &= \frac{1}{1 - \sum_{i\geq 0} \alpha_{i,1} x^{i+1}} \frac{1}{1 - y\frac{\sum_{i\geq 0}\alpha_{i,0}x^{i+1}}{1-\sum_{i\geq 0}\alpha_{i,1}x^{i+1}}}, \end{aligned}$$

which implies

$$(d(t), h(t)) = \left(\frac{1}{1 - \sum_{i\geq 0} \alpha_{i,1} t^{i+1}}, \frac{\sum_{i\geq 0} \alpha_{i,0} t^{i+1}}{1 - \sum_{i\geq 0} \alpha_{i,1} t^{i+1}} \right).$$

Example 3.2 As an example, if ρ- sequence is the zero sequence, $\rho_j = 0$ $(j = 0, 1, \ldots)$, and A-matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

then the corresponding Riordan array $(d_{n,k})_{n,k\in\mathbb{N}_0} = (d(t), h(t))$ satisfies the recurrence relation

$$d_{n+1,k+1} = d_{n,k} + d_{n,k+1} + d_{n-1,k}.$$

From Proposition 3.2, we find the bivariate generating function of $(d(t), h(t))$ is

$$A(x, y) = \frac{1}{1 - x - xy - x^2 y} = \frac{1}{1-x} \frac{1}{1 - xy\frac{1+x}{1-x}}.$$

Hence, the corresponding Riordan array is

$$(d(t), h(t)) = \left(\frac{1}{1-t}, t\frac{1+t}{1-t} \right),$$

which is the Delannoy matrix with its matrix form

$$(d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{N}_0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 5 & 5 & 1 & 0 & \cdots \\ 1 & 7 & 13 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is known that $d_{n,k} = D(n-k, k)$ for $n \geq k \geq 0$ and 0 otherwise, where $D(n, k)$ is the Delannoy number that gives the number of lattice paths from $(0, 0)$ to (n, k) via steps $(1, 0)$, $(0, 1)$, and $(1, 1)$.

4 Applications of characterization matrices

4.1 Recursive polynomial sequences

First we apply the P-matrix characterization to derive the recurrence relations of some recursive polynomial sequences. Let $R^{-1} = (f(t), g(t))$ be the inverse of $R = (d(t), h(t))$. From (11) of Proposition 1.1, there holds

$$PR^{-1} = R^{-1}U. \tag{47}$$

Let V be an infinite vector $V = (v_0(x), v_1(x), \ldots)^T$, and let $v(x,t)$ be the ordinary GF with the parameter x. We apply the operators on two sides of (47) to V and obtain

$$P(R^{-1}V) = R^{-1}(UV). \tag{48}$$

Thus, we obtain

Theorem 4.1 *Let $v(x;t)$ be the ordinary generating function of the infinite vector $V = (v_0(x), v_1(x), \ldots)^T$, and let $R = (d(t), h(t))$ be a Riordan array with inverse $R^{-1} = (f(t), g(t))$ and characterization matrix P. Then there holds*

$$f(0)v_0(x)\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} [t^1]\, f(t)v(x;g(t)) \\ [t^2]\, f(t)v(x;g(t)) \\ [t^3]\, f(t)v(x;g(t)) \\ \vdots \end{bmatrix} = R^{-1}\begin{bmatrix} v_1(x) \\ v_2(x) \\ \vdots \end{bmatrix}, \tag{49}$$

or equivalently,

$$\begin{aligned} & f(0)v_0(x)Z(t) + \frac{A(t)}{t}\left(f(t)v(x;g(t)) - f(0)v_0(x)\right) \\ = \quad & \frac{f(t)}{g(t)}\left(v(x;g(t)) - v_0(x)\right), \end{aligned} \tag{50}$$

where $Z(t)$ and $A(t)$ are the characterization sequences of R.

Proof. Denote by M_Z and M_A the first column vector and the remaining part of P, respectively. Then $P = [M_z\ M_A]$. Since the generating function of $R^{-1}V$ is

$$\mathcal{G}\left(R^{-1}V\right) = \mathcal{G}\left((f(t), f(t)g(t), f(t)(g(t))^2, \ldots)\begin{bmatrix} v_0(x) \\ v_1(x) \\ \vdots \end{bmatrix}\right) = f(t)v(x;g(t)),$$

where $v(x;g(t)) = \sum_{i\geq 0} v_i(x)(g(t))^i$, we may write the left-hand side of (48) as $[M_Z\ M_A](R^{-1}V)$, which implies (49) and (50).

■

Example 4.1 In particular, if $V := (1, x, x^2, \ldots)^T$, then its ordinary generating function is $v(x;t) = 1/(1-xt)$, and (49) yields

$$\begin{bmatrix} a_0 & 0 & 0 & \cdots \\ a_1 - x & a_0 & 0 & \cdots \\ a_2 & a_1 - x & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} [t^1]\, f(t)/(1-xg(t)) \\ [t^2]\, f(t)/(1-xg(t)) \\ [t^2]\, f(t)/(1-xg(t)) \\ \vdots \end{bmatrix} = f(0)\begin{bmatrix} x - z_0 \\ -z_1 \\ -z_2 \\ \vdots \end{bmatrix}. \tag{51}$$

Denote

$$p_n(x) := [t^n]\frac{f(t)}{(1-xg(t))} \equiv \sum_{k=0}^{n} d_{n,k}x^k,$$

where $d_{n,k} = [t^n]f(t)(g(t))^k$ is the (n,k) entry of Riordan array $(d(t),h(t))$, and $p_0(x) = d_{0,0} = f(0)$. Then (51) yields the following recurrence relation of polynomial sequence $\{p_n(x)\}_{n\geq 0}$ by comparing the nth elements of two sides' vectors of (51):

$$a_0 p_n(x) = -z_{n-1}f(0) - a_{n-1}p_1(x) = -a_{n-2}p_2(x) - \cdots - a_2 p_{n-2}(x) - (a_1 - x)p_{n-1}(x) \tag{52}$$

for $n \geq 2$, and $a_0 p_1(x) = f(0)(x - z_0)$. Recurrence relation (52) is studied in [29]. From (50) there holds

$$f(0)Z(t) + \frac{A(t)}{t}\left(\frac{f(t)}{1-xg(t)} - f(0)\right) = \frac{f(t)}{g(t)}\left(\frac{1}{1-xg(t)} - 1\right).$$

If $V := (1, x/1!, x^2/2!, \ldots)^T$, then $v(x;t) = e^{xt}$. From (50) there holds

$$f(0)Z(t) + \frac{A(t)}{t}\left(f(t)e^{xg(t)} - f(0)\right) = \frac{f(t)}{g(t)}\left(e^{xg(t)} - 1\right). \tag{53}$$

If $V := (1, x, x^2/2, x^3/3\ldots)^T$, then $v(x;t) = 1 - \ln(1-xt)$. From (50) there holds

$$\begin{aligned} & f(0)Z(t) + \frac{A(t)}{t}\left(f(t)(1-\ln(1-xg(t))) - f(0)\right) \\ = \; & \frac{f(t)}{g(t)}\left((1-\ln(1-xg(t))) - 1\right). \end{aligned} \tag{54}$$

From (53) and (54), we may obtain recurrence relations of polynomial sequences $\{p_n(x) = [t^n]f(t)e^{xg(t)}\}_{n\geq 0}$ and $\{p_n(x) = [t^n]f(t)(1-\ln(1-xg(t)))\}_{n\geq 0}$, respectively.

4.2 Weighted row sums of Riordan arrays

In [25] a method to study combinatorial sequences by using Stieltjes matrices is provided. The key step of the method is the Riordan array decomposition of the sequences. For the sake of the readers' convenience, we provide a survey of the method presented in [25]. Let $\{a_n\}_{n\geq 0}$ be a given combinatorial sequence, called the *target sequence*, with generating function $a(t)$. The sequence can be considered as row sums of a Riordan array $L = (d(t), h(t))$ in various subgroups of the Riordan group. $(d(t), h(t))$ is called the Riordan array subgroup decomposition of the target sequence, if the matrix satisfies

$$(d(t), h(t)) \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}, \tag{55}$$

or equivalently,

$$d(t)\frac{1}{1-h(t)} = a(t). \tag{56}$$

If we find the Stieltjes matrix S_L of L, then we can find a bijection with an ECO succession rule to give a combinatorial interpretation to the original given sequence. We now extend

the method to a more generalized case. More precisely, we consider a target sequence as row weighted sums of a Riordan array $L = (d(t), h(t))$ associated with a weight set $\{b_n\}_{n\geq 0}$, called the Riordan array decomposition of the target sequence associated with $\{b_n\}$, where the Riordan array satisfies

$$(d(t), h(t)) \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}, \tag{57}$$

or equivalently,

$$d(t)b(h(t)) = c(t). \tag{58}$$

In particular, if $b_n = 1$ for $n = 0, 1, \ldots$, then (57) and (58) become (55) and (56), respectively. Our approach is based on the observation that S_L is the characterization matrix that has the structure of the matrix in (11) and on the application of Proposition 1.1. Our goal is to find a decomposition of $\{c_n\}_{n\geq 0}$ by using a Riordan array $(d(t), h(t))$ such that its nth row sum with weights $\{b_n\}_{n\geq 0}$ is c_n. Let $b(t)$ be the generating function of $\{b_n\}_{n\geq 0}$. For the sake of convenience, we consider $\{b_n\}_{n\geq 0}$ as the Z-sequence of $(d(t), t)$ for a suitable $d(t)$. From Proposition 1.1, we have $(d(t), h(t))b(t) = d(t)b(h(t)) = (d(t) - 1)/t$, or equivalently,

$$b(h(t)) = \frac{d(t) - 1}{td(t)}. \tag{59}$$

From the fundamental theorem of Riordan arrays, $(d(t), h(t))b(t) = d(t)b(h(t))$. We set $d(t)b(h(t)) = c(t)$. Solving this equation and equation (59), we obtain $d(t) = 1 + tc(t)$ and $h(t) = b^{-1}(c(t)/(1+tc(t))$. The weighted sum of the nth row of obtained $(d(t), h(t))$ is equal to c_n, $0, 1, \ldots$.

4.3 Computation of generalized Stirling number sequences

The third application of matrix characterizations is the computation of exponential Riordan arrays' elements. Here, an exponential Riordan array is an infinite lower triangular array whose column k ($k = 0, 1, \ldots$) has the exponential generating function $d(t)(h(t))^k/k!$. From expression (12) in Theorem 2 of [14] with $\alpha\beta \neq 0$, we have the generating function of the generalized Stirling numbers shown below:

$$\frac{1}{k!}(1 + \alpha t)^{\gamma/\alpha} \left(\frac{(1 + \alpha t)^{\beta/\alpha} - 1}{\beta} \right)^k = \sum_{n\geq 0} S(n, k) \frac{t^n}{n!}, \tag{60}$$

where $S(n, k) \equiv S(n, k, \alpha, \beta, \gamma)$.

In [7] an interesting algorithm is provided in computation of $d_{n,k}$, the entries of exponential Riordan array, by using two different sequences, c-sequence and r-sequence. Denote by $c(t)$ and $r(t)$ the c-sequence and r-sequence, respectively. The following relationships between $c(t)$ and $d(t)$ and $r(t)$ and $h(t)$ are given in [7]:

$$c(h(t)) = d'(t)/d(t), \qquad r(h(t)) = h'(t). \tag{61}$$

Let us consider the exponential Riordan array $[S(n,k)] \equiv [S(n,k,\alpha,\beta,\gamma)]$ of the generalized Stirling numbers, whose generating function is given in (62). Thus $[S(n,k)] = (d(t), h(t))$, where

$$d(t) = (1+\alpha t)^{\gamma/\alpha}, \quad h(t) = \frac{(1+\alpha t)^{\beta/\alpha} - 1}{\beta}. \tag{62}$$

It is obvious that the compositional inverse of $h(t)$ is

$$\bar{h}(t) = \frac{(1+\beta t)^{\alpha/\beta} - 1}{\alpha}. \tag{63}$$

From (61) we obtain the generating functions of the c-sequence and of the r-sequence

$$c(x) \equiv c(x,\alpha,\beta,r) = \left.\frac{d'(x)}{d(x)}\right|_{x=\bar{h}(x)} = \left.\frac{r}{1+\alpha x}\right|_{x=\bar{h}(x)} = r(1+\beta x)^{-\alpha/\beta} \tag{64}$$

$$r(x) \equiv r(x,\alpha,\beta,r) = h'(x)|_{x=\bar{h}(x)} = \left.(1+\alpha t)^{(\beta/\alpha)-1}\right|_{x=\bar{h}(x)} = (1-\beta x)^{1-\alpha/\beta}. \tag{65}$$

An algorithm based on [7] can be designed as follows.

Algorithm 4.1 Let $c(x)$ and $r(x)$ be the generating functions of the c-sequence, $\{c_k\}_{k\geq 0}$, and the r-sequence $\{r_k\}_{k\geq 0}$, shown in (64) and (65), respectively. Then the generalized Stirling numbers defined by (60) can be evaluated using the recursive formulas

$$\begin{aligned} S(n+1,0) &= \gamma \sum_{i\geq 0} i! \binom{-\alpha/\beta}{i} \beta^i S(n,i) = \gamma \sum_{i\geq 0} i! \binom{(\alpha/\beta)+i-1}{i} (-\beta)^i S(n,i), \\ S(n+1,k) &= S(n,k-1) \\ &\quad + \frac{1}{k!} \sum_{i\geq k} i! \left(\gamma \binom{-\alpha/\beta}{i-k} + (-1)^{i-k+1} \beta^k \binom{1-(\alpha/\beta)}{i-k+1} \right) \beta^{i-k} S(n,i), \end{aligned} \tag{66}$$

where $S(0,0) = 1$ and $S(n,k) \equiv S(n,k,\alpha,\beta,\gamma)$.

In particular, if $\alpha = \beta \neq 0$, then

$$\begin{aligned} c(x) &= \gamma(1+\beta x)^{-1} = \gamma - \gamma\beta x + \gamma\beta^2 x^2 - \gamma\beta^3 x^3 + \cdots \\ r(x) &= 1. \end{aligned} \tag{67}$$

Thus, we obtain a recursive formula for the computation of $S(n,k) = S(n,k,\beta,\beta,\gamma)$:

$$S(n+1,0) = \gamma \sum_{i=0}^{n} i!(-\beta)^i S(n,i), \quad S(n,n) = 1 \ (n \geq 0) \tag{68}$$

$$S(n+1,k) = S(n,k-1) + \frac{r}{k!} \sum_{i=k}^{n} i!(-\beta)^{i-k} S(n,i) \ (n \geq k \geq 1). \tag{69}$$

Example 4.2 Since the Stirling numbers of the second kind $S(n,k,0,1,0)$ are defined by the exponential Riordan array $(1, e^t - 1)$, one may set $c(x) = 0$ and $r(x) = 1 + x$. Thus Algorithm 3.5 gives

$$\begin{aligned} &S(0,0) = 1, S(n,0) = 0 \ (n \geq 1), S(n,n) = S(n-1,n-1) = 1 \ (n \geq 1), \\ &S(n+1,k) = S(n,k-1) + kS(n,k) \ (n \geq k \geq 1). \end{aligned}$$

Similarly, for the Stirling numbers of the first kind $[S(n,k,1,0,0)] = (1, \ln(1+t))$, we have the corresponding

$$c(x) = 0, \quad r(x) = e^{-x}.$$

Thus,

$$S(0,0) = 1, S(n,0) = 0 \ (n \geq 1), S(n,n) = S(n-1,n-1) = 1 \ (n \geq 1),$$
$$S(n+1,k) = S(n,k-1) + \sum_{i=k}^{n} (-1)^{i-k+1} \binom{i}{k-1} S(n,i) \ (n \geq k \geq 1).$$

Furthermore, Algorithm 4.1 produces some well-known identities and may give rise to some new identities involving the generalized Stirling numbers.

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