Sequences of Numbers Meet the Generalized Gegenbauer-Humbert Polynomials

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Abstract

Here we present a connection between a sequence of numbers generated by a linear recurrence relation of order 2 and sequences of the generalized Gegenbauer-Humbert polynomials. Many new and known formulas of Fibonacci, Lucas, Pell, Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values are given. The applications of the relationship to the construction of identities of number and polynomial value sequences defined by linear recurrence relations are also discussed.

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Key Words and Phrases: sequence of order 2, linear recurrence relation, Fibonacci sequence, Chebyshev polynomial, the

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generalized Gegenbauer-Humbert polynomial sequence, Lucas number, Pell number.

1 Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and/or classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called sequence of order 2 if it satisfies the linear recurrence relation of order 2:

$$a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{1}$$

for some non-zero constants p and q and initial conditions a_0 and a_1. In Mansour [17], the sequence $\{a_n\}_{n\geq 0}$ defined by (1) is called Horadam's sequence, which was introduced in 1965 by Horadam [12]. [17] also obtained the generating functions for powers of Horadam's sequence. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [8], Hsu [13], Strang [20], Wilf [21], etc.) In [3], Benjamin and Quinn presented many elegant combinatorial meanings of the sequence defined by recurrence relation (1). For instance, a_n counts the number of ways to tile an n-board (i.e., board of length n) with squares (representing 1s) and dominoes (representing 2s) where each tile, except the initial one has a color. In addition, there are p colors for squares and q colors for dominoes. In particular, Aharonov, Beardon, and Driver (see [1]) have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions $a_0 = 0$ and $a_1 = 1$, called the primary solution, can be expressed in terms of Chebyshev polynomial values. For instance, the authors show $F_n = i^{-n}U_n(i/2)$ and $L_n = 2i^{-n}T_n(i/2)$, where F_n and L_n respectively are Fibonacci numbers and Lucas numbers, and T_n and U_n are Chebyshev polynomials of the first kind and the second kind, respectively. Some identities drawn from those relations were given by Beardon in [2]. Marr and Vineyard in [18] use the relationship to establish explicit expression of five-diagonal Toeplitz determinants. In [11], the first two authors presented a new method to construct an explicit formula of $\{a_n\}$ generated by (1). For the sake of the reader's convenience, we cite this result as follows.

Proposition 1.1 *([11]) Let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation (1), and let α and β be two roots of of quadratic equation $x^2 - px - q = 0$. Then*

$$a_n = \begin{cases} \left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & if\ \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & if\ \alpha = \beta. \end{cases} \tag{2}$$

A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, for example, [8], Gould [9], Lidl, Mullen, and Turnwald[16], the first two of authors with Hsu [10])

$$\Phi(t) \equiv (C - 2xt + yt^2)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x)t^n, \tag{3}$$

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (3), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [10])

$$\begin{aligned}
&P_n^{1,1,1}(x) = U_n(x),\ \textit{Chebyshev polynomial of the second kind},\\
&P_n^{1/2,1,1}(x) = \psi_n(x),\ \textit{Legendre polynomial},\\
&P_n^{1,-1,1}(x) = P_{n+1}(x),\ \textit{Pell polynomial},\\
&P_n^{1,-1,1}\left(\frac{x}{2}\right) = F_{n+1}(x),\ \textit{Fibonacci polynomial},\\
&P_n^{1,2,1}\left(\frac{x}{2}\right) = \Phi_{n+1}(x),\ \textit{Fermat polynomial of the first kind},\\
&P_n^{1,2a,2}(x) = D_n(x,a),\ \textit{Dickson polynomial of the second}\\
&\qquad\qquad\qquad\qquad \textit{kind},\ a \neq 0,\ (\textit{see},\ \textit{for example},\ [16]),
\end{aligned}$$

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called Gegenbauer polynomials (see [8]). More results on the Gegenbauer-Humbert-type polynomials can be found in [14] by Hsu and in [15] by the second author and Hsu, etc.

Similarly, for a class of the generalized Gegenbauer-Humbert polynomial sequences defined by

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x) \tag{4}$$

for all $n \geq 2$ with initial conditions

$$P_0^{\lambda,y,C}(x) = \Phi(0) = C^{-\lambda},$$
$$P_1^{\lambda,y,C}(x) = \Phi'(0) = 2\lambda x C^{-\lambda-1},$$

the following theorem has been obtained in [11]

Theorem 1.2 *([11]) Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion (3) can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \quad (5)$$

In this paper, we shall use an alternative form of (2) to establish a relationship between the number sequences defined by recurrence relation (1) and the generalized Gegenbauer-Humbert polynomial sequences defined by (4). Our results are suitable for all such number sequences defined by (1) with arbitrary initial conditions a_0 and a_1, which includes the results in [1] and [2] as our special cases. Many new and known formulas of Fibonacci, Lucas, Pell, Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values and applications of the established relationship to the construction of identities of number and polynomial value sequences will be presented in Section 3.

2 Main results

We now modify the explicit formula of the number sequences defined by linear recurrence relations of order 2. If $\alpha \neq \beta$, the first formula in (2) can be written as

$$\begin{aligned} a_n &= \frac{a_1(\alpha^n - \beta^n) - a_0\alpha\beta(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\ &= \frac{a_1(\alpha^n - \beta^n) + a_0 q(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}, \end{aligned}$$

where the last step is due to α and β being solutions of $t^2 - pt - q = 0$. Noting that $\alpha^2 - p\alpha = \alpha^2 - (\alpha+\beta)\alpha = -\alpha\beta = q$ and $\alpha(\alpha - p) = -\alpha\beta = \beta(\beta - p)$, we may further write the above last expression of a_n as

$$\begin{aligned} a_n &= \frac{a_1(\alpha^n - \beta^n) + a_0(\alpha^2 - p\alpha)(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\ &= \frac{a_1(\alpha^n - \beta^n) + a_0(\alpha^2 - p\alpha)\alpha^{n-1} - a_0(\beta^2 - p\beta)\beta^{n-1}}{\alpha - \beta} \\ &= \frac{a_0(\alpha^{n+1} - \beta^{n+1}) + (a_1 - a_0 p)(\alpha^n - \beta^n)}{\alpha - \beta}. \end{aligned} \tag{6}$$

Denote $r(x) = x + \sqrt{x^2 - Cy}$ and $s(x) = x - \sqrt{x^2 - Cy}$. Comparing expressions (6) and (5), we have reason to consider the following transform: for a non-zero real or complex number k, we set

$$\alpha := \frac{r(x)}{k} \quad \text{and} \quad \beta := \frac{s(x)}{k} \tag{7}$$

for a certain x depends on α, β and k, which we will find out later. Denote $\alpha+\beta = p$ and $\alpha\beta = -q$, i.e., α and β are roots of $t^2 - pt - q$. By adding the two equations in (7) side by side, we obtain $2x = kp$. Thus, when $x = \frac{kp}{2}$, equations in (6) hold. Meanwhile, by using $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 + 4q$, we have

$$r(x) - s(x) = 2\sqrt{x^2 - Cy} = k(\alpha - \beta) = k\sqrt{p^2 + 4q},$$

where $x = kp/2$. Therefore, we obtain

$$2\sqrt{\left(\frac{kp}{2}\right)^2 - Cy} = k\sqrt{p^2 + 4q},$$

which implies

$$k = \pm\sqrt{\frac{Cy}{-q}}. \tag{8}$$

We first consider the case of $k = \sqrt{-Cy/q}$.

We now substitute $r(x) = k\alpha$, $s(x) = k\beta$, $x = kp/2$, and $k = \sqrt{-Cy/q}$ into (6) and simplify as follows.

$$
\begin{aligned}
a_n &= \frac{a_0\left(\left(\frac{r(x)}{k}\right)^{n+1}-\left(\frac{s(x)}{k}\right)^{n+1}\right)+(a_1-a_0p)\left(\left(\frac{r(x)}{k}\right)^{n}-\left(\frac{s(x)}{k}\right)^{n}\right)}{\frac{1}{k}(r(x)-s(x))}\\
&= \frac{a_0(r^{n+1}(x)-s^{n+1}(x))+k(a_1-a_0p)(r^n(x)-s^n(x))}{k^n(r(x)-s(x))}\\
&= a_0C^{n+2}\left(\sqrt{\frac{-q}{Cy}}\right)^n P_n^{1,y,C}\left(\frac{kp}{2}\right)\\
&\quad +(a_1-a_0p)C^{n+1}\left(\sqrt{\frac{-q}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{kp}{2}\right)\\
&= a_0C^{n+2}\left(\sqrt{\frac{-q}{Cy}}\right)^n P_n^{1,y,C}\left(\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right)\\
&\quad +(a_1-a_0p)C^{n+1}\left(\sqrt{\frac{-q}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right). \qquad (9)
\end{aligned}
$$

Similarly, for $k=-\sqrt{-Cy/q}$, we have

$$
\begin{aligned}
a_n &= a_0C^{n+2}\left(-\sqrt{\frac{-q}{Cy}}\right)^n P_n^{1,y,C}\left(-\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right)\\
&\quad +(a_1-a_0p)C^{n+1}\left(-\sqrt{\frac{-q}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(-\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right). \qquad (10)
\end{aligned}
$$

Therefore, we obtain our main result.

Theorem 2.1 *Let sequence $\{a_n\}$ be defined by $a_n=pa_{n-1}+qa_{n-2}$ ($n\geq 2$) with initial conditions a_0 and a_1. Then, a_n can be presented as (9) and (10). In particular, for $(y,C)=(1,1),(-1,1),(2,1), and (2a,2) (a\neq 0)$, respectively, we have*

$$
\begin{aligned}
a_n &= a_0\left(\sqrt{-q}\right)^n U_n\left(\frac{p}{2\sqrt{-q}}\right)+(a_1-a_0p)\left(\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right),\\
a_n &= a_0\left(\sqrt{q}\right)^n P_{n+1}\left(\frac{p}{2\sqrt{q}}\right)+(a_1-a_0p)\left(\sqrt{q}\right)^{n-1} P_n\left(\frac{p}{2\sqrt{q}}\right)\ ,\\
a_n &= a_0\left(\sqrt{q}\right)^n F_{n+1}\left(\frac{p}{\sqrt{q}}\right)+(a_1-a_0p)\left(\sqrt{q}\right)^{n-1} F_n\left(\frac{p}{\sqrt{q}}\right),\\
a_n &= a_0\left(\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(p\sqrt{\frac{2}{-q}}\right)+(a_1-a_0p)\left(\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(p\sqrt{\frac{2}{-q}}\right),\\
a_n &= a_0 2^{n+2}\left(\sqrt{\frac{-q}{4a}}\right)^n D_n\left(p\sqrt{\frac{a}{-q}},a\right)\\
&\quad +(a_1-a_0p)2^{n+1}\left(\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(p\sqrt{\frac{a}{-q}},a\right),
\end{aligned}
$$

and

$$
\begin{aligned}
a_n &= a_0\left(-\sqrt{-q}\right)^n U_n\left(\frac{-p}{2\sqrt{-q}}\right)+(a_1-a_0p)\left(-\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{-p}{2\sqrt{-q}}\right),\\
a_n &= a_0\left(-\sqrt{q}\right)^n P_{n+1}\left(\frac{-p}{2\sqrt{q}}\right)+(a_1-a_0p)\left(-\sqrt{q}\right)^{n-1} P_n\left(\frac{-p}{2\sqrt{q}}\right),\\
\\
a_n &= a_0\left(-\sqrt{q}\right)^n F_{n+1}\left(\frac{-p}{\sqrt{q}}\right)+(a_1-a_0p)\left(-\sqrt{q}\right)^{n-1} F_n\left(\frac{-p}{\sqrt{q}}\right),\\
a_n &= a_0\left(-\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(-p\sqrt{\frac{2}{-q}}\right)+(a_1-a_0p)\left(-\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(-p\sqrt{\frac{2}{-q}}\right),\\
a_n &= a_0 2^{n+2}\left(-\sqrt{\frac{-q}{4a}}\right)^n D_n\left(-p\sqrt{\frac{a}{-q}},a\right)\\
&\quad +(a_1-a_0p)2^{n+1}\left(-\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(-p\sqrt{\frac{a}{-q}},a\right),
\end{aligned}
$$

where $U_n(x)$, $P_n(x)$, $F_n(x)$, $\Phi_n(x)$, and $D_n(x,a)$ are the nth degree Chebyshev polynomial of the second find, Pell polynomial, Fibonacci *polynomial, Fermat polynomial, and Dickson polynomial of the second kind, respectively.*

For the special cases of a_0 and a_1, we have the following corollaries.

Corollary 2.2 *Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = 0$ and $a_1 = d$. Then*

$$
\begin{aligned}
a_n &= d\left(\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right),\\
a_n &= d\left(\sqrt{q}\right)^{n-1} P_n\left(\frac{p}{2\sqrt{q}}\right),\\
a_n &= d\left(\sqrt{q}\right)^{n-1} F_n\left(\frac{p}{\sqrt{q}}\right),\\
a_n &= d\left(\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(p\sqrt{\frac{2}{-q}}\right),\\
a_n &= d2^{n+1}\left(\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(p\sqrt{\frac{a}{-q}}, a\right),
\end{aligned}
$$

and

$$
\begin{aligned}
a_n &= d\left(-\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{-p}{2\sqrt{-q}}\right),\\
a_n &= d\left(-\sqrt{q}\right)^{n-1} P_n\left(\frac{-p}{2\sqrt{q}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_n &= d\left(-\sqrt{q}\right)^{n-1} F_n\left(\frac{-p}{\sqrt{q}}\right),\\
a_n &= d\left(-\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(-p\sqrt{\frac{2}{-q}}\right),\\
a_n &= d2^{n+1}\left(-\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(-p\sqrt{\frac{a}{-q}}, a\right).
\end{aligned}
$$

Corollary 2.3 *Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = c$ and $a_1 = pc$. Then*

$$
\begin{aligned}
a_n &= c\left(\sqrt{-q}\right)^n U_n\left(\frac{p}{2\sqrt{-q}}\right),\\
a_n &= c\left(\sqrt{q}\right)^n P_{n+1}\left(\frac{p}{2\sqrt{q}}\right),\\
a_n &= c\left(\sqrt{q}\right)^n F_{n+1}\left(\frac{p}{\sqrt{q}}\right),\\
a_n &= c\left(\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(p\sqrt{\frac{2}{-q}}\right),\\
a_n &= c2^{n+2}\left(\sqrt{\frac{-q}{4a}}\right)^n D_n\left(p\sqrt{\frac{a}{-q}}, a\right),
\end{aligned}
$$

and

$$
\begin{aligned}
a_n &= c\left(-\sqrt{-q}\right)^n U_n\left(\frac{-p}{2\sqrt{-q}}\right),\\
a_n &= c\left(-\sqrt{q}\right)^n P_{n+1}\left(\frac{-p}{2\sqrt{q}}\right),\\
a_n &= c\left(-\sqrt{q}\right)^n F_{n+1}\left(\frac{-p}{\sqrt{q}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_n &= c\left(-\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(-p\sqrt{\frac{2}{-q}}\right),\\
a_n &= c2^{n+2}\left(-\sqrt{\frac{-q}{4a}}\right)^n D_n\left(-p\sqrt{\frac{a}{-q}}, a\right).
\end{aligned}
$$

If $a_1 = d = 1$, then Corollary 2.2 gives the primary solutions of recurrence relation (1) in terms of the nth degree Chebyshev polynomial of the second kind, Pell polynomial, Fibonacci polynomial, Fermat polynomial, and Dickson polynomial of the second kind, respectively. For instance, if $p = q = 1$, then a_n are Fibonacci numbers F_n. Thus,

$$
\begin{aligned}
F_n &= (i)^{n-1}\, U_{n-1}\left(\frac{1}{2i}\right) = (i)^{n-1}\, U_{n-1}\left(-\frac{i}{2}\right), \\
F_n &= P_n\left(\frac{1}{2}\right), \\
F_n &= F_n\,(1)\,, \\
F_n &= \left(\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(-\sqrt{2}i\right), \\
F_n &= 2^{n+1}\left(\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(-\sqrt{a}i, a\right),
\end{aligned}
$$

and

$$
\begin{aligned}
F_n &= (-i)^{n-1}\, U_{n-1}\left(\frac{i}{2}\right), \\
F_n &= (-1)^{n-1}\, P_n\left(-\frac{1}{2}\right), \\
F_n &= (-1)^{n-1}\, F_n\,(-1)\,, \\
F_n &= \left(-\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\sqrt{2}i\right), \\
F_n &= 2^{n+1}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(\sqrt{a}i, a\right),
\end{aligned}
$$

where $F_n = (i)^{n-1}\, U_{n-1}\left(-\frac{i}{2}\right)$ was shown in [1], and $F_n = (-i)^{n-1}\, U_{n-1}\left(\frac{i}{2}\right)$ was given by Chen and Louck in [5]. From the above expressions of F_n we may obtain many identities. For instance, we have

$$
P_n\left(\frac{1}{2}\right) = (-1)^{n-1}\, P_n\left(-\frac{1}{2}\right) = F_n(1) = (-1)^{n-1} F_n(-1),
$$

$$
\begin{aligned}
& (i)^{n-1}\, U_{n-1}\left(-\frac{i}{2}\right) = (-i)^{n-1}\, U_{n-1}\left(\frac{i}{2}\right) \\
= & \left(\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(-\sqrt{2}i\right) = \left(-\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\sqrt{2}i\right),
\end{aligned}
$$

etc.

We now give another special case of Theorem 2.1 for the sequence defined by (1) with initial cases $a_0 = 2$ and a_1.

Corollary 2.4 *Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = 2$ and $a_1 = p$. Then*

$$\begin{aligned}
a_n &= 2\left(\sqrt{-q}\right)^n U_n\left(\frac{p}{2\sqrt{-q}}\right) - p\left(\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right),\\
a_n &= 2\left(\sqrt{q}\right)^n P_{n+1}\left(\frac{p}{2\sqrt{q}}\right) - p\left(\sqrt{q}\right)^{n-1} P_n\left(\frac{p}{2\sqrt{q}}\right),\\
a_n &= 2\left(\sqrt{q}\right)^n F_{n+1}\left(\frac{p}{\sqrt{q}}\right) - p\left(\sqrt{q}\right)^{n-1} F_n\left(\frac{p}{\sqrt{q}}\right),\\
a_n &= 2\left(\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(p\sqrt{\frac{2}{-q}}\right) - p\left(\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(p\sqrt{\frac{2}{-q}}\right),\\
a_n &= 2^{n+3}\left(\sqrt{\frac{-q}{4a}}\right)^n D_n\left(p\sqrt{\frac{a}{-q}}, a\right)\\
&\quad -p2^{n+1}\left(\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(p\sqrt{\frac{a}{-q}}, a\right),
\end{aligned}$$

and

$$\begin{aligned}
a_n &= 2\left(-\sqrt{-q}\right)^n U_n\left(\frac{-p}{2\sqrt{-q}}\right) - p\left(-\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{-p}{2\sqrt{-q}}\right),\\
a_n &= 2\left(-\sqrt{q}\right)^n P_{n+1}\left(\frac{-p}{2\sqrt{q}}\right) - p\left(-\sqrt{q}\right)^{n-1} P_n\left(\frac{-p}{2\sqrt{q}}\right),
\end{aligned}$$

$$
\begin{aligned}
a_n &= 2\,(-\sqrt{q})^n\, F_{n+1}\left(\frac{-p}{\sqrt{q}}\right) - p\,(-\sqrt{q})^{n-1}\, F_n\left(\frac{-p}{\sqrt{q}}\right),\\
a_n &= 2\left(-\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(-p\sqrt{\frac{2}{-q}}\right) - p\left(-\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(-p\sqrt{\frac{2}{-q}}\right),\\
a_n &= 2^{n+3}\left(-\sqrt{\frac{-q}{4a}}\right)^n D_n\left(-p\sqrt{\frac{a}{-q}}, a\right)\\
&\quad -p2^{n+1}\left(-\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(-p\sqrt{\frac{a}{-q}}, a\right).
\end{aligned}
$$

In addition, we have

$$
a_n = 2\left(\sqrt{-q}\right)^n T_n\left(\frac{p}{2\sqrt{-q}}\right) \tag{11}
$$

and

$$
a_n = 2\left(-\sqrt{-q}\right)^n T_n\left(-\frac{p}{2\sqrt{-q}}\right), \tag{12}
$$

where $T_n(x)$ are Chebyshev polynomials of the first kind.

Proof. It is sufficient to prove (11) and (12). From the first formula shown in Corollary 2.4 and the recurrence relation $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, one easily sees

$$
\begin{aligned}
a_n &= \left(\sqrt{-q}\right)^n \left[2U_n\left(\frac{p}{2\sqrt{-q}}\right) - \frac{p}{\sqrt{-q}} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right)\right]\\
&= \left(\sqrt{-q}\right)^n \left[2U_n\left(\frac{p}{2\sqrt{-q}}\right) - \left(U_n\left(\frac{p}{2\sqrt{-q}}\right) + U_{n-2}\left(\frac{p}{2\sqrt{-q}}\right)\right)\right]\\
&= \left(\sqrt{-q}\right)^n \left[U_n\left(\frac{p}{2\sqrt{-q}}\right) - U_{n-2}\left(\frac{p}{2\sqrt{-q}}\right)\right].
\end{aligned}
$$

From the basic relation between Chebyshev polynomials of the first and the second kinds (see for example, (1.7) in [19] by Mason and Handscomb), $U_n(x) - U_{n-2}(x) = 2T_n(x)$, the last expression of a_n implies (11). (12) can be proved similarly.

■

As an example, the Lucas number sequence $\{L_n\}$ defined by (1) with $p = q = 1$ and initial conditions $L_0 = 2$ and $L_1 = 1$ has the explicit formula for its general term:

$$L_n = 2i^n T_n \left(-\frac{i}{2}\right) = 2(-i)^n T_n \left(\frac{i}{2}\right). \tag{13}$$

3 Examples and applications

We first give some examples of Corollary 2.2 for sequences $\{a_n\}$ that are primary solutions of (1).

Example 1 If $p = 2$ and $q = 1$, then a_n defined by (1) with initial conditions $a_0 = 0$, $a_1 = 1$ are Pell numbers P_n. Thus, from Corollary 2.2, we have

$$\begin{aligned}
P_n &= (i)^{n-1} U_{n-1}(-i) = (-i)^{n-1} U_{n-1}(i), \\
P_n &= P_n(1) = (-1)^{n-1} P_n(-1), \\
P_n &= F_n(2) = (-1)^{n-1} F_n(-2), \\
P_n &= \left(\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(-2\sqrt{2}i\right) = \left(-\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(2\sqrt{2}i\right), \\
P_n &= 2^{n+1}\left(\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(-2\sqrt{a}i, a\right) \\
&= 2^{n+1}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(2\sqrt{a}i, a\right).
\end{aligned}$$

Example 2 If $p = 1$ and $q = 2$, then a_n defined by (1) with initial conditions $a_0 = 0$ $a_1 = 1$ are Jacobsthal numbers J_n (see Bergum, Bennett, Horadam, and Moore [4]). Thus Corollary 2.2 gives the expressions of J_n as follows.

$$\begin{aligned}
J_n &= \left(\sqrt{2}i\right)^{n-1} U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right) = \left(-\sqrt{2}i\right)^{n-1} U_{n-1}\left(\frac{i}{2\sqrt{2}}\right),\\
J_n &= \left(\sqrt{2}\right)^{n-1} P_n\left(\frac{1}{2\sqrt{2}}\right) = \left(-\sqrt{2}\right)^{n-1} P_n\left(-\frac{1}{2\sqrt{2}}\right),\\
J_n &= \left(\sqrt{2}\right)^{n-1} F_n\left(\frac{1}{\sqrt{2}}\right) = \left(-\sqrt{2}\right)^{n-1} F_n\left(-\frac{1}{\sqrt{2}}\right),\\
J_n &= i^{n-1}\Phi_n\left(-pi\right) = (-i)^{n-1}\,\Phi_n\left(pi\right),\\
J_n &= 2^{n+1}\left(\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{p\sqrt{a}i}{\sqrt{2}}, a\right)\\
&= 2^{n+1}\left(-\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{p\sqrt{a}i}{\sqrt{2}}, a\right).
\end{aligned}$$

Example 3 If $p = 3$ and $q = -2$, then a_n defined by (1) with initial conditions $a_0 = 0$ $a_1 = 1$ are Mersenne numbers $M_n = 2^n - 1$. From Corollary 2.2, we have

$$\begin{aligned}
M_n &= \left(\sqrt{2}\right)^{n-1} U_{n-1}\left(\frac{3}{2\sqrt{2}}\right) = \left(-\sqrt{2}\right)^{n-1} U_{n-1}\left(\frac{-3}{2\sqrt{2}}\right),\\
M_n &= \left(\sqrt{2}i\right)^{n-1} P_n\left(-\frac{3i}{2\sqrt{2}}\right) = \left(-\sqrt{2}i\right)^{n-1} P_n\left(\frac{3i}{2\sqrt{2}}\right),\\
M_n &= \left(\sqrt{2}i\right)^{n-1} F_n\left(-\frac{3i}{\sqrt{2}}\right) = \left(-\sqrt{2}i\right)^{n-1} F_n\left(\frac{3i}{\sqrt{2}}\right),\\
M_n &= \Phi_n\left(3\right) = (-1)^{n-1}\Phi_n(-3),\\
M_n &= 2^{n+1}\left(\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{3\sqrt{a}}{\sqrt{2}}, a\right)\\
&= 2^{n+1}\left(-\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{3\sqrt{a}}{\sqrt{2}}, a\right).
\end{aligned}$$

Next, we give several examples of non-primary solutions of (1) by using Corollary 2.4.

Example 4 If $p = 1$ and $q = 1$, then a_n defined by (1) with initial conditions $a_0 = 2$, $a_1 = 1$ are Lucas numbers L_n. Thus besides (13) we have

$$
\begin{aligned}
L_n &= 2i^n U_n\left(-\frac{i}{2}\right) - i^{n-1}U_{n-1}\left(-\frac{i}{2}\right) \\
&= 2(-i)^n U_n\left(\frac{i}{2}\right) - (-i)^{n-1}U_{n-1}\left(\frac{i}{2}\right), \\
L_n &= 2P_{n+1}\left(\frac{1}{2}\right) - P_n\left(\frac{1}{2}\right) \\
&= 2(-1)^n P_{n+1}\left(-\frac{1}{2}\right) - (-1)^{n-1}P_n\left(-\frac{1}{2}\right), \\
L_n &= 2F_{n+1}(1) - F_n(1) = 2(-1)^n F_{n+1}(-1) - (-1)^{n-1}F_n(-1), \\
L_n &= 2\left(\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(-\sqrt{2}i\right) - \left(\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(-\sqrt{2}i\right) \\
&= 2\left(-\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\sqrt{2}i\right) - \left(-\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(\sqrt{2}i\right), \\
L_n &= 2^{n+3}\left(\frac{i}{\sqrt{4a}}\right)^n D_n\left(-\sqrt{a}i, a\right) - 2^{n+1}\left(\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(-\sqrt{a}i, a\right) \\
&= 2^{n+3}\left(-\frac{i}{\sqrt{4a}}\right)^n D_n\left(\sqrt{a}i, a\right) - 2^{n+1}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(\sqrt{a}i, a\right).
\end{aligned}
$$

Example 5 If $p = 2$ and $q = 1$, then a_n defined by (1) with initial conditions $a_0 = 2$, $a_1 = 2$ are Pell-Lucas numbers A_n (see Example 2 in [11]). Thus, from Corollary 2.4, we obtain

$$A_n = 2i^n T_n(-i) = 2(-i)^n T_n(i)$$

and

$$
\begin{aligned}
A_n &= 2i^n U_n(-i) - 2i^{n-1}U_{n-1}(-i) = 2i^n U_n(-i) - 2i^{n-1}U_{n-1}(-i), \\
A_n &= 2P_{n+1}(1) - 2P_n(1) = 2(-1)^n P_{n+1}(-1) - p(-1)^{n-1}P_n(-1), \\
A_n &= 2F_{n+1}(2) - 2F_n(2) = 2(-1)^n F_{n+1}(-2) - p(-1)^{n-1}F_n(-2), \\
A_n &= 2\left(-\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(2\sqrt{2}i\right) - 2\left(-\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(2\sqrt{2}i\right) \\
&= 2\left(\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(-2\sqrt{2}i\right) - 2\left(\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(-2\sqrt{2}i\right),
\end{aligned}
$$

$$\begin{aligned}
A_n &= 2^{n+3}\left(\frac{i}{\sqrt{4a}}\right)^n D_n\left(-2\sqrt{a}i,a\right)-2^{n+2}\left(\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(-2\sqrt{a}i,a\right)\\
&= 2^{n+3}\left(-\frac{i}{\sqrt{4a}}\right)^n D_n\left(2\sqrt{a}i,a\right)-2^{n+2}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(2\sqrt{a}i,a\right).
\end{aligned}$$

Example 6 If $p=1$ and $q=2$, then a_n defined by (1) with initial conditions $a_0=2$, $a_1=1$ are Jacobsthal-Lucas numbers B_n (see Example 2 in [11]). Thus,

$$B_n=2\left(\sqrt{2}i\right)^n T_n\left(-\frac{i}{2\sqrt{2}}\right)=2\left(-\sqrt{2}i\right)^n T_n\left(\frac{i}{2\sqrt{2}}\right),$$

and

$$\begin{aligned}
B_n &= 2\left(\sqrt{2}i\right)^n U_n\left(-\frac{i}{2\sqrt{2}}\right)-\left(\sqrt{2}i\right)^{n-1} U_{n-1}\left(-\frac{i}{2\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}i\right)^n U_n\left(\frac{i}{2\sqrt{2}}\right)-\left(-\sqrt{2}i\right)^{n-1} U_{n-1}\left(\frac{i}{2\sqrt{2}}\right),\\
B_n &= 2\left(\sqrt{2}\right)^n P_{n+1}\left(\frac{1}{2\sqrt{2}}\right)-\left(\sqrt{2}\right)^{n-1} P_n\left(\frac{1}{2\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}\right)^n P_{n+1}\left(-\frac{1}{2\sqrt{2}}\right)-\left(-\sqrt{2}\right)^{n-1} P_n\left(-\frac{1}{2\sqrt{2}}\right),\\
B_n &= 2\left(\sqrt{2}\right)^n F_{n+1}\left(\frac{1}{\sqrt{2}}\right)-\left(\sqrt{2}\right)^{n-1} F_n\left(\frac{1}{\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}\right)^n F_{n+1}\left(-\frac{1}{\sqrt{2}}\right)-\left(-\sqrt{2}\right)^{n-1} F_n\left(-\frac{1}{\sqrt{2}}\right),\\
B_n &= 2i^n\Phi_{n+1}\left(-i\right)-i^{n-1}\Phi_n\left(-i\right)=2(-i)^n\Phi_{n+1}\left(i\right)-(-i)^{n-1}\Phi_n\left(i\right),\\
B_n &= 2^{n+3}\left(\frac{i}{\sqrt{2a}}\right)^n D_n\left(-\frac{\sqrt{a}i}{\sqrt{2}},a\right)\\
&\quad -2^{n+1}\left(\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{\sqrt{a}i}{\sqrt{2}},a\right)\\
&= 2^{n+3}\left(-\frac{i}{\sqrt{2a}}\right)^n D_n\left(\frac{\sqrt{a}i}{\sqrt{2}},a\right)\\
&\quad -2^{n+1}\left(-\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{\sqrt{a}i}{\sqrt{2}},a\right).
\end{aligned}$$

Example 7 If $p = 3$ and $q = -2$, then a_n defined by (1) with initial conditions $a_0 = 2$, $a_1 = 3$ are Fermat numbers f_n (see [7]). Thus, from Corollary 2.4, we obtain

$$f_n = 2\left(\sqrt{2}\right)^n T_n\left(\frac{3}{2\sqrt{2}}\right) = 2\left(-\sqrt{2}\right)^n T_n\left(-\frac{3}{2\sqrt{2}}\right)$$

and

$$\begin{aligned}
f_n &= 2\left(\sqrt{2}\right)^n U_n\left(\frac{3}{2\sqrt{2}}\right) - 3\left(\sqrt{2}\right)^{n-1} U_{n-1}\left(\frac{3}{2\sqrt{2}}\right) \\
&= 2\left(-\sqrt{2}\right)^n U_n\left(-\frac{3}{2\sqrt{2}}\right) - 3\left(-\sqrt{2}\right)^{n-1} U_{n-1}\left(-\frac{3}{2\sqrt{2}}\right), \\
f_n &= 2\left(\sqrt{2}i\right)^n P_{n+1}\left(-\frac{3i}{2\sqrt{2}}\right) - 3\left(\sqrt{2}i\right)^{n-1} P_n\left(-\frac{3i}{2\sqrt{2}}\right) \\
&= 2\left(-\sqrt{2}i\right)^n P_{n+1}\left(\frac{3i}{2\sqrt{2}}\right) - 3\left(-\sqrt{2}i\right)^{n-1} P_n\left(\frac{3i}{2\sqrt{2}}\right), \\
f_n &= 2\left(\sqrt{2}i\right)^n F_{n+1}\left(-\frac{3i}{\sqrt{2}}\right) - 3\left(\sqrt{2}i\right)^{n-1} F_n\left(-\frac{3i}{\sqrt{2}}\right) \\
&= 2\left(-\sqrt{2}i\right)^n F_{n+1}\left(\frac{3i}{\sqrt{2}}\right) - 3\left(-\sqrt{2}i\right)^{n-1} F_n\left(\frac{3i}{\sqrt{2}}\right), \\
f_n &= 2\Phi_{n+1}(3) - 3\Phi_n(3) = 2(-1)^n\Phi_{n+1}(-3) - 3(-1)^{n-1}\Phi_n(-3), \\
f_n &= 2^{n+3}\left(\frac{1}{\sqrt{2a}}\right)^n D_n\left(\frac{3\sqrt{a}}{\sqrt{2}}, a\right) \\
&\quad -32^{n+1}\left(\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{3\sqrt{a}}{\sqrt{2}}, a\right) \\
&= 2^{n+3}\left(-\frac{1}{\sqrt{2a}}\right)^n D_n\left(-\frac{3\sqrt{a}}{\sqrt{2}}, a\right) \\
&\quad -32^{n+1}\left(-\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{3\sqrt{a}}{\sqrt{2}}, a\right).
\end{aligned}$$

Using the relationship established above we may obtain some identities of number sequences and polynomial value sequences. Theorem 3.2 in [11] presented a generalized Gegenbauer-Humbert polynomial sequence identity

$$P_n^{1,y,C}(x) = \alpha(x)P_{n-1}^{1,y,C}(x) + C^{-2}(2x - \alpha(x)C)\left(\beta(x)\right)^{n-1}, \tag{14}$$

where $P_n^{1,y,C}(x)$ satisfies the recurrence relation of order 2 $P_n^{1,y,C} = pP_{n-1}^{1,y,C} + qP_{n-2}^{1,y,C}$ with coefficients $p(x)$ and $q(x)$, and $\alpha(x)+\beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Clearly (see (19) and (20) in [11]),

$$\begin{aligned} \alpha &= \frac{1}{C}\left\{x + \sqrt{x^2 - Cy}\right\} \text{ and} \qquad (15) \\ \beta &= \frac{1}{C}\left\{x - \sqrt{x^2 - Cy}\right\}. \qquad (16) \end{aligned}$$

For $y = -1$ and $C = 1$, we have $P_n^{1,-1,1}(x) = F_{n+1}(2x)$, where $F_n(x)$ are Fibonacci polynomials, and we can write (14) as

$$F_{n+1}(2x) = \alpha(x)F_n(2x) + (2x - \alpha(x))\left(\beta(x)\right)^{n-1} = \alpha(x)F_n(2x) + \left(\beta(x)\right)^n, \qquad (17)$$

where $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$. If $x = 1/2$, then $F_n(1) = F_n$, Fibonacci numbers, and

$$\alpha\left(\frac{1}{2}\right) = \frac{1+\sqrt{5}}{2}, \text{ and } \beta\left(\frac{1}{2}\right) = \frac{1-\sqrt{5}}{2}.$$

Thus (17) yields the identity

$$F_{n+1} = \frac{1+\sqrt{5}}{2}F_n + \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

or equivalently,

$$\frac{1-\sqrt{5}}{2}F_{n+1} + F_n = \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Similarly, if $x = 1$, then $F_n(2) = P_n$, Pell numbers, and

$$\alpha(1) = 1 + \sqrt{2}, \text{ and } \beta(1) = 1 - \sqrt{2}.$$

Thus (17) yields the identity

$$P_{n+1} = (1+\sqrt{2})P_n + \left(1 - \sqrt{2}\right)^n,$$

or equivalently,

$$(1-\sqrt{2})P_{n+1}+P_n=\left(1-\sqrt{2}\right)^{n+1}.$$

Substituting $x=1/(2\sqrt{2})$ into (17) and noting $F_n(1/\sqrt{2})=J_n/(\sqrt{2})^n$, where J_n are Jacobsthal numbers, we obtain the identity

$$J_{n+1}-2J_n=(-1)^n.$$

When $x=-3i/(2\sqrt{2})$, $F_n(-3i/(2\sqrt{2}))=M_n/(\sqrt{2}i)^{n-1}$, Mersenne numbers. Hence (17) gives $M_{n+1}-M_n=2^n$.

Conversely, one may use the expressions of various number sequences in terms of the generalized Gegenbauer-Humbert polynomial sequences to construct the identities of the different generalized Gegenbauer-Humbert polynomial values such as the formulas shown in the example after Corollary 2.3.

References

[1] D. Aharonov, A. Beardon, and K. Driver, Fibonacci, Chebyshev, and orthogonal polynomials, Amer. Math. Monthly. 122 (2005) 612–630.

[2] A. Beardon, Fibonacci meets Chebyshev, The Mathematical Gazetle, 91 (2007), 251-255.

[3] A. T. Benjamin and J. J. Quinn, Proofs that really count. The art of combinatorial proof. The Dolciani Mathematical Expositions, 27. Mathematical Association of America, Washington, DC, 2003.

[4] G. E. Bergum, L. Bennett, A. F. Horadam, and S. D. Moore, Jacobsthal Polynomials and a Conjecture Concerning Fibonacci-Like Matrices, Fibonacci Quart. 23 (1985), 240-248.

[5] W. Y. C. Chen and J. D. Louck, The combinatorial power of the companion matrix, Linear Algebra Appl. 232 (1996), 261–278.

[6] T. Chow and J. West, Forbidden subsequences and Chebyshev polynomials. Discrete Math. 204 (1999), no. 1-3, 119–128.

[7] H. Civciv and R. Türkmen, Notes on the (s,t)-Lucas and Lucas matrix sequences. Ars Combin. 89 (2008), 271-285.

[8] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.

[9] H. W. Gould, Inverse series relations and other expansions involving Humbert polynomials, Duke Math. J. 32 (1965), 697–711.

[10] T. X. He, L. C. Hsu, P. J.-S. Shiue, A symbolic operator approach to several summation formulas for power series II, Discrete Math. 308 (2008), no. 16, 3427–3440.

[11] T. X. He and P. J.-S. Shiue, On sequences of numbers and polynomials defined by linear recurrence relations of order 2, International Journal of Mathematics and Mathematical Sciences, to appear.

[12] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, Fibonacci Quart. 3 (1965), 161–176.

[13] L. C. Hsu, Computational Combinatorics (Chinese), First edition, Shanghai Scientific & Techincal Publishers, Shanghai, 1983.

[14] L. C. Hsu, On Stirling-type pairs and extended Gegenbauer-Humbert-Fibonacci polynomials. Applications of Fibonacci numbers, Vol. 5 (St. Andrews, 1992), 367–377, Kluwer Acad. Publ., Dordrecht, 1993.

[15] L. C. Hsu and P. J.-S. Shiue, Cycle indicators and special functions. Ann. Comb. 5 (2001), no. 2, 179–196.

[16] R. Lidl, G. L. Mullen, and G. Turnwald, Dickson polynomials. Pitman Monographs and Surveys in Pure and Applied Mathematics, 65, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993.

[17] T. Mansour, A formula for the generating functions of powers of Horadam's sequence, Australas. J. Combin. 30 (2004), 207–212.

[18] R. B. Marr and G. H. Vineyard, Five-diagonal Toeplitz determinants and their relation to Chebyshev polynomials, SIAM Matrix Anal. Appl. 9 (1988), 579-586.

[19] J. C. Mason and D. C. Handscomb, Chebyshev Polynomials, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[20] G. Strang, Linear algebra and its applications. Second Edition, Academic Press (Harcourt Brace Jovanovich, Publishers), New York-London, 1980.

[21] H. S. Wilf, Generatingfunctionology, Academic Press, New York, 1990.