Sequences of Numbers Meet the Generalized Gegenbauer-Humbert Polynomials

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Abstract

Here we present a connection between a sequence of numbers generated by a linear recurrence relation of order 2 and sequences of the generalized Gegenbauer-Humbert polynomials. Many new and known formulas of Fibonacci, Lucas, Pell, Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values are given. The applications of the relationship to the construction of identities of number and polynomial value sequences defined by linear recurrence relations are also discussed.

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Key Words and Phrases: sequence of order 2, linear recurrence relation, Fibonacci sequence, Chebyshev polynomial, the

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generalized Gegenbauer-Humbert polynomial sequence, Lucas number, Pell number.

1 Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and/or classified by linear recurrence relations with certain orders. A number sequence \( \{a_n\} \) is called sequence of order 2 if it satisfies the linear recurrence relation of order 2:

\[
a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2,
\]

for some non-zero constants \( p \) and \( q \) and initial conditions \( a_0 \) and \( a_1 \). In Mansour [17], the sequence \( \{a_n\}_{n \geq 0} \) defined by (1) is called Horadam’s sequence, which was introduced in 1965 by Horadam [12]. [17] also obtained the generating functions for powers of Horadam’s sequence. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [8], Hsu [13], Strang [20], Wilf [21], etc.) In [3], Benjamin and Quinn presented many elegant combinatorial meanings of the sequence defined by recurrence relation (1). For instance, \( a_n \) counts the number of ways to tile an \( n \)-board (i.e., board of length \( n \)) with squares (representing 1s) and dominoes (representing 2s) where each tile, except the initial one has a color. In addition, there are \( p \) colors for squares and \( q \) colors for dominoes. In particular, Aharonov, Beardon, and Driver (see [1]) have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions \( a_0 = 0 \) and \( a_1 = 1 \), called the primary solution, can be expressed in terms of Chebyshev polynomial values. For instance, the authors show \( F_n = i^{-n}U_n(i/2) \) and \( L_n = 2i^{-n}T_n(i/2) \), where \( F_n \) and \( L_n \) respectively are Fibonacci numbers and Lucas numbers, and \( T_n \) and \( U_n \) are Chebyshev polynomials of the first kind and the second kind, respectively. Some identities drawn from those relations were given by Beardon in [2]. Marr and Vineyard in [18] use the relationship to establish explicit expression of five-diagonal Toeplitz determinants. In [11], the first two authors presented a new method to construct an explicit formula of \( \{a_n\} \) generated by (1). For the sake of the reader’s convenience, we cite this result as follows.
Proposition 1.1 ([11]) Let \( \{a_n\} \) be a sequence of order 2 satisfying linear recurrence relation (1), and let \( \alpha \) and \( \beta \) be two roots of quadratic equation \( x^2 - px - q = 0 \). Then

\[
a_n = \begin{cases} 
\left( \frac{a_1 - \beta a_0}{\alpha - \beta} \right) \alpha^n - \left( \frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \beta^n, & \text{if } \alpha \neq \beta; \\
na_1 \alpha^{n-1} - (n-1)a_0 \alpha^n, & \text{if } \alpha = \beta.
\end{cases}
\] (2)

A sequence of the generalized Gegenbauer-Humbert polynomials \( \{P_{\lambda,y,C}^n(x)\}_{n \geq 0} \) is defined by the expansion (see, for example, [8], Gould [9], Lidl, Mullen, and Turnwald [16], the first two of authors with Hsu [10])

\[
\Phi(t) = (C - 2xt + y t^2)^{-\lambda} = \sum_{n \geq 0} P_{\lambda,y,C}^n(x) t^n,
\] (3)

where \( \lambda > 0 \), \( y \) and \( C \neq 0 \) are real numbers. As special cases of (3), we consider \( P_{\lambda,y,C}^n(x) \) as follows (see [10])

\[
\begin{align*}
P_{n}^{1,1,1}(x) &= U_n(x), \text{ Chebyshev polynomial of the second kind,} \\
P_{n}^{1/2,1,1}(x) &= \psi_n(x), \text{ Legendre polynomial,} \\
P_{n}^{1,-1,1}(x) &= P_{n+1}(x), \text{ Pell polynomial,} \\
P_{n}^{1,-1,1}(\frac{x}{2}) &= F_{n+1}(x), \text{ Fibonacci polynomial,} \\
P_{n}^{1,2,1}(\frac{x}{2}) &= \Phi_{n+1}(x), \text{ Fermat polynomial of the first kind,} \\
P_{n}^{1,2a,2}(x) &= D_n(x,a), \text{ Dickson polynomial of the second kind, } a \neq 0, \text{ (see, for example, [16]),}
\end{align*}
\]

where \( a \) is a real parameter, and \( F_n = F_n(1) \) is the Fibonacci number. In particular, if \( y = C = 1 \), the corresponding polynomials are called Gegenbauer polynomials (see [8]). More results on the Gegenbauer-Humbert-type polynomials can be found in [14] by Hsu and in [15] by the second author and Hsu, etc.

Similarly, for a class of the generalized Gegenbauer-Humbert polynomial sequences defined by

\[
P_{n}^{\lambda,y,C}(x) = 2x \frac{\lambda + n - 1}{C_n} P_{n-1}^{\lambda,y,C}(x) - y \frac{2\lambda + n - 2}{C_n} P_{n-2}^{\lambda,y,C}(x) \] (4)
for all $n \geq 2$ with initial conditions

$$P_{0}^{\lambda, y, C}(x) = \Phi(0) = C^{-\lambda},$$
$$P_{1}^{\lambda, y, C}(x) = \Phi'(0) = 2\lambda x C^{-\lambda - 1},$$

the following theorem has been obtained in [11]

**Theorem 1.2 ([11])** Let $x \neq \pm \sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_{n}^{1, y, C}(x)\}_{n \geq 0}$ defined by expansion (3) can be expressed as

$$P_{n}^{1, y, C}(x) = C^{-n-2} \frac{(x + \sqrt{x^2 - Cy})^{n+1} - (x - \sqrt{x^2 - Cy})^{n+1}}{2\sqrt{x^2 - Cy}}. \quad (5)$$

In this paper, we shall use an alternative form of (2) to establish a relationship between the number sequences defined by recurrence relation (1) and the generalized Gegenbauer-Humbert polynomial sequences defined by (4). Our results are suitable for all such number sequences defined by (1) with arbitrary initial conditions $a_0$ and $a_1$, which includes the results in [1] and [2] as our special cases. Many new and known formulas of Fibonacci, Lucas, Pell, Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values and applications of the established relationship to the construction of identities of number and polynomial value sequences will be presented in Section 3.

## 2 Main results

We now modify the explicit formula of the number sequences defined by linear recurrence relations of order 2. If $\alpha \neq \beta$, the first formula in (2) can be written as

$$a_n = \frac{a_1(\alpha^n - \beta^n) - a_0 \alpha \beta (\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} = \frac{a_1(\alpha^n - \beta^n) + a_0 \alpha \beta (\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta},$$
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where the last step is due to $\alpha$ and $\beta$ being solutions of $t^2 - pt - q = 0$. Noting that $\alpha^2 - p\alpha = \alpha^2 - (\alpha + \beta)\alpha = -\alpha\beta = q$ and $\alpha(\alpha - p) = -\alpha\beta = \beta(\beta - p)$, we may further write the above last expression of $a_n$ as

$$a_n = \frac{a_1(\alpha^n - \beta^n) + a_0(\alpha^2 - p\alpha)(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}$$

$$= \frac{a_1(\alpha^n - \beta^n) + a_0(\alpha^2 - p\alpha)\alpha^{n-1} - a_0(\beta^2 - p\beta)\beta^{n-1}}{\alpha - \beta}$$

$$= \frac{a_0(\alpha^{n+1} - \beta^{n+1}) + (a_1 - a_0p)(\alpha^n - \beta^n)}{\alpha - \beta}. \quad (6)$$

Denote $r(x) = x + \sqrt{x^2 - Cy}$ and $s(x) = x - \sqrt{x^2 - Cy}$. Comparing expressions (6) and (5), we have reason to consider the following transform: for a non-zero real or complex number $k$, we set

$$\alpha := \frac{r(x)}{k} \text{ and } \beta := \frac{s(x)}{k} \quad (7)$$

for a certain $x$ depends on $\alpha$, $\beta$ and $k$, which we will find out later. Denote $\alpha + \beta = p$ and $\alpha\beta = -q$, i.e., $\alpha$ and $\beta$ are roots of $t^2 - pt - q$. By adding the two equations in (7) side by side, we obtain $2x = kp$. Thus, when $x = \frac{kp}{2}$, equations in (6) hold. Meanwhile, by using $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 + 4q$, we have

$$r(x) - s(x) = 2\sqrt{x^2 - Cy} = k(\alpha - \beta) = k\sqrt{p^2 + 4q},$$

where $x = \frac{kp}{2}$. Therefore, we obtain

$$2\sqrt{\left(\frac{kp}{2}\right)^2 - Cy} = k\sqrt{p^2 + 4q},$$

which implies

$$k = \pm \sqrt{\frac{Cy}{-q}}. \quad (8)$$

We first consider the case of $k = \sqrt{-Cy/q}$.

We now substitute $r(x) = k\alpha$, $s(x) = k\beta$, $x = \frac{kp}{2}$, and $k = \sqrt{-Cy/q}$ into (6) and simplify as follows.
\[ a_n = a_0 \left( \frac{r(x)}{k} \right)^{n+1} - \left( \frac{s(x)}{k} \right)^{n+1} + (a_1 - a_0 p) \left( \frac{r(x)}{k} \right)^n - \left( \frac{s(x)}{k} \right)^n \]

\[ = a_0 \left( r^{n+1}(x) - s^{n+1}(x) \right) + \frac{k}{k} \left( r(x) - s(x) \right) \]

\[ = a_0 C^{n+2} \left( \sqrt{-q} \right)^n \left( \frac{kp}{2} \right) P_{n}^{1,y,C} \left( \frac{kp}{2} \right) \]

\[ + (a_1 - a_0 p) C^{n+1} \left( \sqrt{-q} \right)^{n-1} \left( \frac{kp}{2} \right) P_{n-1}^{1,y,C} \left( \frac{kp}{2} \right) \]

\[ = a_0 C^{n+2} \left( \frac{p}{2} \sqrt{Cy} \right) \]

\[ + (a_1 - a_0 p) C^{n+1} \left( \frac{p}{2} \sqrt{Cy} \right) \left( \sqrt{-q} \right)^{n-1} \left( \frac{p}{2} \sqrt{Cy} \right) \left( \sqrt{-q} \right) \].

\((9)\)

Similarly, for \( k = -\sqrt{-Cy/q} \), we have

\[ a_n = a_0 C^{n+2} \left( -\sqrt{-q} \right)^n \left( \frac{p}{2} \sqrt{-q} \right) \left( \frac{p}{2} \sqrt{Cy} \right) \left( \frac{p}{2} \sqrt{-q} \right) \]

\[ + (a_1 - a_0 p) C^{n+1} \left( -\sqrt{-q} \right)^{n-1} \left( \frac{p}{2} \sqrt{Cy} \right) \left( \frac{p}{2} \sqrt{-q} \right) \].

\((10)\)

Therefore, we obtain our main result.

**Theorem 2.1** Let sequence \( \{a_n\} \) be defined by \( a_n = pa_{n-1} + qa_{n-2} \) \( (n \geq 2) \) with initial conditions \( a_0 \) and \( a_1 \). Then, \( a_n \) can be presented as \((9)\) and \((10)\). In particular, for \((y, C) = (1, 1), (-1, 1), (2, 1), \text{and} (2a, 2) (a \neq 0)\), respectively, we have
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\[
an = a_0 (\sqrt{-q})^n U_n \left( \frac{p}{2\sqrt{-q}} \right) + (a_1 - a_0p) (\sqrt{-q})^{n-1} U_{n-1} \left( \frac{p}{2\sqrt{-q}} \right),
\]
\[
a_n = a_0 (\sqrt{q})^n P_{n+1} \left( \frac{p}{2\sqrt{q}} \right) + (a_1 - a_0p) (\sqrt{q})^{n-1} P_n \left( \frac{p}{2\sqrt{q}} \right),
\]
\[
a_n = a_0 (\sqrt{q})^n F_{n+1} \left( \frac{p}{\sqrt{q}} \right) + (a_1 - a_0p) (\sqrt{q})^{n-1} F_n \left( \frac{p}{\sqrt{q}} \right),
\]
\[
a_n = a_0 \left( \frac{-q}{4a} \right)^n \Phi_{n+1} \left( -p \sqrt{\frac{a}{-q}} \right) + (a_1 - a_0p) \left( \frac{-q}{2} \right)^{n-1} \Phi_n \left( -p \sqrt{\frac{a}{-q}} \right),
\]
\[
a_n = a_0 2^{n+2} \left( \frac{-q}{4a} \right)^n D_n \left( -p \sqrt{\frac{a}{-q}, a} \right) + (a_1 - a_0p) 2^{n+1} \left( \frac{-q}{4a} \right)^{n-1} D_{n-1} \left( -p \sqrt{\frac{a}{-q}, a} \right),
\]

and

\[
an = a_0 (\sqrt{-q})^n U_n \left( \frac{-p}{2\sqrt{-q}} \right) + (a_1 - a_0p) (\sqrt{-q})^{n-1} U_{n-1} \left( \frac{-p}{2\sqrt{-q}} \right),
\]
\[
a_n = a_0 (\sqrt{-q})^n P_{n+1} \left( \frac{-p}{2\sqrt{-q}} \right) + (a_1 - a_0p) (\sqrt{-q})^{n-1} P_n \left( \frac{-p}{2\sqrt{-q}} \right),
\]
\[
a_n = a_0 (\sqrt{-q})^n F_{n+1} \left( \frac{-p}{\sqrt{-q}} \right) + (a_1 - a_0p) (\sqrt{-q})^{n-1} F_n \left( \frac{-p}{\sqrt{-q}} \right),
\]
\[
a_n = a_0 \left( \frac{-q}{2} \right)^n \Phi_{n+1} \left( -p \sqrt{\frac{a}{-q}} \right) + (a_1 - a_0p) \left( \frac{-q}{2} \right)^{n-1} \Phi_n \left( -p \sqrt{\frac{a}{-q}} \right),
\]
\[
a_n = a_0 2^{n+2} \left( \frac{-q}{4a} \right)^n D_n \left( -p \sqrt{\frac{a}{-q}, a} \right) + (a_1 - a_0p) 2^{n+1} \left( \frac{-q}{4a} \right)^{n-1} D_{n-1} \left( -p \sqrt{\frac{a}{-q}, a} \right),
\]

where \( U_n(x) \), \( P_n(x) \), \( F_n(x) \), \( \Phi_n(x) \), and \( D_n(x, a) \) are the \( n \)th degree Chebyshev polynomial of the second kind, Pell polynomial, Fibonacci
polynomial, Fermat polynomial, and Dickson polynomial of the second kind, respectively.

For the special cases of $a_0$ and $a_1$, we have the following corollaries.

**Corollary 2.2** Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = 0$ and $a_1 = d$. Then

\[
\begin{align*}
    a_n &= d \left( \sqrt{-q} \right)^{n-1} U_{n-1} \left( \frac{p}{2\sqrt{-q}} \right), \\
    a_n &= d \left( \sqrt{q} \right)^{n-1} P_n \left( \frac{p}{2\sqrt{q}} \right), \\
    a_n &= d \left( \sqrt{q} \right)^{n-1} F_n \left( \frac{p}{\sqrt{q}} \right), \\
    a_n &= d \left( \sqrt{-q} \right)^{n-1} \Phi_n \left( p\sqrt{\frac{2}{-q}} \right), \\
    a_n &= d2^{n+1} \left( \sqrt{-q} \right)^{n-1} D_{n-1} \left( p\sqrt{\frac{a}{-q}}, a \right),
\end{align*}
\]

and

\[
\begin{align*}
    a_n &= d \left( -\sqrt{-q} \right)^{n-1} U_{n-1} \left( \frac{-p}{2\sqrt{-q}} \right), \\
    a_n &= d \left( -\sqrt{q} \right)^{n-1} P_n \left( \frac{-p}{2\sqrt{q}} \right), \\
    a_n &= d \left( -\sqrt{q} \right)^{n-1} F_n \left( \frac{-p}{\sqrt{q}} \right), \\
    a_n &= d \left( -\sqrt{-q} \right)^{n-1} \Phi_n \left( -p\sqrt{\frac{2}{-q}} \right), \\
    a_n &= d2^{n+1} \left( -\sqrt{-q} \right)^{n-1} D_{n-1} \left( -p\sqrt{\frac{a}{-q}}, a \right).
\end{align*}
\]
Corollary 2.3 Let sequence \( \{a_n\} \) be defined by \( a_n = pa_{n-1} + qa_{n-2} \) \((n \geq 2)\) with initial conditions \( a_0 = c \) and \( a_1 = pc \). Then

\[
\begin{align*}
a_n &= c (\sqrt{-q})^n U_n \left( \frac{p}{2\sqrt{-q}} \right), \\
a_n &= c (\sqrt{q})^n P_{n+1} \left( \frac{p}{2\sqrt{q}} \right), \\
a_n &= c (\sqrt{q})^n F_{n+1} \left( \frac{p}{\sqrt{q}} \right), \\
a_n &= c \left( \sqrt{-q} \right)^n \Phi_{n+1} \left( \frac{p\sqrt{2}}{-q} \right), \\
a_n &= c 2^{n+2} \left( \frac{-q}{4a} \right)^n D_n \left( p\sqrt{\frac{a}{-q}}, a \right),
\end{align*}
\]

and

\[
\begin{align*}
a_n &= c (-\sqrt{-q})^n U_n \left( \frac{-p}{2\sqrt{-q}} \right), \\
a_n &= c (-\sqrt{q})^n P_{n+1} \left( \frac{-p}{2\sqrt{q}} \right), \\
a_n &= c (-\sqrt{q})^n F_{n+1} \left( \frac{-p}{\sqrt{q}} \right), \\
a_n &= c \left( \sqrt{-q} \right)^n \Phi_{n+1} \left( -p\sqrt{\frac{2}{-q}} \right), \\
a_n &= c 2^{n+2} \left( \frac{-q}{4a} \right)^n D_n \left( -p\sqrt{\frac{a}{-q}}, a \right).
\end{align*}
\]

If \( a_1 = d = 1 \), then Corollary 2.2 gives the primary solutions of recurrence relation (1) in terms of the \( n \)th degree Chebyshev polynomial of the second kind, Pell polynomial, Fibonacci polynomial, Fermat polynomial, and Dickson polynomial of the second kind, respectively. For instance, if \( p = q = 1 \), then \( a_n \) are Fibonacci numbers \( F_n \). Thus,
\[ F_n = (i)^{n-1} U_{n-1} \left( \frac{1}{2i} \right) = (i)^{n-1} U_{n-1} \left( -\frac{i}{2} \right), \]
\[ F_n = P_n \left( \frac{1}{2} \right), \]
\[ F_n = F_n(1), \]
\[ F_n = \left( \frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n \left( -\sqrt{2}i \right), \]
\[ F_n = 2^{n+1} \left( \frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} \left( -\sqrt{ai}, a \right), \]

and

\[ F_n = (-i)^{n-1} U_{n-1} \left( \frac{i}{2} \right), \]
\[ F_n = (-1)^{n-1} P_n \left( -\frac{1}{2} \right), \]
\[ F_n = (-1)^{n-1} F_n(-1), \]
\[ F_n = \left( -\frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n \left( \sqrt{2}i \right), \]
\[ F_n = 2^{n+1} \left( -\frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} \left( \sqrt{ai}, a \right), \]

where \( F_n = (i)^{n-1} U_{n-1} \left( -\frac{i}{2} \right) \) was shown in [1], and \( F_n = (-i)^{n-1} U_{n-1} \left( \frac{i}{2} \right) \) was given by Chen and Louck in [5]. From the above expressions of \( F_n \) we may obtain many identities. For instance, we have

\[ P_n \left( \frac{1}{2} \right) = (-1)^{n-1} P_n \left( -\frac{1}{2} \right) = F_n(1) = (-1)^{n-1} F_n(-1), \]

\[ (i)^{n-1} U_{n-1} \left( -\frac{i}{2} \right) = (-i)^{n-1} U_{n-1} \left( \frac{i}{2} \right) \]
\[ = \left( \frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n \left( -\sqrt{2}i \right) = \left( -\frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n \left( \sqrt{2}i \right), \]
We now give another special case of Theorem 2.1 for the sequence defined by (1) with initial cases $a_0 = 2$ and $a_1$.

**Corollary 2.4** Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = 2$ and $a_1 = p$. Then

\[
a_n = 2 (\sqrt{-q})^n U_n \left( \frac{p}{2\sqrt{-q}} \right) - p (\sqrt{-q})^{n-1} U_{n-1} \left( \frac{p}{2\sqrt{-q}} \right),
\]

\[
a_n = 2 (\sqrt{q})^n P_{n+1} \left( \frac{p}{2\sqrt{q}} \right) - p (\sqrt{q})^{n-1} P_n \left( \frac{p}{2\sqrt{q}} \right),
\]

\[
a_n = 2 (\sqrt{q})^n F_{n+1} \left( \frac{p}{\sqrt{q}} \right) - p (\sqrt{q})^{n-1} F_n \left( \frac{p}{\sqrt{q}} \right),
\]

\[
a_n = 2 \left( \sqrt{-q} \right)^n \Phi_{n+1} \left( p\sqrt{\frac{2}{q}} \right) - p \left( \sqrt{-q} \right)^{n-1} \Phi_n \left( p\sqrt{\frac{2}{q}} \right),
\]

\[
a_n = 2^{n+3} \left( \sqrt{-q} \right)^n D_n \left( \frac{p\sqrt{\frac{a}{-q}}}{4a} \right) - p2^{n+1} \left( \sqrt{-q} \right)^{n-1} D_{n-1} \left( p\sqrt{\frac{a}{-q}} \right),
\]

and

\[
a_n = 2 \left( -\sqrt{-q} \right)^n U_n \left( \frac{-p}{2\sqrt{-q}} \right) - p \left( -\sqrt{-q} \right)^{n-1} U_{n-1} \left( \frac{-p}{2\sqrt{-q}} \right),
\]

\[
a_n = 2 \left( -\sqrt{q} \right)^n P_{n+1} \left( \frac{-p}{2\sqrt{q}} \right) - p \left( -\sqrt{q} \right)^{n-1} P_n \left( \frac{-p}{2\sqrt{q}} \right),
\]
\[ a_n = 2 \left( -\sqrt{-q} \right)^n F_{n+1} \left( -\frac{p}{\sqrt{-q}} \right) - p \left( -\sqrt{-q} \right)^{n-1} F_n \left( \frac{-p}{\sqrt{-q}} \right), \]
\[ a_n = 2 \left( -\sqrt{-q/2} \right)^n \Phi_{n+1} \left( -p \sqrt{\frac{2}{-q}} \right) - p \left( -\sqrt{-q/2} \right)^{n-1} \Phi_n \left( -p \sqrt{\frac{2}{-q}} \right), \]
\[ a_n = 2^{n+3} \left( -\sqrt{-q/4a} \right)^n D_n \left( -p \sqrt{\frac{a}{-q}}, a \right) - p 2^{n+1} \left( -\sqrt{-q/4a} \right)^{n-1} D_{n-1} \left( -p \sqrt{\frac{a}{-q}}, a \right). \]

In addition, we have
\begin{equation}
\tag{11}
a_n = 2 \left( \sqrt{-q} \right)^n T_n \left( \frac{p}{2\sqrt{-q}} \right)
\end{equation}
and
\begin{equation}
\tag{12}
a_n = 2 \left( -\sqrt{-q} \right)^n T_n \left( -\frac{p}{2\sqrt{-q}} \right),
\end{equation}
where \( T_n(x) \) are Chebyshev polynomials of the first kind.

Proof. It is sufficient to prove (11) and (12). From the first formula shown in Corollary 2.4 and the recurrence relation \( U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \), one easily sees
\[ a_n = (\sqrt{-q})^n \left[ 2U_n \left( \frac{p}{2\sqrt{-q}} \right) - \frac{p}{\sqrt{-q}} U_{n-1} \left( \frac{p}{2\sqrt{-q}} \right) \right] \]
\[ = (\sqrt{-q})^n \left[ 2U_n \left( \frac{p}{2\sqrt{-q}} \right) - \left( U_n \left( \frac{p}{2\sqrt{-q}} \right) + U_{n-2} \left( \frac{p}{2\sqrt{-q}} \right) \right) \right] \]
\[ = (\sqrt{-q})^n \left[ U_n \left( \frac{p}{2\sqrt{-q}} \right) - U_{n-2} \left( \frac{p}{2\sqrt{-q}} \right) \right]. \]

From the basic relation between Chebyshev polynomials of the first and the second kinds (see for example, (1.7) in [19] by Mason and Handscomb), \( U_n(x) - U_{n-2}(x) = 2T_n(x) \), the last expression of \( a_n \) implies (11). (12) can be proved similarly.
As an example, the Lucas number sequence \( \{L_n\} \) defined by (1) with \( p = q = 1 \) and initial conditions \( L_0 = 2 \) and \( L_1 = 1 \) has the explicit formula for its general term:

\[
L_n = 2i^n T_n \left( -\frac{i}{2} \right) = 2(-i)^n T_n \left( \frac{i}{2} \right). \tag{13}
\]

### 3 Examples and applications

We first give some examples of Corollary 2.2 for sequences \( \{a_n\} \) that are primary solutions of (1).

**Example 1** If \( p = 2 \) and \( q = 1 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 0 \), \( a_1 = 1 \) are Pell numbers \( P_n \). Thus, from Corollary 2.2, we have

\[
\begin{align*}
P_n &= (i)^n U_{n-1} (-i) = (-i)^{n-1} U_{n-1} (i), \\
P_n &= P_n (1) = (-1)^{n-1} P_n (-1), \\
P_n &= F_n (2) = (-1)^{n-1} F_n (-2), \\
P_n &= \left( \frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n (-2\sqrt{2}i) = \left( -\frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n (2\sqrt{2}i), \\
P_n &= 2^{n+1} \left( \frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} (2\sqrt{2}i, a) \\
&= 2^{n+1} \left( -\frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} (2\sqrt{2}i, a).
\end{align*}
\]

**Example 2** If \( p = 1 \) and \( q = 2 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 0 \), \( a_1 = 1 \) are Jacobsthal numbers \( J_n \) (see Bergum, Bennett, Horadam, and Moore [4]). Thus Corollary 2.2 gives the expressions of \( J_n \) as follows.
\[ J_n = \left( \sqrt{2}i \right)^{n-1} U_{n-1} \left( \frac{-i}{2\sqrt{2}} \right) = \left( -\sqrt{2}i \right)^{n-1} U_{n-1} \left( \frac{i}{2\sqrt{2}} \right), \]
\[ J_n = \left( \sqrt{2} \right)^{n-1} P_n \left( \frac{1}{2\sqrt{2}} \right) = \left( -\sqrt{2} \right)^{n-1} P_n \left( -\frac{1}{2\sqrt{2}} \right), \]
\[ J_n = \left( \sqrt{2} \right)^{n-1} F_n \left( \frac{1}{\sqrt{2}} \right) = \left( -\sqrt{2} \right)^{n-1} F_n \left( -\frac{1}{\sqrt{2}} \right), \]
\[ J_n = \left( \sqrt{2}i \right)^{n-1} \Phi_n \left( \frac{-p}{q} \right) = \left( -i \right)^{n-1} \Phi_n \left( \frac{q}{p} \right), \]
\[ J_n = 2^{n+1} \left( \frac{i}{\sqrt{2a}} \right)^{n-1} D_{n-1} \left( -\frac{p\sqrt{a}i}{\sqrt{2}}, a \right), \]
\[ = 2^{n+1} \left( -\frac{i}{\sqrt{2a}} \right)^{n-1} D_{n-1} \left( \frac{p\sqrt{a}i}{\sqrt{2}}, a \right). \]

Example 3 If \( p = 3 \) and \( q = -2 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 0 \) \( a_1 = 1 \) are Mersenne numbers \( M_n = 2^n - 1 \). From Corollary 2.2, we have

\[ M_n = \left( \sqrt{2} \right)^{n-1} U_{n-1} \left( \frac{3}{2\sqrt{2}} \right) = \left( -\sqrt{2} \right)^{n-1} U_{n-1} \left( \frac{-3}{2\sqrt{2}} \right), \]
\[ M_n = \left( \sqrt{2}i \right)^{n-1} P_n \left( \frac{-3i}{2\sqrt{2}} \right) = \left( -\sqrt{2}i \right)^{n-1} P_n \left( \frac{3i}{2\sqrt{2}} \right), \]
\[ M_n = \left( \sqrt{2}i \right)^{n-1} F_n \left( \frac{-3i}{\sqrt{2}} \right) = \left( -\sqrt{2}i \right)^{n-1} F_n \left( \frac{3i}{\sqrt{2}} \right), \]
\[ M_n = \Phi_n \left( 3 \right) = (-1)^{n-1} \Phi_n \left( -3 \right), \]
\[ M_n = 2^{n+1} \left( \frac{1}{\sqrt{2a}} \right)^{n-1} D_{n-1} \left( \frac{3\sqrt{a}}{\sqrt{2}}, a \right) \]
\[ = 2^{n+1} \left( -\frac{1}{\sqrt{2a}} \right)^{n-1} D_{n-1} \left( -\frac{3\sqrt{a}}{\sqrt{2}}, a \right). \]

Next, we give several examples of non-primary solutions of (1) by using Corollary 2.4.

Example 4 If \( p = 1 \) and \( q = 1 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 2 \) \( a_1 = 1 \) are Lucas numbers \( L_n \). Thus besides (13) we have
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\[ \begin{align*}
L_n &= 2i^n U_n \left( -\frac{i}{2} \right) - i^{n-1} U_{n-1} \left( -\frac{i}{2} \right) \\
&= 2(-i)^n U_n \left( \frac{i}{2} \right) - (-i)^{n-1} U_{n-1} \left( \frac{i}{2} \right), \\
L_n &= 2P_{n+1} \left( \frac{1}{2} \right) - P_n \left( \frac{1}{2} \right) \\
&= 2(-1)^n P_{n+1} \left( -\frac{1}{2} \right) - (-1)^{n-1} P_n \left( -\frac{1}{2} \right), \\
L_n &= 2F_{n+1} (1) - F_n (1) = 2(-1)^n F_{n+1} (-1) - (-1)^{n-1} F_n (-1), \\
L_n &= 2 \left( \frac{i}{\sqrt{2}} \right)^n \Phi_{n+1} (-\sqrt{2}i) - \left( \frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n (-\sqrt{2}i) \\
&= 2 \left( \frac{i}{\sqrt{2}} \right)^n \Phi_{n+1} (\sqrt{2}i) - \left( \frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n (\sqrt{2}i), \\
L_n &= 2^{n+3} \left( \frac{i}{\sqrt{4a}} \right)^n D_n (\sqrt{a}i, a) - 2^{n+1} \left( \frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} (-\sqrt{a}i, a) \\
&= 2^{n+3} \left( -\frac{i}{\sqrt{4a}} \right)^n D_n (\sqrt{a}i, a) - 2^{n+1} \left( -\frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} (\sqrt{a}i, a).
\end{align*} \]

\textbf{Example 5} If \( p = 2 \) and \( q = 1 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 2 \), \( a_1 = 2 \) are Pell-Lucas numbers \( A_n \) (see Example 2 in [11]). Thus, from Corollary 2.4, we obtain

\[ A_n = 2i^n T_n (-i) = 2 (-i)^n T_n (i) \]

and

\[ \begin{align*}
A_n &= 2i^n U_n (-i) - 2i^{n-1} U_{n-1} (-i) = 2i^n U_n (-i) - 2i^{n-1} U_{n-1} (-i), \\
A_n &= 2P_{n+1} (1) - 2P_n (1) = 2 (-1)^n P_{n+1} (-1) - p (-1)^{n-1} P_n (-1), \\
A_n &= 2F_{n+1} (2) - 2F_n (2) = 2 (-1)^n F_{n+1} (-2) - p (-1)^{n-1} F_n (-2), \\
A_n &= 2 \left( -\frac{i}{\sqrt{2}} \right)^n \Phi_{n+1} (2\sqrt{2}i) - 2 \left( -\frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n (2\sqrt{2}i) \\
&= 2 \left( \frac{i}{\sqrt{2}} \right)^n \Phi_{n+1} (2\sqrt{2}i) - 2 \left( \frac{i}{\sqrt{2}} \right)^{n-1} \Phi_n (2\sqrt{2}i),
\end{align*} \]
\[ A_n = 2^{n+3} \left( \frac{i}{\sqrt{4a}} \right)^n D_n (-2\sqrt{ai}, a) - 2^{n+2} \left( \frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} (-2\sqrt{ai}, a) \]
\[ = 2^{n+3} \left( -\frac{i}{\sqrt{4a}} \right)^n D_n (2\sqrt{ai}, a) - 2^{n+2} \left( -\frac{i}{\sqrt{4a}} \right)^{n-1} D_{n-1} (2\sqrt{ai}, a) . \]

**Example 6** If \( p = 1 \) and \( q = 2 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 2 \), \( a_1 = 1 \) are Jacobsthal-Lucas numbers \( B_n \) (see Example 2 in [11]). Thus,

\[ B_n = 2 \left( \sqrt{2}i \right)^n T_n \left( -\frac{i}{2\sqrt{2}} \right) = 2 \left( -\sqrt{2}i \right)^n T_n \left( \frac{i}{2\sqrt{2}} \right), \]

and

\[ B_n = 2 \left( \sqrt{2}i \right)^n U_n \left( -\frac{i}{2\sqrt{2}} \right) - \left( \sqrt{2}i \right)^{n-1} U_{n-1} \left( \frac{i}{2\sqrt{2}} \right), \]
\[ = 2 \left( -\sqrt{2}i \right)^n U_n \left( \frac{i}{2\sqrt{2}} \right) - \left( -\sqrt{2}i \right)^{n-1} U_{n-1} \left( \frac{i}{2\sqrt{2}} \right) , \]
\[ B_n = 2 \left( \sqrt{2} \right)^n P_{n+1} \left( \frac{1}{2\sqrt{2}} \right) - \left( \sqrt{2} \right)^{n-1} P_n \left( \frac{1}{2\sqrt{2}} \right) , \]
\[ = 2 \left( -\sqrt{2} \right)^n P_{n+1} \left( -\frac{1}{2\sqrt{2}} \right) - \left( -\sqrt{2} \right)^{n-1} P_n \left( -\frac{1}{2\sqrt{2}} \right) , \]
\[ B_n = 2 \left( \sqrt{2} \right)^n F_{n+1} \left( \frac{1}{\sqrt{2}} \right) - \left( \sqrt{2} \right)^{n-1} F_n \left( \frac{1}{\sqrt{2}} \right) , \]
\[ = 2 \left( -\sqrt{2} \right)^n F_{n+1} \left( -\frac{1}{\sqrt{2}} \right) - \left( -\sqrt{2} \right)^{n-1} F_n \left( -\frac{1}{\sqrt{2}} \right) , \]
\[ B_n = 2i^n \Phi_{n+1} (-i) - i^{n-1} \Phi_n (-i) = 2(-i)^n \Phi_{n+1} (i) - (-i)^{n-1} \Phi_n (i) , \]
\[ B_n = 2^{n+3} \left( \frac{i}{\sqrt{2a}} \right)^n D_n \left( -\frac{\sqrt{ai}}{\sqrt{2}}, a \right) \]
\[ -2^{n+1} \left( \frac{i}{\sqrt{2a}} \right)^{n-1} D_{n-1} \left( -\frac{\sqrt{ai}}{\sqrt{2}}, a \right) \]
\[ = 2^{n+3} \left( -\frac{i}{\sqrt{2a}} \right)^n D_n \left( \frac{\sqrt{ai}}{\sqrt{2}}, a \right) \]
\[ -2^{n+1} \left( -\frac{i}{\sqrt{2a}} \right)^{n-1} D_{n-1} \left( \frac{\sqrt{ai}}{\sqrt{2}}, a \right) . \]
Example 7 If \( p = 3 \) and \( q = -2 \), then \( a_n \) defined by (1) with initial conditions \( a_0 = 2, a_1 = 3 \) are Fermat numbers \( f_n \) (see [7]). Thus, from Corollary 2.4, we obtain

\[
f_n = 2 \left( \sqrt{2} \right)^n T_n \left( \frac{3}{2\sqrt{2}} \right) = 2 \left( -\sqrt{2} \right)^n T_n \left( -\frac{3}{2\sqrt{2}} \right)
\]

and

\[
f_n = 2 \left( \sqrt{2} \right)^n U_n \left( \frac{3}{2\sqrt{2}} \right) - 3 \left( \sqrt{2} \right)^{n-1} U_{n-1} \left( \frac{3}{2\sqrt{2}} \right)
\]
\[
= 2 \left( -\sqrt{2} \right)^n U_n \left( -\frac{3}{2\sqrt{2}} \right) - 3 \left( -\sqrt{2} \right)^{n-1} U_{n-1} \left( -\frac{3}{2\sqrt{2}} \right),
\]
\[
f_n = 2 \left( \sqrt{2}i \right)^n P_{n+1} \left( -\frac{3i}{2\sqrt{2}} \right) - 3 \left( \sqrt{2}i \right)^{n-1} P_n \left( -\frac{3i}{2\sqrt{2}} \right)
\]
\[
= 2 \left( -\sqrt{2}i \right)^n P_{n+1} \left( \frac{3i}{2\sqrt{2}} \right) - 3 \left( -\sqrt{2}i \right)^{n-1} P_n \left( \frac{3i}{2\sqrt{2}} \right),
\]
\[
f_n = 2 \left( \sqrt{2}i \right)^n F_{n+1} \left( -\frac{3i}{\sqrt{2}} \right) - 3 \left( \sqrt{2}i \right)^{n-1} F_n \left( -\frac{3i}{\sqrt{2}} \right)
\]
\[
= 2 \left( -\sqrt{2}i \right)^n F_{n+1} \left( \frac{3i}{\sqrt{2}} \right) - 3 \left( -\sqrt{2}i \right)^{n-1} F_n \left( \frac{3i}{\sqrt{2}} \right),
\]
\[
f_n = 2\Phi_{n+1} (3) - 3\Phi_n (3) = 2(-1)^n\Phi_{n+1} (-3) - 3(-1)^{n-1}\Phi_n (-3),
\]
\[
f_n = 2^{n+3} \left( \frac{1}{\sqrt{2}a} \right)^n D_n \left( \frac{3\sqrt{a}}{\sqrt{2}}, a \right)
\]
\[
- 32^{n+1} \left( \frac{1}{\sqrt{2}a} \right)^{n-1} D_{n-1} \left( \frac{3\sqrt{a}}{\sqrt{2}}, a \right)
\]
\[
= 2^{n+3} \left( \frac{1}{\sqrt{2}a} \right)^n D_n \left( -\frac{3\sqrt{a}}{\sqrt{2}}, a \right)
\]
\[
- 32^{n+1} \left( \frac{1}{\sqrt{2}a} \right)^{n-1} D_{n-1} \left( -\frac{3\sqrt{a}}{\sqrt{2}}, a \right).
\]

Using the relationship established above we may obtain some identities of number sequences and polynomial value sequences. Theorem 3.2 in [11] presented a generalized Gegenbauer-Humbert polynomial sequence identity

\[
P_n^{1,y,C}(x) = \alpha(x)P_{n-1}^{1,y,C}(x) + C^{-2}(2x - \alpha(x)C) (\beta(x))^{n-1}, \quad (14)
\]
where $P_{n}^{1,y,C}(x)$ satisfies the recurrence relation of order 2 $P_{n}^{1,y,C} = pP_{n-1}^{1,y,C} + qP_{n-2}^{1,y,C}$ with coefficients $p(x)$ and $q(x)$, and $\alpha(x) + \beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Clearly (see (19) and (20) in [11]),

$$\alpha = \frac{1}{C} \left\{ x + \sqrt{x^2 - Cy} \right\} \quad \text{and} \quad \beta = \frac{1}{C} \left\{ x - \sqrt{x^2 - Cy} \right\}. \quad \text{(15)}$$

For $y = -1$ and $C = 1$, we have $P_{n}^{1,-1,1}(x) = F_{n+1}(2x)$, where $F_n(x)$ are Fibonacci polynomials, and we can write (14) as

$$F_{n+1}(2x) = \alpha(x)F_n(2x) + (2x - \alpha(x)) \beta(x) = \alpha(x)F_n(2x) + \beta(x)^n, \quad \text{(17)}$$

where $\alpha(x) = x + \sqrt{x^2 + 1}$ and $\beta(x) = x - \sqrt{x^2 + 1}$. If $x = 1/2$, then $F_n(1) = F_n$, Fibonacci numbers, and

$$\alpha \left( \frac{1}{2} \right) = \frac{1 + \sqrt{5}}{2}, \quad \text{and} \quad \beta \left( \frac{1}{2} \right) = \frac{1 - \sqrt{5}}{2}.$$  

Thus (17) yields the identity

$$F_{n+1} = \frac{1 + \sqrt{5}}{2} F_n + \left( \frac{1 - \sqrt{5}}{2} \right)^n,$$

or equivalently,

$$\frac{1 - \sqrt{5}}{2} F_{n+1} + F_n = \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$  

Similarly, if $x = 1$, then $F_n(2) = P_n$, Pell numbers, and

$$\alpha (1) = 1 + \sqrt{2}, \quad \text{and} \quad \beta (1) = 1 - \sqrt{2}.$$  

Thus (17) yields the identity

$$P_{n+1} = (1 + \sqrt{2}) P_n + \left( 1 - \sqrt{2} \right)^n,$$

or equivalently,
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\[(1 - \sqrt{2})P_{n+1} + P_n = \left(1 - \sqrt{2}\right)^{n+1}.\]

Substituting \(x = 1/(2\sqrt{2})\) into (17) and noting \(F_n(1/\sqrt{2}) = J_n/(\sqrt{2})^n\), where \(J_n\) are Jacobsthal numbers, we obtain the identity

\[J_{n+1} - 2J_n = (-1)^n.\]

When \(x = -3i/(2\sqrt{2})\), \(F_n(-3i/(2\sqrt{2})) = M_n/((\sqrt{2}i)^{n-1}\), Mersenne numbers. Hence (17) gives \(M_{n+1} - M_n = 2^n\).

Conversely, one may use the expressions of various number sequences in terms of the generalized Gegenbauer-Humbert polynomial sequences to construct the identities of the different generalized Gegenbauer-Humbert polynomial values such as the formulas shown in the example after Corollary 2.3.

References


