10.1080/0003681YYxxxxxxxx 1563-504X 0003-6811 00 00 2008 January

# Characterization of Compactly Supported Refinable Splines With Integer Matrix

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v3.3 released May 2008

August 8, 2009

**Abstract**

Let $M$ be an integer matrix with absolute values of all its eigenvalues being greater than 1. We give a characterization of compactly supported $M$-refinable splines $f$ and the conditions that the shifts of $f$ form a Riesz basis.

spline; refinement; blockwise polynomial; Riesz basis; simplex decomposition 42C40 (39A70, 41A15, 41A30, 65D07, 65D18, 65T60)

## 1 Introduction and Main Results

Let $M \in \mathbb{Z}^{n\times n}$ be an integer matrix with absolute values of all its eigenvalues being greater than 1. A function $f$ defined on $\mathbb{R}^n$ is $M$-refinable if there exists a finite sequence $\{h_j\}$ such that

$$f(x) = \sum_{j\in\mathbb{Z}^n} h_j f(Mx - j) \tag{1}$$

In [1], Lawton et al considered the one-dimensional setting of the scaling coefficient $M$ being an integer greater than 1. They gave a characterization of the refinable univariate splines and proved that only the shifts of B-spline with the smallest support form a Riesz basis. In [2] Sun extended the partial result of [1] to $M = mI$ using Box-splines, where $m \in \mathbb{Z}, m > 1$, and $I$ is the

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identity matrix, namely, an $M = mI$-refinable and blockwise polynomial with compact support is a finite linear combination of a box-spline and its translates. In [3],Y.Guan et al. further gave a characterization of $M = mI$-refinable and blockwise polynomial with compact support forming a Riesz basis. More relative results can be found in the survey [4, 5] by Goodman et al. In this paper, we generalize the results of [1, 2, 3] to the setting of a certain class of scaling matrices, namely. We shall derive a characterization of functions (1) when $M$ is a matrix with integer entries.

In the following, the multi-index notational system is adopted. First, throughout the paper, all vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ are column vectors.

Let $\omega \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$ with components $\omega_j \in \mathbb{R}$ and $z_j := \exp(i\omega_j)$ ($j = 1, 2, \dots, n$), respectively, where $i = \sqrt{-1}$. Denote the transpose of vector k and matrix M by $k'$ and $M'$, respectively. We also write $z = (\exp(i\omega_1), \exp(i\omega_2), \dots, \exp(i\omega_n))' = (z_1, z_2, \cdots, z_n)'$ as $z \equiv \exp(i\omega)$ for convenience when it is clear in the content. For $k \in \mathbb{Z}^n$, we denote $z^k = z^{k'} := \prod_{j=1}^n z_j^{k_j}$. For an integer matrix $M$, we denote $z^M := \exp(i(M\omega))$. Obviously, $z^{kM} = z^{M'k'} = \exp(ik'M\omega)$.

A trigonometric polynomial $R(\omega)$ is said to be $M$-closed if $R(M\omega)/R(\omega)$ is a trigonometric polynomial too.

Let $s \geq n$ and $A = (a_1, a_2, \cdots, a_s)$ a nonsingular matrix with integer entries and column vectors $a_j \in \mathbb{Z}^n, j = 1, 2, \cdots, s$. By means of Fourier transform, we can define box splines $B_A(x)$ of dimension $n$ as follows:

$$\hat{B}_A(\omega) = \prod_{j=1}^{s} \frac{\exp(ia_j'\omega) - 1}{ia_j'\omega} \tag{2}$$

A function $\phi$ is called a blockwise polynomial if there exists a simplex decomposition $\{\Delta_j\}_1^L$ of $\phi$, such that $\phi$ is a polynomial on every simplex. A standard simplex is defined as $\Delta^0 = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n; 0 \leq x_j \leq 1, \sum_{j=1}^n x_j \leq 1\}$, and a simplex $\Delta$ is an affine transform of the standard simplex, $\Delta = A\Delta^0 + c$, where $A$ is nonsingular and $c \in \mathbb{R}^n$. We say that $\{\Delta_j\}_{j=1}^L$ is a simplex decomposition of a bounded set $E$ if $\bigcup_{j=1}^L \Delta_j \supseteq E$, where $\Delta_j$ is a simplex for every $1 \leq j \leq L$, and $\Delta_j \bigcap \Delta_l$ has Lebesgue measure zero when $j \neq l$. For the adjacent $\Delta_j$ and $\Delta_l$, let $E$ be their $n-1$ dimensional common boundary lying on the plane $\pi$. Then we call $\pi$ a singular hyperplane of $f$ if $f(x)|_{\Delta_j}$ and $f(x)|_{\Delta_l}$ are different polynomials. Hence, all the planes passing through the $n-1$ dimensional boundaries of the simplex support of a block polynomial function are also called singular hyperplane of the blockwise polynomial.

Let $s \geq n$, $a_1, a_2, \cdots, a_s \in \mathbb{Z}^n$. We say a matrix $A = (a_1, a_2, \cdots, a_s)$ is *unimodular* if any matrix generated by any $n$ linearly independent column vectors of the matrix $A$ has determinant value $\pm 1$.

**Theorem 1.1.** *Let $n \geq 2$. Suppose $\phi$ is a compact support blockwise polynomial, $D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$, and $M$ is a matrix with integer entries and the absolute values of all its eigenvalues are greater than 1. Then we have the following results:*

*a). $\phi$ satisfies equation (1) if and only if it can be written as*

$$\phi(x) = P(D)\left(\sum_j r_j B_A(x - j - (M^k - I)^{-1} l)\right) \tag{3}$$

*and*

1. *$B_A(x)$ is a box spline defined on $A = (a_1, a_2, \cdots, a_s)$, where $A$ satisfies $\prod_{j=1}^s (Ma_j)'\omega = c_1 \prod_{j=1}^s a_j'\omega$ for some constant $c_1$;*
2. *$k$ is a positive integer and satisfies $(M')^k e_j = \lambda_j e_j$ for the normal vector $e_j$ of every singular hyperplane of $\phi$;*
3. *$l \in \mathbb{Z}^n$ satisfies $(M - I)(M^k - I)^{-1} l \in \mathbb{Z}^n$;*
4. *$P$ is a polynomial, and satisfies $P(M'\omega) = c_2 P(\omega)$, where $P(\omega)$ and $\prod_{j=1}^s a_j'\omega$ don't have common factor, and $c_2$ is a constant;*
5. *$R(z) = R(\exp(i\omega)) = \sum_j r_j z^j$ satisfies that $R(z)\prod_{j=1}^s (z^{a_j} - 1)$ is $M'$-closed trigonometric polynomial;*

*b). Furthermore, integer shifts of $\phi$ form Riesz basis if and only if $P$ is polynomial of zero degree, $A$ is unimodular, and $R(z)$ is a monomial.*

□

## 2 The Proof of Main Results

A polynomial $P$ is called a principal homogeneous polynomial if there exists a natural number $k$ and $a_j \in \mathbb{R}^n$, $1 \le j \le k$, such that $P(\omega) = \prod_{j=1}^k a_j{}'\omega$. In addition, for real $b_j$ and complex $a_j$, we call $\sum_j a_j \exp(ib_j\omega)$ a generalized trigonometric polynomial. Clearly, the following result holds.

**Lemma 2.1.** *The fourier transform of $\phi(Mx - k)$ is $|det(M)|^{-1} \exp(-i\omega' \cdot (M^{-1}k))\hat{\phi}((M^{-1})'\omega)$.*

**Proposition 2.2.** *Let $f$ be a blockwise polynomial with compact support satisfying Equation (1). Then there exists an integer $k \in \mathbb{Z}$ such that $(M')^k e_j = \lambda_j e_j$, where $\lambda_j \in \mathbb{R}$, holds for the normal vectors $e_j$ of all singular hyperplanes of $f$.*

*Proof.* Let $E = \{e_j\}$ be a finite set of the normal vectors $e_j$ of all singular hyperplanes of $f$. $\forall e_j \in E$, there exists a hyperplane $e_j'x - c_j = 0$ on which $f$ is singular. And expression (1) implies that both sides of the equation have the same singularities. Hence, there exists an integer $l$ on the right-hand side such that $f$ is singular at the hyperplane $e_j'(Mx - l) - c_j = 0 = (M'e_j)'x - e_j'l - c_j$. Thus, $M'e_j$ is also the normal vector of a singular hyperplane of $f$. $M$ is one-to-one mapping from $E$ to itself because $M$ is not singular. Hence, from the finiteness of $E$ there exist an integer $k$ such that $(M')^k e_j = \lambda_j e_j$ for every $e_j \in E$. Furthermore, since both $M$ and $e_j$ are real, $\lambda_j$ is also real. □

**Proposition 2.3.** *Let $f$ be an $M-$refinable blockwise polynomial with compact support, then its fourier transform $\hat{f}$ can be written as $\hat{f}(\omega)=\sum\frac{q_n(z)}{p_n(\omega)}$, where $p_n(\omega)$ are the principal homogeneous polynomials and $q_n(z)$ are the generalized trigonometric polynomials. In addition, there exist $k$ such that $p_n((M')^{-k}\omega)=c_np_n(\omega)$ for some constants $c_n, n=1,2,\cdots$.*

*Proof.* Let $\{e_j\}$ be the finite set of the normal vectors of singular hyperplanes of $f$. From proposition 2.2, there exists an integer $k$ such that $(M')^ke_j=\lambda_je_j$ for every $e_j$. Denote by $\alpha_1,\alpha_2,\cdots,\alpha_m$ the different eigenvalue of $(M')^k$, where $m\le n$, and by $V_1,V_2,\cdots,V_m$ the corresponding eigenspaces of $\alpha_1,\alpha_2,\cdots,\alpha_m$. For any $j$, since $e_j$ is the eigenvector of $(M')^k$, there exists $l_j$ such that $e_j\in V_{l_j}$. Obviously, $f$ is compactly supported and its support must be a polyhedral in $\mathbb{R}^n$. So every boundary of the polyhedral must be on a singular plane of $f$ and set $\{e_j\}$ spans $\mathbb{R}^n$. Furthermore, every $V_j$ has a basis $E_j$ consisting of the elements of $\{e_j\}$. Hence, we can write $E_j=(e_{j,i}),e_{j,i}\in\{e_l|e_l\in V_j\}$, $i=1,2,\cdots,m_j$. Therefore, we obtain a basis $E=(E_1,E_2,\cdots,E_m)$ of $V=V_1+V_2+\cdots+V_m$, which consists of the elements of $\{e_j\}$. Let $\tilde{E}=(\tilde{E}_1,\tilde{E}_2,\cdots,\tilde{E}_m)$, here $E_l'\tilde{E}_l=I$, $l=1,2,\cdots,m$, and $E_j'\tilde{E}_l=0$ when $j\neq l$, so $E'\tilde{E}=I$. For every $l=1,2,\ldots,m$, $\tilde{E}_l$ spans a space denoted by $\tilde{V}_l$ with $\tilde{V}_l\perp V_j$ when $j\neq l$. Obviously, $V=\tilde{V}_1+\tilde{V}_2+\cdots+\tilde{V}_m$.

For an arbitrary $1\le j\le m$, the intersections of $\tilde{V}_j$ and the singular plane of $f(x)$, whose normal vector belongs to $\{e_l|e_l\in V_j\}$, form a polyhedral partition of $\tilde{V}_j$. Furthermore, we can establish a simplex partition $\{\tilde{\Delta}_{j,l}\}$ from the polyhedral decomposition of $\tilde{V}_j$, so that we obtain a new polyhedral partition $\tilde{\Delta}=\{\bigoplus_{j=1}^m(\sum_l\tilde{\Delta}_{j,l})\}$ of $V$.

$\forall l_1,l_2,\cdots,l_m$, we claim that $f(x)$ is a polynomial in the domain $\bigoplus_{s=1}^m\tilde{\Delta}_{s,l_s}$. In fact, since $f(x)$ is a spline, we only need to prove domain $\bigoplus_{s=1}^m\tilde{\Delta}_{s,l_s}$ is not divided by any singular hyperplane of $f(x)$, that is, all the point in $\bigoplus_{s=1}^m\tilde{\Delta}_{s,l_s}$ are on the same side of any singular hyperplane of $f(x)$. Let $e_j$ be an arbitrary normal vector in $V_{l_j}$, and let the corresponding singular hyperplanes be $<e_j,x>=c_{j,i}$, $i=1,2,\cdots$. Denote $x=\sum_{s=1}^m\tilde{E}_s\alpha_s$ and $y=\sum_{s=1}^m\tilde{E}_s\beta_s$, where $\tilde{E}_s\alpha_s,\tilde{E}_s\beta_s\in\tilde{\Delta}_{s,l_s}, s=1,2,\cdots,m$. $\forall\ x,y\in\bigoplus_{s=1}^m\tilde{\Delta}_{s,l_s}$ and $i=1,2,\ldots,$ we have $(<x,e_j>-c_{j,i})(<y,e_j>-c_{j,i})=(<\tilde{E}_{l_j}\alpha_{l_j},e_j>-c_{j,i})(<\tilde{E}_{l_j}\beta_{l_j},e_j>-c_{j,i})\ge 0$. The last inequality can be obtained from the simplex partition of $\tilde{V}_{l_j}$.

If $x\in V$, we may write $x=\tilde{E}\beta$ with some $\beta\in\mathbb{R}^n$. Furthermore we have $\tilde{\Delta}_{j,l}=\tilde{E}\Delta_{j,l}$, $\tilde{\Delta}=\tilde{E}\Delta$ and $f(\tilde{E}\beta)$ is a polynomial on $\bigotimes_{j=1}^m\Delta_{j,l_j}$. Thus,

$$\begin{aligned}\hat{f} &= \int_{\mathbb{R}^n} f(x)\exp(-ix'\omega)dx\\ &= \det(\tilde{E})\int_{\mathbb{R}^n} f(\tilde{E}\beta)\exp(-i\beta'\tilde{E}'\omega))d\beta\end{aligned}$$

Let $\xi = (\xi'_1, \xi'_2, \cdots, \xi'_m)' = \tilde{E}'\omega = (\tilde{E}_1, \tilde{E}_2, \cdots, \tilde{E}_m)'\omega$. Then

$$\begin{aligned}
\hat{f} &= \det(\tilde{E}) \int_{\bigotimes_{j=1}^m (\sum_l \Delta_{j,l})} f(\tilde{E}\beta) \exp(-i\beta'\xi) d\beta \\
&= \det(\tilde{E}) \int_{\sum_{l_1} \cdots \sum_{l_m} \bigotimes_{j=1}^m \Delta_{j,l_j}} f(\tilde{E}\beta) \exp(-i\beta'\xi) d\beta \\
&= \det(\tilde{E}) \sum_{l_1} \cdots \sum_{l_m} \int_{\bigotimes_{j=1}^m \Delta_{j,l_j}} f(\tilde{E}\beta) \exp(-i\beta'\xi) d\beta
\end{aligned}$$

If $f$ is a polynomial on $\bigotimes_{j=1}^m \Delta_{j,l_j}$ we have

$$\begin{aligned}
\hat{f} &= \det(\tilde{E}) \sum_{l_1} \cdots \sum_{l_m} \int_{\Delta_{1,l_1}} \cdots \int_{\Delta_{m,l_m}} \sum_n a_n(l_1, l_2, \cdots, l_m) \beta^n \exp(-i\beta'\xi) d\beta \\
&= \det(\tilde{E}) \sum_{l_1} \cdots \sum_{l_m} \sum_n a_n(l_1, l_2, \cdots, l_m) \prod_{j=1}^m \int_{\Delta_{j,l_j}} \beta_j^{n_j} \exp(-i\beta'_j\xi_j) d\beta_j
\end{aligned}$$

After simplifying above sum, from the lemma 1 in [2], there exist principle homogenous polynomials $p_{k,j}(\xi_j)$ in terms of the variants $\xi_j$ and generalized trigonometric polynomials $q_{k,j}(\exp(-i\xi_j))$ such that

$$\begin{aligned}
\hat{f} &= \sum_n \prod_{j=1}^m \frac{q_{n,j}(\exp(-i\xi_j))}{p_{n,j}(\xi_j)} \\
&= \sum_n \prod_{j=1}^m \frac{q_{n,j}(\exp(-i\tilde{E}'_j\omega))}{p_{n,j}(\tilde{E}'_j\omega)} \\
&= \sum_n \frac{q_n(\exp(-i\omega))}{\prod_{j=1} p_{n,j}(\tilde{E}'_j\omega)},
\end{aligned}$$

where $q_n(\exp(-i\omega)) = \prod_{j=1}^m q_{n,j}(\exp(-i\tilde{E}'_j\omega))$ are the generalized trigonometric polynomials.

From the above discussion we know $(M')^k E = E\lambda$, where $\lambda$ is a diagonal matrix, and $(M')^k E_j = \alpha_j E_j$. Furthermore, $M^k \tilde{E} = \tilde{E}\lambda$ and $M^k \tilde{E}_j = \alpha_j \tilde{E}_j$. Let $p_n(\omega) = \prod_j p_{n,j}(\tilde{E}'_j\omega)$, then $p_n((M')^{-k}\omega) = \prod_j p_{n,j}(\tilde{E}'_j(M')^{-k}\omega) = \prod_j p_{n,j}((M^{-k}\tilde{E}_j)'\ \omega) = \prod_j p_{n,j}(\alpha_j^{-1}\tilde{E}'_j\omega) = c_n \prod_j p_{n,j}(\tilde{E}'_j\omega) = c_n p_n(\omega)$. Obviously, $p_n(\omega)$ is a principal homogeneous polynomial because $p_{n,j}(\omega)$ are principal homogeneous polynomials. $\square$

**Lemma 2.4.** *Let $P_j$ $(j = 1, 2)$ be two nonzero polynomials, and let $T_j$ $(j = 1, 2)$ be two nonzero generalized trigonometric polynomials. If $P_j$ and $T_j$ $(j = 1, 2)$ satisfy $P_1(\omega)T_1(\omega) = P_2(\omega)T_2(\omega)$, then $P_1(\omega) = CP_2(\omega)$ and $T_1(\omega) = C^{-1}T_2(\omega)$ for some complex number $C$.*

*Proof.* Let $\hat{f}_1 = P_1(\omega)T_1(\omega)$, $\hat{f}_2 = P_2(\omega)T_2(\omega)$ be two generalized functions, where $T_1(\omega) = \sum_{j=1}^K c_{1,j} \exp(-ia_j\omega)$, $T_2(\omega) = \sum_{j=1}^L c_{2,j} \exp(-ib_j\omega)$, $\{a_j\}_{j=1}^K$

are $K$ different real numbers, and $\{b_j\}_{j=1}^L$ are $L$ different real numbers. Denote $Q_1(i\omega) = P_1(\omega)$, $Q_2(i\omega) = P_2(\omega)$, and $D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$. Then $f_1 = Q_1(D)\sum_{j=1}^K c_{1,j}\delta(x-a_j)$ and $f_2 = Q_2(D)\sum_{j=1}^L c_{2,j}\delta(x-b_j)$. Thus, for an arbitrary infinitely differentiable function $\phi$ with compact support, we have $0 = \langle \hat{f}_1 - \hat{f}_2, \hat{\phi}\rangle = \langle f_1 - f_2, \phi\rangle = \langle \sum_{j=1}^K c_{1,j}\delta(x-a_j), Q_1(D)\phi\rangle - \langle \sum_{j=1}^L c_{2,j}\delta(x-b_j), Q_2(D)\phi\rangle$. So $\langle \sum_{j=1}^K c_{1,j}\delta(x-a_j), Q_1(D)\phi\rangle = \langle \sum_{j=1}^L c_{2,j}\delta(x-b_j), Q_2(D)\phi\rangle$. Since $\phi$ is arbitrary, we obtain $L = K, c_{1,j} = C^{-1}c_{2,j} (j = 1, 2, \cdots, K), Q_1 = CQ_2$ for some constant C, so the lemma is proved. □

**Lemma 2.5.** *Let $M \in \mathbb{Z}^{n\times n}$ be a matrix of integer entries, and let all of its eigenvalue be real and lager than 1. Suppose $T(\omega)$ is a nonzero generalized trigonometric polynomial, and $H(\omega)$ is a nonzero trigonometric polynomial defined on $\mathbb{R}^n$. If*

$$T(M\omega) = H(\omega)T(\omega) \tag{4}$$

*then* $\exp(-il(M-I)^{-1}\omega)T(\omega)$ *is a trigonometric polynomial for some* $l \in \mathbb{Z}^n$.

*Proof.* One can write

$$T(\omega) = \sum_j \exp(ix_j'\omega)T_j(\omega) = \sum_k \exp(iy_k'\omega)Q_k(\omega) \tag{5}$$

where $T_j(\omega)$ is a trigonometric polynomial, $x_j - x_{\tilde{j}} \notin \mathbb{Z}^n$ for $j \neq \tilde{j}$, and $Q_k(M\omega)$ is a trigonometric polynomial with $M'(y_k - y_{\tilde{k}}) \notin \mathbb{Z}^n$ when $k \neq \tilde{k}$. So from (4) and (5) we have

$$\sum_k \exp(iy_k'M\omega)Q_k(M\omega) = \sum_j \exp(ix_j'\omega)H(\omega)T_j(\omega). \tag{6}$$

For a given $k$, suppose $M'y_k - x_j \in \mathbb{Z}^n$, then for all $\tilde{j} \neq j$ we have $M'y_k - x_{\tilde{j}} \notin \mathbb{Z}^n$ because $x_j - x_{\tilde{j}} \in \mathbb{Z}^n$. Similarly, there is only one $y_k$ satisfying $M'y_k - x_j \in \mathbb{Z}^n$ for all $x_j$. So the numbers of the elements in sets $\{x_j\}$ and $\{y_k\}$ are equal. From $H \neq 0$ and (6), we have

$$\exp(iy_k'M\omega)Q_k(M\omega) = \exp(ix_j'\omega)H(\omega)T_j(\omega) \tag{7}$$

In addition, from (5) we have

$$T(\omega) = \sum_j \exp(ix_j'\omega)T_j(\omega), \tag{8}$$

where $\{x_j\}$ satisfies $M'(x_j - x_{\tilde{j}}) \notin \mathbb{Z}^n$ when $j \neq \tilde{j}$. Hence for all $x_j$, $\exists x_{\tilde{j}}$ and $s \in \mathbb{Z}^n$ satisfy $M'x_j = s + x_{\tilde{j}}$ and

$$\exp(ix_j'M\omega)T_j(M\omega) = \exp(ix_{\tilde{j}}'\omega)H(\omega)T_{\tilde{j}}(\omega). \tag{9}$$

Define map $\mathcal{M}x_j = x_{\tilde{j}}$, where $x_{\tilde{j}}$ is chosen as above. Then $\mathcal{M}$ is a well-defined one-to-one map on $\{x_j\}$. We also define $X_s = \{\mathcal{M}^k x_s; k = 1, 2, \cdots\}$ for every $x_s$. Then $X_s = X_{\tilde{s}}$ or $X_s \bigcap X_{\tilde{s}} = \emptyset$. Thus we can choose finite numbers of $X_l$ such that $\{x_j\} = \bigcup_l X_l$ and $X_l \bigcap X_{\tilde{l}} = \emptyset$. Therefore the lemma is true if we can prove that $X_l$ is a singleton for every $l$, and there exists only one $X_l$ in the decomposition of $\{x_j\}$.

First we prove that there is only one element in $X_l$ by using the method of contradiction. Assume $X_l = \{x_1, x_2, \cdots, x_k\}$, $k \geq 2$. Then for $1 \leq s \leq k$, $\exists \tau_s \in \mathbb{Z}^n$ such that

$$T_s(M\omega) = \exp(i\tau_s'\omega)H(\omega)T_{s+1}(\omega) \tag{10}$$

Let $T_1(\omega) = T_{k+1}(\omega)$. Then

$$T_s(M^k\omega) = \exp(i\tilde{\tau_s}'\omega)T_s(\omega)\prod_{j=0}^{k-1} H(M^j\omega) \tag{11}$$

where $\tilde{\tau}_s \in \mathbb{Z}^n$. Denote $\tilde{H}(\omega) = \prod_{j=0}^{k-1} H(M^j\omega)$ and $\tilde{M} = M^k$. We have

$$T_s(\tilde{M}\omega) = \exp(i\tilde{\tau_s}'\omega)T_s(\omega)\tilde{H}(\omega) \tag{12}$$

Let $e_1, e_2, \cdots, e_n$ be the linearly independent eigenvectors of $\tilde{M}^{-1}$. And the corresponding eigenvalues are denoted by $\rho_1, \rho_2, \cdots, \rho_n$ that satisfy $1 > |\rho_1| = |\rho_2| = \cdots = |\rho_t| > |\rho_{t+1}| \geq \cdots \geq |\rho_n| > 0$. Thus, the claim is obtained from the fact of that $M$ is nonsingular, and the absolute value of all its eigenvalues are greater than 1. Hence there exists a invertible transform $\omega = \sum_j \alpha_j e_j = E\alpha$, where $E = (e_1, e_2, \cdots, e_n)$. Let the Taylor expansion of $T_s(\omega)$ with the remainder be written as

$$T_s(\omega) = T_s(E\alpha) = p_1(\alpha) + p_2(\alpha) + p_3(\alpha), \tag{13}$$

where $p_1(\alpha) + p_2(\alpha) \neq 0$ is homogeneous polynomial with degree $K$, in which $p_1(\alpha)$ will be described later, and $p_2(\alpha)$ is the difference of the Taylor expansion of $T_s(\omega)$ and $p_1(\alpha)$, and the remainder $|p_3(\alpha)| \leq C\,|\alpha|^{K+1}$. Assume in $p_1(\alpha) + p_2(\alpha)$ the degrees of $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_j}$ are nonzero, and denote $\tilde{i}_1 = \min\{i_1, i_2, \cdots, i_j\}$. Hence, for $p_1(\alpha) + p_2(\alpha)$, the degrees of $\alpha_1, \alpha_2, \cdots, \alpha_{\tilde{i}-1}$ are zero, where $p_1(\alpha)$ is a polynomial in terms of the variants, whose corresponding eigenvalue's absolute value is $\left|\rho_{\tilde{i}_1}\right|(\alpha_i \to e_i \to \rho_i)$. For convenience, assume $\tilde{i}_1 = 1$, then $p_1(\alpha)$ is a homogeneous polynomial with degree $K$ of $\alpha_1, \alpha_2, \cdots, \alpha_t$ and $p_2(\alpha)$ is also a homogeneous polynomial with degree $K$, but its every monomial has nonzero degree of $\alpha_{t+1}, \cdots, \alpha_n$. Let $\rho \in \mathbb{C}^n$, write $(\rho \cdot \alpha) = (\rho_1\alpha_1, \rho_2\alpha_2, \cdots, \rho_n\alpha_n)^T$. Therefore,

$$T_s(\tilde{M}^{-p}(\omega)) = T_s(E(\rho^p \cdot \alpha)) = \rho_1^{Kp}p_1(\alpha) + p_2(E(\rho^p \cdot \alpha)) + O(|\rho_1|^{p(K+1)}),$$

which implies

$$\lim_{p\to\infty} \frac{T_s(\tilde{M}^{-p}(\omega))}{\rho_1^{Kp}} = \lim_{p\to\infty} \frac{T_s(E(\rho^p \cdot \alpha))}{\rho_1^{Kp}} = p_1(\alpha). \tag{14}$$

Indeed the above formula holds due to $p_2(E(\rho^p \cdot \alpha))/\rho_1^{pK} \to 0$ as $p \to \infty$, which can be proved as follows: For every monomial of $p_2$, the degree of $\alpha_{t+1}, \cdots, \alpha_n$ is non-zero. Write $p_2(E(\rho^p \cdot \alpha)) = \sum_j \beta_j(\rho^p \cdot \alpha)^j$, then $\left|p_2(E(\rho^p \cdot \alpha))/\rho_1^{pK}\right| = \left|\sum_j \beta_j(\frac{\rho^p}{\rho_1^p} \cdot \alpha)^j\right| \leq \sum_j \left|\beta_j \alpha^j\right| \left|\frac{\rho_{t+1}}{\rho_1}\right|^p \to 0$ as $p \to \infty$.

For $1 \leq s \leq k$, from (12) we have

$$\begin{aligned} T_s(\omega) &= T_s(E\alpha) = \exp(i\tau_s' E(\rho \cdot \alpha))\tilde{H}(\tilde{M}^{-1}E\alpha)T_s(\tilde{M}^{-1}E\alpha) \\ &= \exp(i\tau_s' E((\sum_{j=1}^{p} \rho^j) \cdot \alpha))T_s(\tilde{M}^{-p}E\alpha) \prod_{j=1}^{p} \tilde{H}(\tilde{M}^{-j}E\alpha) \end{aligned}$$

Since $\tilde{H}(\omega)$ is a trigonometric polynomial, we have

$$\begin{aligned} & T_s(\omega) \exp(i\tau_1' E((\sum_{j=1}^{p} \rho^j) \cdot \alpha))T_1(\tilde{M}^{-p}E\alpha) \\ = \; & T_1(\omega) \exp(i\tau_s' E((\sum_{j=1}^{p} \rho^j) \cdot \alpha))T_s(\tilde{M}^{-p}E\alpha) \end{aligned}$$

a.e. Divide the two sides of the above formula by $\rho_1^{Kp}$, and let $p \to \infty$. From (14) there exist $\beta_1, \beta_s \in \mathbb{R}^n$ as well as polynomials $P_s$ and $P_1$ such that

$$\exp(i\beta_s'\omega)P_s(E^{-1}\omega)T_1(\omega) = \exp(i\beta_1'\omega)P_1(E^{-1}\omega)T_s(\omega) \tag{15}$$

for $2 \leq s \leq k$. From lemma 3 in [2] , there exist $j_s \in \mathbb{Z}^n$ and a constant $c_s$ such that $P_s(\omega) = C_s P_1(\omega)$ and $T_s(\omega) = C_s \exp(ij_s\omega)T_1(\omega)$ for $1 \leq s \leq k$. Without losing the generality, let $j_s = 0$ by selecting appropriate $x_j$. Thus,

$$\begin{aligned} & C_s \exp(ix_s'\omega)T_1(\omega) = \exp(ix_s'\omega)T_s(\omega) \\ = \; & \exp(ix_{s+1}' M^{-1}\omega)H(M^{-1}\omega)T_{s+1}(M^{-1}\omega) \\ = \; & \exp(ix_{s+1}' M^{-1}\omega)H(M^{-1}\omega)T_1(M^{-1}\omega)C_{s+1}. \end{aligned}$$

From (9) and the definition of $\mathcal{M}$, there exists a fixed $j \in \mathbb{Z}^n$ such that $-\left(M^{-1}\right)' x_{s+1} + x_s = \left(M^{-1}\right)' j$ for all $1 \leq s \leq k$. Therefore, using $x_{k+1} = x_1$ yields $-x_{s+1} + M'x_s = j$ for all $1 \leq s \leq k$. By a direct calculating, $x_s = (M' - I)^{-1}j$ satisfies the above request and it is only one solution when

$$\begin{pmatrix} M' & -I & & \\ & \ddots & \ddots & \\ & & \ddots & -I \\ -I & & & M' \end{pmatrix}$$

is nonsingular. Furthermore, $M'(x_s - x_{\tilde{s}}) \in \mathbb{Z}^n$ for all $s$ and $\tilde{s}$. This is inconsistent with $M'(x_s - x_{\tilde{s}}) \notin \mathbb{Z}^n$ as $s \neq \tilde{s}$, so the assumption is wrong, that is, there is only one element for every $X_l$.

Using the above argument and from (9) we obtain

$$\exp(ix_j'\omega)T_j(\omega) = \exp(ix_j'M^{-1}\omega)H(M^{-1}\omega)T_j(M^{-1}\omega) \tag{16}$$

similarly as (15) , we also have

$$\exp(i\beta_s'\omega)P_s(E'\omega)T_1(\omega) = \exp(i\beta_1'\omega)P_1(E'\omega)T_s(\omega). \tag{17}$$

Hence, from lemma 3 in [2], $T_j(\omega) = C_j \exp(ik_j\omega)T_1(\omega)$, $k_j \in \mathbb{Z}^n$. Choosing an appropriate $x_j$ one may have $k_j = 0$. Then, for all $j$, from (16) we have $\exp(ix_j'\omega)T_1(\omega) = \exp(ix_j'M^{-1}\omega)H(M^{-1}\omega)T_1(M^{-1}\omega)$, which implies $x_j - x_1 \in \mathbb{Z}^n$, this contradiction completes the proof of the lemma.

□

**Proposition 2.6.** *Let $M$ be a matrix of integer entries, and all of its eigenvalue be real and lager than 1. Then for a given nonzero trigonometric polynomial $H(z)$, if $q_1(z^M) = c_1H(z)q_1(z)$ and $q_2(z^M) = c_2H(z)q_2(z)$ then $c_1 = c_2$ and there exist a const $c$ such that $q_1(z) = cq_2(z)$, where $c_1$ and $c_2$ are two nonzero real constants, and $q_1(z)$ and $q_2(z)$ are nonzero generalized trigonometric polynomials.*

*Proof.* From the condition, $\forall k \in \mathbb{N}$

$$q_1(z) = c_1^k q_1(z^{M^{-k}}) \prod_{j=1}^{k} H(z^{M^{-j}})$$

and

$$q_2(z) = c_2^k q_2(z^{M^{-k}}) \prod_{j=1}^{k} H(z^{M^{-j}}).$$

Thus, $q_1(z)c_2^k q_2(z^{M^{-k}}) = c_1^k q_1(z^{M^{-k}})q_2(z)$.

Let $\omega = E\alpha$, where $E = (e_1, e_2, \cdots, e_n)$ is a nonsigular matrix and $Me_j = \lambda_j e_j$, $j = 1, 2, \cdots, n$, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. Then

$$z^{M^{-k}} = \exp(iM^{-k}\omega) = \exp(iM^{-k}E\alpha) = \exp(iE(\alpha/\lambda^k)).$$

Using Taylor series of $q_1, q_2$ we have for $m \in \mathbb{N}$

$$q_1(z)\sum_j a_j\alpha^j(c_2\lambda^{-j})^{2m} = q_2(z)\sum_j b_j\alpha^j(c_1\lambda^{-j})^{2m}$$

Divide the both sides of the above equation by $\lambda_{max}^{2m}$, where $\lambda_{max} = \max_j \{|c_2\lambda^{-j}|, |c_1\lambda^{-j}|\}$, and let $m \to \infty$, then we obtain polynomials $p_1(\alpha), p_2(\alpha)$ and the equation

$$q_1(z)p_2(\alpha) = q_2(z)p_1(\alpha).$$

From lemma 2.4 we have $q_1(z) = cq_2(z)$. Furthermore, we have $c_1 = c_2$, which completes the proof. □

**Lemma 2.7.** *Let $T$ be a trigonometric polynomial and $T(\omega) = 0$ on plane $a_j'\omega = 0$, $j = 1, 2, \cdots, N$. Then there exist trigonometric polynomial $R(\omega)$ and $\alpha_j \in \mathbb{R}$, $j = 1, 2, \cdots, N$ such that $T(\omega) = R(\omega)\prod_{j=1}^{N}(\exp(i\alpha_j a_j'\omega) - 1)$, where $\alpha_j a_j \in \mathbb{Z}^n$, $j = 1, 2, \cdots, N$.*

*Proof.* This lemma can be proved by a similar argument of the proof of theorem 1 in [2]. □

In the following we give the proof of Theorem 1.1.

Proof: Sufficiency of the first part: from the fourier transform of (3) , it is easy to get that $\hat{\phi}(M'\omega)/\hat{\phi}(\omega)$ is a trigonometric polynomial because $\phi$ satisfies equation (1).

Necessity of the first part: By proposition 2.2, there exists k such that $(M')^k e_j = \lambda_j e_j$ for the normal vector $e_j$ of an arbitrary singular hyper-plane. Let $\tilde{M} = M^k$, then $f$ is $\tilde{M}$-refinable. By proposition 2.3

$$\hat{\phi}(\omega) = \sum_{j\in\Lambda} \frac{T_j(\omega)}{P_j(\omega)} = \sum_{s\geq s_0} \sum_{\deg P_j = s} \frac{T_j(\omega)}{P_j(\omega)} \tag{18}$$

where $s_0 \geq 0$ and $\sum_{\deg P_j = s_0} \frac{T_j(\omega)}{P_j(\omega)} \neq 0$, and $\{P_j^{-1}(\omega)\}_{\deg P_j = s}$ is linearly independent $(s = s_0, s_0 + 1, s_0 + 2, \cdots)$. In addition,

$$\sum_{s>s_0} \sum_{\deg P_j = s} \frac{T_j(r\omega)}{P_j(r\omega)} r^{s_0} \to 0$$

and

$$\sum_{s>s_0} \sum_{\deg P_j = s} \frac{T_j(r(\tilde{M}')^{-1}\omega)}{P_j(r(\tilde{M}')^{-1}\omega)} r^{s_0} \to 0$$

as $r \to +\infty$ a.e. for all $\omega$ on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

From (1), there exists trigonometric polynomial $H(\omega)$ such that $\hat{\phi}(\omega) = H((\tilde{M}')^{-1}\omega)\hat{\phi}((\tilde{M}')^{-1}\omega)$, and from proposition 2.3 it satisfies

$$\sum_{\deg P_j = s_0} \left( \frac{T_j(r\omega)}{P_j(\omega)} - \frac{H(r(\tilde{M}')^{-1}\omega)T_j(r(\tilde{M}')^{-1}\omega)}{P_j((\tilde{M}')^{-1}\omega)} \right) \tag{19}$$

$$= \sum_{\deg P_j = s_0} \left( \frac{T_j(r\omega) - \lambda_j H(r(\tilde{M}')^{-1}\omega)T_j(r(\tilde{M}')^{-1}\omega)}{P_j(\omega)} \right) \to 0 \tag{20}$$

as $r \to +\infty$ a.e. $\omega \in \mathbb{S}^{n-1}$. Write

$$T_j(\omega) - \lambda_j H((\tilde{M}')^{-1}\omega)T_j((\tilde{M}')^{-1}\omega) \quad = \textstyle\sum_k c_{j,k} \exp(iy_k'\omega)$$

and denote

$$D_k(\omega) = \sum_j \frac{c_{j,k}}{P_j(\omega)}$$

Hence $\sum_k D_k(\omega)\exp(iy_k'\omega r) \to 0$ as $r \to \infty$ for $y_k'\omega \neq y_{\tilde{k}}'\omega$ a.e. $\omega \in \mathbb{S}^n$. By using lemma 2 of [2], we have $D_k(\omega) = 0$ a.e. $\omega \in \mathbb{S}^n$. Since $\{p_j^{-1}(\omega)\}$ is linearly independent, we have $c_{j,k} = 0$ for all $j, k$. Hence $T_j(\omega) = \lambda_j H((\tilde{M}')^{-1}\omega)T_j((\tilde{M}')^{-1}\omega)$. Thus,

$$\sum_{\deg P_j = s_0} \left( \frac{T_j(r\omega)}{P_j(\omega)} - \frac{H(r(\tilde{M}')^{-1}\omega)T_j(r(\tilde{M}')^{-1}\omega)}{P_j((\tilde{M}')^{-1}\omega)} \right) = 0$$

Recursively, for degree of $P_j = s_0+1, s_0+2, \cdots$, $T_j(\omega) = \lambda_j H((\tilde{M}')^{-1}\omega)T_j((\tilde{M}')^{-1}\omega)$, so that for all $T_j$ we have $T_j(\omega) = \lambda_j H((\tilde{M}')^{-1}\omega)T_j((\tilde{M}')^{-1}\omega)$. By using lemma 2.5 and proposition 2.6, there exists a trigonometric polynomial $\tilde{T}(\omega)$ such that $T_j(\omega) = c_j \exp(il'((\tilde{M}') - I)^{-1}\omega)\tilde{T}(\omega) = c_j \exp(il'((M')^k - I)^{-1}\omega)\tilde{T}(\omega)$, which implies

$$\hat{\phi}(\omega) = \sum_{s \geq s_0} \sum_{\deg P_j = s} \frac{c_j}{P_j(\omega)} \exp(il'((M')^k - I)^{-1}\omega)\tilde{T}(\omega).$$

Let $P(\omega)/Q(\omega) = \sum_{s\geq s_0} \sum_{\deg P_j = s} \frac{c_j}{P_j(\omega)}$, where $Q(\omega)$ is a principle homogeneous polynomial, and $P(\omega)$ and $Q(\omega)$ do not have common factors. Hence

$$Q(\omega)\hat{\phi}(\omega) = \exp(il'((M')^k - I)^{-1}\omega)\tilde{T}(\omega)P(\omega)$$

Let $Q(\omega) = \prod_{j=1}^N a_j'\omega$, $0 \neq a_j \in \mathbb{R}^n$. Because $|\exp(il'((M')^k - I)^{-1}\omega)| = 1$, and there is no common factors for $P$ and $Q$, we obtain $\tilde{T}(\omega) = 0$ on the hyperplanes of $a_j'\omega = 0$, $j = 1, 2, \cdots, N$. From lemma 2.7 we know that there exists a trigonometric polynomial $R(\omega)$ and $\tilde{a}_j = \alpha_j a_j \in \mathbb{Z}^n$ such that $\tilde{T}(\omega) = R(\omega)\prod_{j=1}^N(\exp(i\tilde{a}_j'\omega) - 1)$. Hence we have

$$\hat{\phi}(\omega) = \exp(il'((M')^k - I)^{-1}\omega)P(\omega)R(\omega)\prod_{j=1}^N \frac{(\exp(i\tilde{a}_j'\omega) - 1)}{i\tilde{a}_j'\omega}.$$

Thus, (3) holds. And from (1), there exists a trigonometric polynomial $H(\omega)$, such that

$$\begin{aligned} H(\omega) &= \frac{\hat{\phi}((M')\omega)}{\hat{\phi}(\omega)} \\ &= \frac{P((M')\omega)}{P(\omega)} \frac{R(\exp(i(M')\omega))}{R((\exp(i\omega)))} \exp(il'((M')^k - I)^{-1}(M' - I)\omega) \\ &\quad \times \prod_{j=1}^N \left( \frac{(\exp(i\tilde{a}_j'(M')\omega) - 1)}{(\exp(i\tilde{a}_j'\omega) - 1)} \frac{a_j'\omega}{a_j'(M')\omega} \right) \end{aligned}$$

Let $\tilde{l} = (M - I)((M^k) - I)^{-1}l$, $\tilde{R}(\omega) = R(\exp(i\omega))\prod_{j=1}^N(\exp(i\tilde{a}_j'\omega) - 1)$, By using lemma 3 of [2], for $\tilde{l} \in \mathbb{Z}^n$

$$P(M'\omega)\prod_{j=1}^N a_j'\omega = CP(\omega)\prod_{j=1}^N a_j'M'\omega \tag{21}$$

$$\tilde{R}(M'\omega)) = C^{-1}H(\omega)\exp(i\tilde{l}'\omega)\tilde{R}(\omega)$$

From (21) and noting that $Q(\omega) = \prod_{j=1}^{N} a'_j\omega$ and $P(\omega)$ do not have common factor, we obtain

$$P(M'\omega) = \tilde{c}_1 P(\omega) \qquad \prod_{j=1}^{N} \tilde{a}'_j\omega = \tilde{c}_2 \prod_{j=1}^{N} \tilde{a}'_j M'\omega$$

Sufficiency of the second part: The sufficiency holds from (3) and lemma 3 of [3].

Necessity of the second part:

First we prove that if there is a nonzero constant term in $P(\omega)$, then $P(\omega)$ must be degree of zero. Let $E = (e_1, e_2, \cdots, e_n)$ be a nonsingular matrix consisting of the eigenvector of $M'$, namely, $M'e_j = \rho_j e_j$. Write $\omega = E\alpha$, then $P(M'\omega) = cP(\omega) = P(M'E\alpha) = P(E(\rho\cdot\alpha)) = cP(E\alpha)$, so that $P((\rho\cdot\alpha)) = cP(\alpha)$. Suppose $P(\alpha) = \sum_j \beta_j\alpha^j$, then we have $c\beta_j = \beta_j\rho^j$ for all $j$. Since $\beta_0 \neq 0$, we have $c = 1$. And from $|\rho_i| > 1$ we have $\beta_j = 0$ for $j \neq 0$.

Because $\sum_j B(x-j)$ is a constant, we know$\sum_j \phi(x-j)$ is also a constant. Thus, from [6, theorem 5.1](or [7, theorem 1.1]) the shifts of $\phi$ can not form a Riesz basis when there isn't constant term in $P(D)$. So $P(D)$ must be a polynomial of degree zero.

Similar to the proof of theorem 2 in [3], one can prove $A$ is unimodular and $R(z)$ does not have any root on $T^n = \{z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid |z_1| = |z_2| = \cdots = |z_n| = 1\}$. Using the same argument of the proof of proposition 9 in [3], it can be proved that all prime factors of $\prod_{j=1}^{s}(z^{a_j} - 1)$ have roots on $T^n$, which implies that $R(\exp(i\omega))$ have no any prime factor of $\prod_{j=1}^{s}(z^{a_j} - 1)$. From this fact and noting $R(z)\prod_{j=1}^{s}(z^{a_j} - 1)$ is $M'$-closed, we know $R(\exp(i\omega))$ is also $M'$-closed. Furthermore, similar to the proof of proposition 6 and 7 in [3], we have learnt that $R(z)$ is a monomial.

## Acknowledgments

We wish to thank the editor and referees for their helpful comments and suggestions.

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