Characterization of Compactly Supported Refinable Splines With Integer Matrix

Yu-jing Guan\textsuperscript{a}∗ and Tian-xiao He\textsuperscript{b}
\textsuperscript{a}School of Mathematics, Jilin University, Changchun, Jilin 130023, PR China;
\textsuperscript{b}Department of Mathematics and Computer Science
Illinois Wesleyan University, Bloomington, IL 61702-2900, USA
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Abstract
Let $M$ be an integer matrix with absolute values of all its eigenvalues being greater than 1. We give a characterization of compactly supported $M$-refinable splines $f$ and the conditions that the shifts of $f$ form a Riesz basis.

1 Introduction and Main Results
Let $M \in \mathbb{Z}^{n \times n}$ be an integer matrix with absolute values of all its eigenvalues being greater than 1. A function $f$ defined on $\mathbb{R}^n$ is $M$-refinable if there exists a finite sequence $\{h_j\}$ such that
\[ f(x) = \sum_{j \in \mathbb{Z}^n} h_j f(Mx - j) \quad (1) \]

In [1], Lawton et al considered the one-dimensional setting of the scaling coefficient $M$ being an integer greater than 1. They gave a characterization of the refinable univariate splines and proved that only the shifts of B-spline with the smallest support form a Riesz basis. In [2] Sun extended the partial result of [1] to $M = ml$ using Box-splines, where $m \in \mathbb{Z}, m > 1$, and $l$ is the

∗∗The author was supported by the 985 project of Jilin University, Email: guanyj@jlu.edu.cn
identity matrix, namely, an $M = mI$-refinable and blockwise polynomial with compact support is a finite linear combination of a box-spline and its translates. In [3], Y.Guan et al. further gave a characterization of $M = mI$-refinable and blockwise polynomial with compact support forming a Riesz basis. More relative results can be found in the survey [4, 5] by Goodman et al. In this paper, we generalize the results of [1, 2, 3] to the setting of a certain class of scaling matrices, namely. We shall derive a characterization of functions (1) when $M$ is a matrix with integer entries.

In the following, the multi-index notational system is adopted. First, throughout the paper, all vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ are column vectors.

Let $\omega \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$ with components $\omega_j \in \mathbb{R}$ and $z_j := \exp(i\omega_j)$ ($j = 1, 2, \ldots, n$), respectively, where $i = \sqrt{-1}$. Denote the transpose of vector $k$ and matrix $M$ by $k'$ and $M'$, respectively. We also write $z = (\exp(i\omega_1), \exp(i\omega_2), \ldots, \exp(i\omega_n)) = (z_1, z_2, \ldots, z_n)$ as $z \equiv \exp(i\omega)$ for convenience when it is clear in the content. For $k \in \mathbb{Z}^n$, we denote $z^k = z^{k'} := \prod_{j=1}^n z_j^{k_j}$. For an integer matrix $M$, we denote $z^M := \exp(i(M\omega))$. Obviously, $z^M = z^{M'} = \exp(ik'M\omega)$.

A trigonometric polynomial $R(\omega)$ is said to be $M$-closed if $R(M\omega)/R(\omega)$ is a trigonometric polynomial too.

Let $s \geq n$ and $A = (a_1, a_2, \ldots, a_s)$ a nonsingular matrix with integer entries and column vectors $a_j \in \mathbb{Z}^n$, $j = 1, 2, \ldots, s$. By means of Fourier transform, we can define box splines $B_A(x)$ of dimension $n$ as follows:

$$B_A(x) = \prod_{j=1}^s \frac{\exp(ia_j'\omega) - 1}{ia_j'\omega} \quad (2)$$

A function $\phi$ is called a blockwise polynomial if there exists a simplex decomposition $\{\Delta_j\}_{j=1}^L$ of $\phi$, such that $\phi$ is a polynomial on every simplex. A standard simplex is defined as $\Delta^0 = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; 0 \leq x_j \leq 1, \sum_{j=1}^n x_j \leq 1\}$, and a simplex $\Delta$ is an affine transform of the standard simplex, $\Delta = A\Delta^0 + c$, where $A$ is nonsingular and $c \in \mathbb{R}^n$. We say that $\{\Delta_j\}_{j=1}^L$ is a simplex decomposition of a bounded set $E$ if $\bigcup_{j=1}^L \Delta_j \supseteq E$, where $\Delta_j$ is a simplex for every $1 \leq j \leq L$, and $\Delta_j \bigcap \Delta_l$ has Lebesgue measure zero when $j \neq l$. For the adjacent $\Delta_j$ and $\Delta_l$, let $E$ be their $n-1$ dimensional common boundary lying on the plane $\pi$. Then we call $\pi$ a singular hyperplane of $f$ if $f(x)|_{\Delta_j}$ and $f(x)|_{\Delta_l}$ are different polynomials. Hence, all the planes passing through the $n-1$ dimensional boundaries of the simplex support of a block polynomial function are also called singular hyperplane of the blockwise polynomial.

Let $s \geq n$, $a_1, a_2, \ldots, a_s \in \mathbb{Z}^n$. We say a matrix $A = (a_1, a_2, \ldots, a_s)$ is unimodular if any matrix generated by any $n$ linearly independent column vectors of the matrix $A$ has determinant value $\pm 1$.

**Theorem 1.1.** Let $n \geq 2$. Suppose $\phi$ is a compact support blockwise polynomial, $D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$, and $M$ is a matrix with integer entries and the absolute values of all its eigenvalues are greater than 1. Then we have the following results:
a). \( \phi \) satisfies equation (1) if and only if it can be written as

\[
\phi(x) = P(D) \left( \sum_j r_j B_A(x - j - (M^k - I)^{-1}l) \right)
\]

and

1. \( B_A(x) \) is a box spline defined on \( A = (a_1, a_2, \ldots, a_s) \), where \( A \) satisfies \( \prod_{j=1}^s (Ma_j)' \omega = c_1 \prod_{a=1}^s a_a' \omega \) for some constant \( c_1 \);
2. \( k \) is a positive integer and satisfies \( (M')^k e_j = \lambda_j e_j \) for the normal vector \( e_j \) of every singular hyperplane of \( \phi \);
3. \( l \in \mathbb{Z}^n \) satisfies \( (M - I)(M^k - I)^{-1}l \in \mathbb{Z}^n \);
4. \( P \) is a polynomial, and satisfies \( P(M' \omega) = c_2 P(\omega) \), where \( P(\omega) \) and \( \prod_{a=1}^s a_a' \omega \) don’t have common factor, and \( c_2 \) is a constant;
5. \( R(z) = R(\exp(i \omega)) = \sum_j r_j z^j \) satisfies that \( R(z) \prod_{a=1}^s (z^{a_j} - 1) \) is \( M' \)-closed trigonometric polynomial;

b). Furthermore, integer shifts of \( \phi \) form Riesz basis if and only if \( P \) is polynomial of zero degree, \( A \) is unimodular, and \( R(z) \) is a monomial.

\[ \square \]

2 The Proof of Main Results

A polynomial \( P \) is called a principal homogeneous polynomial if there exists a natural number \( k \) and \( a_j \in \mathbb{R}^n, 1 \leq j \leq k \), such that \( P(\omega) = \prod_{a=1}^k a_a' \omega \).

In addition, for real \( b_j \) and complex \( a_j \), we call \( \sum_j a_j \exp(ib_j \omega) \) a generalized trigonometric polynomial. Clearly, the following result holds.

Lemma 2.1. The fourier transform of \( \phi(Mx - k) \) is \( |\det(M)|^{-1} \exp(-i \omega' \cdot (M^{-1}k)) \phi((M^{-1})' \omega) \).

Proposition 2.2. Let \( f \) be a blockwise polynomial with compact support satisfying Equation (1). Then there exists an integer \( k \in \mathbb{Z} \) such that \( (M')^k e_j = \lambda_j e_j \), where \( \lambda_j \in \mathbb{R} \), holds for the normal vectors \( e_j \) of all singular hyperplanes of \( f \).

Proof. Let \( E = \{e_j\} \) be a finite set of the normal vectors \( e_j \) of all singular hyperplanes of \( f \). \( \forall e_j \in E \), there exists a hyperplane \( e'_j x - c_j = 0 \) on which \( f \) is singular. And expression (1) implies that both sides of the equation have the same singularities. Hence, there exists an integer \( l \) on the right-hand side such that \( f \) is singular at the hyperplane \( e'_j (Mx - l) - c_j = 0 = (M'e_j)'x - e'_j l - c_j \). Thus, \( M'e_j \) is also the normal vector of a singular hyperplane of \( f \). \( M \) is one-to-one mapping from \( E \) to itself because \( M \) is not singular. Hence, from the finiteness of \( E \) there exist an integer \( k \) such that \( (M')^k e_j = \lambda_j e_j \) for every \( e_j \in E \). Furthermore, since both \( M \) and \( e_j \) are real, \( \lambda_j \) is also real. \[ \square \]
Proposition 2.3. Let \( f \) be an \( M \)–refinable blockwise polynomial with compact support, then its Fourier transform \( \hat{f} \) can be written as \( \hat{f}(\omega) = \sum_{m} \frac{f_{m}(\frac{\omega}{m})}{m_{m}(\omega)} \) where \( p_{n}(\omega) \) are the principal homogeneous polynomials and \( q_{n}(z) \) are the generalized trigonometric polynomials. In addition, there exist \( k \) such that \( p_{n}(\langle (M')^{-k} \omega \rangle) = c_{n}p_{n}(\omega) \) for some constants \( c_{n}, n = 1, 2, \cdots \)

Proof. Let \( \{e_{j}\} \) be the finite set of the normal vectors of singular hyperplanes of \( f \). From proposition 2.2, there exists an integer \( k \) such that \( \langle (M')^{k} e_{j} \rangle = \lambda_{j} e_{j} \) for every \( e_{j} \). Denote by \( \alpha_{1}, \alpha_{2}, \cdots, \alpha_{m} \) the different eigenvalue of \( (M')^{k} \), where \( m \leq n \), and by \( V_{1}, V_{2}, \cdots, V_{m} \) the corresponding eigenspaces of \( \alpha_{1}, \alpha_{2}, \cdots, \alpha_{m} \). For any \( j \), since \( e_{j} \) is the eigenvector of \( (M')^{k} \), there exists \( l_{j} \) such that \( e_{j} \in V_{l_{j}} \). Obviously, \( f \) is compactly supported and its support must be a polyhedral in \( \mathbb{R}^{n} \). So every boundary of the polyhedral must be on a singular plane of \( f \) and set \( \{e_{j}\} \) spans \( \mathbb{R}^{n} \). Furthermore, every \( V_{l} \) has a basis \( \tilde{E}_{l} \) consisting of the elements of \( \{e_{j}\} \). Hence, we can write \( \tilde{E}_{l} = (e_{j,i}) e_{j,i} \in \{e_{l}|e_{l} \in V_{j}\}, i = 1, 2, \cdots, m_{j} \). Therefore, we obtain a basis \( E = (E_{1}, E_{2}, \cdots, E_{m}) \) of \( V = V_{1} + V_{2} + \cdots + V_{m} \), which consists of the elements of \( \{e_{j}\} \). Let \( \tilde{E} = (\tilde{E}_{1}, \tilde{E}_{2}, \cdots, \tilde{E}_{m}) \), here \( \tilde{E}_{l} \tilde{E}_{l} = I, l = 1, 2, \cdots, m \), and \( \tilde{E}_{j}^{l} \tilde{E}_{l} = 0 \) when \( j \neq l \), so \( \tilde{E}^{l} \tilde{E} = I \). For every \( l = 1, 2, \cdots, m \), \( \tilde{E}_{l} \) spans a space denoted by \( \tilde{V}_{l} \) with \( \tilde{V}_{l} \perp V_{j} \) when \( j \neq l \). Obviously, \( V = \tilde{V}_{1} + \tilde{V}_{2} + \cdots + \tilde{V}_{m} \)

For an arbitrary \( 1 \leq j \leq m \), the intersections of \( \tilde{V}_{j} \) and the singular plane of \( f(x) \), whose normal vector belongs to \( \{e_{l}|e_{l} \in V_{j}\} \), form a polyhedral partition of \( V_{j} \). Furthermore, we can establish a simplex partition \( \{\tilde{\Delta}_{s,l}\} \) from the polyhedral decomposition of \( \tilde{V}_{j} \), so that we obtain a new polyhedral partition \( \Delta = \{\bigoplus_{s=1}^{m} \tilde{\Delta}_{s,l}\} \) of \( V \).

\( \forall 1, t_{2}, \cdots, t_{m}, \) we claim that \( f(x) \) is a polynomial in the domain \( \bigoplus_{s=1}^{m} \tilde{\Delta}_{s,l} \).

In fact, since \( f(x) \) is a spline, we only need to prove domain \( \bigoplus_{s=1}^{m} \tilde{\Delta}_{s,l} \) is not divided by any singular hyperplane of \( f(x) \), that is, all the point in \( \bigoplus_{s=1}^{m} \tilde{\Delta}_{s,l} \) are on the same side of any singular hyperplane of \( f(x) \). Let \( e_{j} \) be an arbitrary normal vector in \( \tilde{V}_{j} \), and let the corresponding singular hyperplanes be \( < e_{j}, x > = c_{j,i}, i = 1, 2, \cdots, \). Denote \( x = \sum_{s=1}^{m} \tilde{E}_{s} \alpha_{s} \) and \( y = \sum_{s=1}^{m} \tilde{E}_{s} \beta_{s} \), where \( \tilde{E}_{s} \alpha_{s}, \tilde{E}_{s} \beta_{s} \in \tilde{\Delta}_{s,l}, s = 1, 2, \cdots, m, \forall x, y \in \bigoplus_{s=1}^{m} \tilde{\Delta}_{s,l} \) and \( i = 1, 2, \cdots, \), we have \( < x, e_{j} > = c_{j,i}(< y, e_{j} > - c_{j,i}) = (< \tilde{E}_{l} \alpha_{i}, e_{j} > - c_{j,i})(< \tilde{E}_{l} \beta_{i}, e_{j} > - c_{j,i}) \geq 0 \). The last inequality can be obtained from the simplex partition of \( \tilde{V}_{j} \).

If \( x \in V \), we may write \( x = \tilde{E} \beta \) with some \( \beta \in \mathbb{R}^{n} \). Furthermore we have \( \tilde{\Delta}_{s,l} = \tilde{E} \Delta_{s,l}, \Delta = \tilde{E} \Delta \) and \( f(\tilde{E} \beta) \) is a polynomial on \( \bigotimes_{s=1}^{m} \tilde{\Delta}_{s,l} \). Thus,

\[
\hat{f} = \int_{\mathbb{R}^{n}} f(x) \exp(-ix'\omega)dx \\
= \det(\tilde{E}) \int_{\mathbb{R}^{n}} f(\tilde{E} \beta) \exp(-i\beta'\tilde{E}'\omega))d\beta
\]
Let \( \xi = (\xi'_1, \xi'_2, \cdots, \xi'_m)' = \hat{E}'\omega = (\hat{E}_1, \hat{E}_2, \cdots, \hat{E}_m)'\omega \). Then

\[
\dot{f} = \det(\hat{E}) \int_{\bigotimes_{j=1}^m (\Delta_{i,j})} f(\hat{E}\beta) \exp(-i\beta'\xi)d\beta
\]
\[
= \det(\tilde{E}) \int_{\sum_{i=1}^m \bigotimes_{j=1}^m \Delta_{i,j}} f(\tilde{E}\beta) \exp(-i\beta'\xi)d\beta
\]
\[
= \det(\tilde{E}) \sum_{l_1} \cdots \sum_{l_m} \int_{\bigotimes_{j=1}^m \Delta_{i,j}} f(\tilde{E}\beta) \exp(-i\beta'\xi)d\beta
\]

If \( f \) is a polynomial on \( \bigotimes_{j=1}^m \Delta_{i,j} \), we have

\[
\dot{f} = \det(\hat{E}) \sum_{l_1} \cdots \sum_{l_m} \int_{\Delta_{i,1}} \cdots \int_{\Delta_{i,m}} \sum_{n} a_n(l_1, l_2, \cdots, l_m) \beta^n \exp(-i\beta'\xi)d\beta
\]
\[
= \det(\tilde{E}) \sum_{l_1} \cdots \sum_{l_m} \sum_{n} a_n(l_1, l_2, \cdots, l_m) \prod_{j=1}^m \int_{\Delta_{i,j}} \beta^n_j \exp(-i\beta'j\xi)d\beta_j
\]

After simplifying above sum, from the lemma 1 in [2], there exist principle homogenous polynomials \( p_{k,j}(\xi_j) \) in terms of the variants \( \xi_j \) and generalized trigonometric polynomials \( q_{n,j}(\exp(-i\xi_j)) \) such that

\[
\dot{f} = \sum_n \prod_{j=1}^m \frac{q_{n,j}(\exp(-i\xi_j))}{p_{n,j}(\xi_j)}
\]
\[
= \sum_n \prod_{j=1}^m \frac{q_{n,j}(\exp(-i\hat{E}_j'\omega))}{p_{n,j}(\hat{E}_j'\omega)}
\]
\[
= \sum_n \prod_{j=1}^m \frac{q_{n,j}(\exp(-i\omega))}{p_{n,j}(\hat{E}_j'\omega)},
\]

where \( q_{n}(\exp(-i\omega)) = \prod_{j=1}^m \pi_{n,j}(\exp(-i\hat{E}_j'\omega)) \) are the generalized trigonometric polynomials.

From the above discussion we know \((M'^k)E = E\lambda\), where \( \lambda \) is a diagonal matrix, and \((M')E_j = \alpha_j E_j\). Furthermore, \( M^k \hat{E} = \hat{E}\lambda \) and \( M^k \hat{E}_j = \alpha_j \hat{E}_j \). Let \( p_n(\omega) = \prod_j \pi_{n,j}(\hat{E}_j'\omega) \), then \( p_n((M'^{-k})\omega) = \prod_j \pi_{n,j}(\hat{E}'_j(M'^{-k})\omega) = \prod_j \pi_{n,j} \)
\((\hat{E}_j')\omega) = \prod_j \pi_{n,j}(\alpha_j^{-1}\hat{E}_j'\omega) = c_n \prod_j \pi_{n,j}(\hat{E}_j'\omega) = c_n \pi_{n}(\omega). \) Obviously, \( p_n(\omega) \) is a principal homogeneous polynomial because \( p_{n,j}(\omega) \) are principal homogeneous polynomials.

**Lemma 2.4.** Let \( P_j \) (\( j = 1, 2 \)) be two nonzero polynomials, and let \( T_j \) (\( j = 1, 2 \)) be two nonzero generalized trigonometric polynomials. If \( P_j \) and \( T_j \) (\( j = 1, 2 \)) satisfy \( P_1(\omega)T_1(\omega) = P_2(\omega)T_2(\omega) \), then \( P_1(\omega) = C P_2(\omega) \) and \( T_1(\omega) = C^{-1} T_2(\omega) \) for some complex number \( C \).

**Proof.** Let \( \hat{f}_1 = P_1(\omega)T_1(\omega), \hat{f}_2 = P_2(\omega)T_2(\omega) \) be two generalized functions, where \( T_1(\omega) = \sum_{j=1}^K c_{1,j} \exp(-ia_{j}\omega), \ T_2(\omega) = \sum_{j=1}^L c_{2,j} \exp(-ib_{j}\omega), \) \( \{a_j\}_{j=1}^K \) and \( \{b_j\}_{j=1}^L \) are constants. Then

\[
\hat{f}_1 = \sum_{j=1}^K c_{1,j} \exp(-i\alpha_j \omega), \ \hat{f}_2 = \sum_{j=1}^L c_{2,j} \exp(-i\beta_j \omega)
\]

where \( \alpha_j, \beta_j \) are constants. Since \( \hat{f}_1 = \hat{f}_2 \), we have \( \{a_j\}_{j=1}^K = \{b_j\}_{j=1}^L \) and \( \alpha_j = \beta_j \) for all \( j \). Therefore, \( P_1(\omega) = C P_2(\omega) \) and \( T_1(\omega) = C^{-1} T_2(\omega) \) for some complex number \( C \).
are $K$ different real numbers, and $\{b_j\}_{j=1}^L$ are $L$ different real numbers. Denote $Q_1(i\omega) = P_1(\omega)$, $Q_2(\omega) = P_2(\omega)$, and $D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$. Then $f_1 = Q_1(D) \sum_{j=1}^K c_{1,j} \delta(x - a_j)$ and $f_2 = Q_2(D) \sum_{j=1}^L c_{2,j} \delta(x - b_j)$. Thus, for an arbitrary infinitely differentiable function $\phi$ with compact support, we have $0 = \langle f_1 - f_2, \phi \rangle = \langle f_1 - f_2, \phi \rangle = \langle \sum_{j=1}^K c_{1,j} \delta(x - a_j), Q_1(D)\phi \rangle - \langle \sum_{j=1}^L c_{2,j} \delta(x - b_j), Q_2(D)\phi \rangle$. So $\langle \sum_{j=1}^K c_{1,j} \delta(x - a_j), Q_1(D)\phi \rangle = \langle \sum_{j=1}^L c_{2,j} \delta(x - b_j), Q_2(D)\phi \rangle$. Since $\phi$ is arbitrary, we obtain $L = K, c_{1,j} = C^{-1}c_{2,j} (j = 1, 2, \ldots, K), Q_1 = CQ_2$ for some constant $C$, so the lemma is proved. \[ \square \]

**Lemma 2.5.** Let $M \in \mathbb{Z}^{n \times n}$ be a matrix of integer entries, and let all of its eigenvalues be real and larger than 1. Suppose $T(\omega)$ is a nonzero generalized trigonometric polynomial, and $H(\omega)$ is a nonzero trigonometric polynomial defined on $\mathbb{R}^n$. If

$$T(M\omega) = H(\omega)T(\omega)$$

then $\exp(-il(M - I)^{-1}\omega)T(\omega)$ is a trigonometric polynomial for some $l \in \mathbb{Z}^n$.

**Proof.** One can write

$$T(\omega) = \sum_j \exp(ix'_j\omega)T_j(\omega) = \sum_k \exp(iy'_k\omega)Q_k(\omega)$$

where $T_j(\omega)$ is a trigonometric polynomial, $x_j - x'_j \notin \mathbb{Z}^n$ for $j \neq \tilde{j}$, and $Q_k(M\omega)$ is a trigonometric polynomial with $M'(y_k - y'_k) \notin \mathbb{Z}^n$ when $k \neq \tilde{k}$. So from (4) and (5) we have

$$\exp(iy'_k M\omega)Q_k(M\omega) = \sum_j \exp(ix'_j\omega)H(\omega)T_j(\omega)$$

For a given $k$, suppose $M'(y_k - x_j) \in \mathbb{Z}^n$, then for all $\tilde{j} \neq j$ we have $M'(y_k - x'_j) \notin \mathbb{Z}^n$ because $x_j - x'_j \notin \mathbb{Z}^n$. Similarly, there is only one $y_k$ satisfying $M'(y_k - x_j) \in \mathbb{Z}^n$ for all $x_j$. So the numbers of the elements in sets $\{x_j\}$ and $\{y_k\}$ are equal. From $H \neq 0$ and (6), we have

$$\exp(iy'_k M\omega)Q_k(M\omega) = \exp(ix'_j\omega)H(\omega)T_j(\omega)$$

In addition, from (5) we have

$$T(\omega) = \sum_j \exp(ix'_j\omega)T_j(\omega),$$

where $\{x_j\}$ satisfies $M'(x_j - x'_j) \notin \mathbb{Z}^n$ when $j \neq \tilde{j}$. Hence for all $x_j$, $\exists x_j$ and $s \in \mathbb{Z}^n$ satisfy $M'x_j = s + x'_j$ and

$$\exp(ix'_j M\omega)T_j(M\omega) = \exp(ix'_j\omega)H(\omega)T_j(\omega).$$
Define map $Mx_j = x_j$, where $x_j$ is chosen as above. Then $M$ is a well-defined one-to-one map on $\{x_j\}$. We also define $X_s = \{M^kx_j; k = 1, 2, \cdots\}$ for every $x_s$. Then $X_s = X_\emptyset$ or $X_s \cap X_\emptyset = \emptyset$. Thus we can choose finite numbers of $X_l$ such that $\{x_j\} = \bigcup_l X_l$ and $X_l \cap X_l = \emptyset$. Therefore the lemma is true if we can prove that $X_l$ is a singleton for every $l$, and there exists only one $X_l$ in the decomposition of $\{x_j\}$.

First we prove that there is only one element in $X_l$ by using the method of contradiction. Assume $X_l = \{x_1, x_2, \cdots, x_k\}$, $k \geq 2$. Then for $1 \leq s \leq k$, \exists \tau_s \in \mathbb{Z}^n$ such that

$$T_s(M\omega) = \exp(i\tau_s\omega)H(\omega)T_{s+1}(\omega) \quad (10)$$

Let $T_1(\omega) = T_{k+1}(\omega)$. Then

$$T_s(M^k\omega) = \exp(i\tau_s\omega)T_s(\omega)\prod_{j=0}^{k-1} H(M^j\omega) \quad (11)$$

where $\tau_s \in \mathbb{Z}^n$. Denote $\tilde{H}(\omega) = \prod_{j=0}^{k-1} H(M^j\omega)$ and $\tilde{M} = M^k$. We have

$$T_\ast(M\omega) = \exp(i\tau_\ast\omega)T_\ast(\omega)\tilde{H}(\omega) \quad (12)$$

Let $e_1, e_2, \cdots, e_n$ be the linearly independent eigenvectors of $M^{-1}$. And the corresponding eigenvalues are denoted by $\rho_1, \rho_2, \cdots, \rho_n$ that satisfy $1 > |\rho_1| = |\rho_2| = \cdots = |\rho_k| > |\rho_{k+1}| \geq \cdots \geq |\rho_n| > 0$. Thus, the claim is obtained from the fact of that $M$ is nonsingular, and the absolute value of all its eigenvalues are greater than 1. Hence there exists a invertible transform $\omega = \sum_j o_j e_j = E\alpha$, where $E = (e_1, e_2, \cdots, e_n)$. Let the Taylor expansion of $T_\ast(\omega)$ with the remainder be written as

$$T_\ast(\omega) = T_\ast(E\omega) = p_1(\alpha) + p_2(\alpha) + p_3(\alpha), \quad (13)$$

where $p_1(\alpha) + p_2(\alpha) \neq 0$ is homogeneous polynomial with degree $K$, in which $p_1(\alpha)$ will be described later, and $p_2(\alpha)$ is the difference of the Taylor expansion of $T_s(\omega)$ and $p_1(\alpha)$, and the remainder $|p_3(\alpha)| \leq C|\alpha|^{K+1}$. Assume in $p_1(\alpha) + p_2(\alpha)$ the degrees of $\alpha_i, \alpha_2, \cdots, \alpha_i$, are nonzero, and denote $\iota_1 = \min\{i_1, i_2, \cdots, i_j\}$. Hence, for $p_1(\alpha) + p_2(\alpha)$, the degrees of $\alpha_1, \alpha_2, \cdots, \alpha_{\iota-1}$ are zero, where $p_1(\alpha)$ is a polynomial in terms of the variants, whose corresponding eigenvalue’s absolute value is $|\rho_1|(|\alpha_i \rightarrow e_i \rightarrow \rho_i$ or $\rho_1|(|\alpha_i \rightarrow e_i \rightarrow \rho_i$. For convenience, assume $\iota_1 = 1$, then $p_1(\alpha)$ is a homogeneous polynomial with degree $K$ of $\alpha_1, \alpha_2, \cdots, \alpha_\iota$ and $p_2(\alpha)$ is also a homogeneous polynomial with degree $K$, but its every monomial has nonzero degree of $\alpha_{\iota+1}, \cdots, \alpha_n$. Let $\rho \in \mathbb{C}^n$, write $(\rho \cdot \alpha) = (\rho_1 \alpha_1, \rho_2 \alpha_2, \cdots, \rho_n \alpha_n)^T$. Therefore,

$$T_s(M^{-p}(\omega)) = T_s(E(\rho \cdot \alpha)) = p_1^{Kp}p_1(\alpha) + p_2(E(\rho \cdot \alpha)) + O(|\rho_1|^{p(K+1)}),$$

which implies

$$\lim_{p \to -\infty} \frac{T_s(M^{-p}(\omega))}{p_1^{Kp}} = \lim_{p \to -\infty} \frac{T_s(E(\rho \cdot \alpha))}{p_1^{Kp}} = p_1(\alpha). \quad (14)$$
Indeed the above formula holds due to \( p_2(E(p^0 \cdot \alpha))/p_1^{pK} \to 0 \) as \( p \to \infty \), which can be proved as follows: For every monomial of \( p_2 \), the degree of \( \alpha_{t+1}, \ldots, \alpha_n \) is non-zero. Write \( p_2(E(p^0 \cdot \alpha)) = \sum_j \beta_j (p^0 \cdot \alpha)^j \), then \( \left| \frac{p_2(E(p^0 \cdot \alpha))}{p_1^{pK}} \right| = \left| \sum_j \beta_j \left( \frac{E_p}{p_1^p} \right)^j \right| \leq \sum_j |\beta_j| \left| \frac{p_{t+1}}{p_1^p} \right|^p \to 0 \) as \( p \to \infty \).

For \( 1 \leq s \leq k \), from (12) we have

\[
T_s(\omega) = T_s(E\alpha) = \exp(i\tau_s^s E(\rho \cdot \alpha))\hat{H}(\hat{M}^{-1}E\alpha)T_s(\hat{M}^{-1}E\alpha)
\]

\[
= \exp(i\tau_s^s E(\sum_{j=1}^p \rho^j \cdot \alpha))T_s(\hat{M}^{-p}E\alpha) \prod_{j=1}^p \hat{H}(\hat{M}^{-j}E\alpha)
\]

Since \( \hat{H}(\omega) \) is a trigonometric polynomial, we have

\[
T_s(\omega) \exp(i\tau_s^s E(\sum_{j=1}^p \rho^j \cdot \alpha))T_1(\hat{M}^{-p}E\alpha)
\]

\[
= T_1(\omega) \exp(i\tau_s^s E(\sum_{j=1}^p \rho^j \cdot \alpha))T_s(\hat{M}^{-p}E\alpha)
\]

a.e. Divide the two sides of the above formula by \( \rho_1^{Kp} \), and let \( p \to \infty \). From (14) there exist \( \beta_1, \beta_s \in \mathbb{R}^n \) as well as polynomials \( P_2 \) and \( P_1 \) such that

\[
\exp(i\beta_s^i\omega)P_s(E^{-1}\omega)\hat{H}(\hat{M}^{-1}E\omega)T_1(\omega) = \exp(i\beta_1^i\omega)P_1(E^{-1}\omega)T_s(\omega)
\]

(15)

for \( 2 \leq s \leq k \). From lemma 3 in [2], there exist \( j_s \in \mathbb{Z}^n \) and a constant \( c_s \) such that \( P_s(\omega) = C_sP_1(\omega) \) and \( T_s(\omega) = C_s \exp(ij_s^i\omega)T_1(\omega) \) for \( 1 \leq s \leq k \). Without losing the generality, let \( j_s = 0 \) by selecting appropriate \( x_j \). Thus,

\[
C_s \exp(ix_s^i\omega)T_1(\omega) = \exp(ix_s^i\omega)T_s(\omega)
\]

\[
= \exp(ix_s^i+1M^{-1}\omega)\hat{H}(\hat{M}^{-1}\omega)T_1(\hat{M}^{-1}\omega)
\]

\[
= \exp(ix_s^i+1M^{-1}\omega)\hat{H}(\hat{M}^{-1}\omega)T_1(\hat{M}^{-1}\omega)c_{s+1}.
\]

From (9) and the definition of \( M \), there exists a fixed \( j \in \mathbb{Z}^n \) such that

\[-(M^{-1})^j x_{s+1} + x_s = (M^{-1})^j \] for all \( 1 \leq s \leq k \). Therefore, using \( x_k+1 = x_1 \) yields

\[-x_{s+1} + M'x_s = j \] for all \( 1 \leq s \leq k \). By a direct calculating, \( x_s = (M' - I)^{-1}j \) satisfies the above request and it is only one solution when

\[
\begin{pmatrix}
M' & -I \\
\vdots & \ddots & \ddots \\
-I & \ddots & -I \\
-1 & \cdots & M'
\end{pmatrix}
\]

is nonsingular. Furthermore, \( M'(x_s - x_{\tilde{s}}) \in \mathbb{Z}^n \) for all \( s \) and \( \tilde{s} \). This is inconsistent with \( M'(x_s - x_{\tilde{s}}) \notin \mathbb{Z}^n \) as \( s \neq \tilde{s} \), so the assumption is wrong, that is, there is only one element for every \( X_i \).
Using the above argument and from (9) we obtain
\[ \exp(ix_j^\prime \omega)T_j(\omega) = \exp(ix_j^\prime M^{-1} \omega)H(M^{-1} \omega)T_j(M^{-1} \omega) \] (16)
similarly as (15), we also have
\[ \exp(i\beta'_j \omega)P_s(E_\omega)T_j(\omega) = \exp(i\beta'_j \omega)P_s(E_\omega)T_j(\omega). \] (17)
Hence, from lemma 3 in [2], \( T_j(\omega) = C_j \exp(ik_j \omega)T_1(\omega), k_j \in \mathbb{Z}^n \). Choosing an appropriate \( x_j \) one may have \( k_j = 0 \). Then, for all \( j \), from (16) we have \( \exp(ix_j^\prime \omega)T_1(\omega) = \exp(ix_j^\prime M^{-1} \omega)H(M^{-1} \omega)T_1(M^{-1} \omega) \), which implies \( x_j - x_1 \in \mathbb{Z}^n \), this contradiction completes the proof of the lemma.

\[ \square \]

**Proposition 2.6.** Let \( M \) be a matrix of integer entries, and all of its eigenvalue be real and larger than 1. Then for a given nonzero trigonometric polynomial \( H(z) \), if \( q_1(z^M) = c_1 H(z)q_1(z) \) and \( q_2(z^M) = c_2 H(z)q_2(z) \) then \( c_1 = c_2 \) and there exist a const \( c \) such that \( q_1(z) = c q_2(z) \), where \( c_1 \) and \( c_2 \) are two nonzero real constants, and \( q_1(z) \) and \( q_2(z) \) are nonzero generalized trigonometric polynomials.

**Proof.** From the condition, \( \forall k \in \mathbb{N} \)
\[ q_1(z) = c_1^k q_1(z^{M^{-k}}) \prod_{j=1}^{k} H(z^{M^{-j}}) \]
and
\[ q_2(z) = c_2^k q_2(z^{M^{-k}}) \prod_{j=1}^{k} H(z^{M^{-j}}). \]
Thus, \( q_1(z)c_2^k q_2(z^{M^{-k}}) = c_1^k q_1(z^{M^{-k}})q_2(z) \).

Let \( \omega = E\alpha \), where \( E = (e_1, e_2, \cdots, e_n) \) is a nonsingular matrix and \( Me_j = \lambda_j e_j, j = 1, 2, \cdots, n, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \). Then
\[ z^{M^{-k}} = \exp(i M^{-k} \omega) = \exp(i M^{-k} E\alpha) = \exp(i E(\alpha/\lambda^k)). \]
Using Taylor series of \( q_1, q_2 \) we have for \( m \in \mathbb{N} \)
\[ q_1(z) \sum_j a_j \alpha_j^j (c_2 \lambda^{-j})^{2m} = q_2(z) \sum_j b_j \alpha_j^j (c_1 \lambda^{-j})^{2m} \]
Divide the both sides of the above equation by \( \lambda_{\text{max}}^{2m} \), where \( \lambda_{\text{max}} = \max_j \{ |c_2 \lambda^{-j}|, |c_1 \lambda^{-j}| \} \), and let \( m \to \infty \), then we obtain polynomials \( p_1(\alpha), p_2(\alpha) \) and the equation
\[ q_1(z)p_1(\alpha) = q_2(z)p_1(\alpha). \]
From lemma 2.4 we have \( q_1(z) = c q_2(z) \). Furthermore, we have \( c_1 = c_2 \), which completes the proof.

\[ \square \]
Lemma 2.7. Let $T$ be a trigonometric polynomial and $T(\omega) = 0$ on plane $a_j^j \omega = 0$, $j = 1, 2, \ldots, N$. Then there exist trigonometric polynomial $R(\omega)$ and $\alpha_j \in \mathbb{R}$, $j = 1, 2, \ldots, N$ such that $T(\omega) = R(\omega) \prod_{j=1}^{N} (\exp(i\alpha_j a_j^j \omega) - 1)$, where $\alpha_j a_j \in \mathbb{Z}^n$, $j = 1, 2, \ldots, N$.

Proof. This lemma can be proved by a similar argument of the proof of theorem 1 in [2].

In the following we give the proof of Theorem 1.1.

Proof: Sufficiency of the first part: from the fourier transform of (3), it is easy to get that $\hat{\phi}(M')/\hat{\phi}(\omega)$ is a trigonometric polynomial because $\phi$ satisfies equation (1).

Necessity of the first part: By proposition 2.2, there exists $k$ such that $(M')^k e_j = \lambda_j e_j$ for the normal vector $e_j$ of an arbitrary singular hyper-plane. Let $\hat{M} = M^k$, then $f$ is $\hat{M}$-refinable. By proposition 2.3

$$\hat{\phi}(\omega) = \sum_{j \in \Lambda} \frac{T_j(\omega)}{P_j(\omega)} = \sum_{s \geq s_0} \sum_{\deg P_j = s} \frac{T_j(\omega)}{P_j(\omega)}$$

where $s_0 \geq 0$ and $\sum_{\deg P_j = s_0} \frac{T_j(\omega)}{P_j(\omega)} \neq 0$, and $\{P_j^{-1}(\omega)\}_{\deg P_j = s}$ is linearly independent ($s = s_0, s_0 + 1, s_0 + 2, \ldots$). In addition,

$$\sum_{s > s_0} \sum_{\deg P_j = s} \frac{T_j(r\omega)}{P_j(r\omega)} r^{s_0} \to 0$$

and

$$\sum_{s > s_0} \sum_{\deg P_j = s} \frac{T_j(r(M')^{-1}\omega)}{P_j(r(M')^{-1}\omega)} r^{s_0} \to 0$$

as $r \to +\infty$ a.e. for all $\omega$ on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

From (1), there exists trigonometric polynomial $H(\omega)$ such that $\hat{\phi}(\omega) = H(M')^{-1}\omega)\hat{\phi}(M')^{-1}\omega)$, and from proposition 2.3 it satisfies

$$\sum_{\deg P_j = s_0} \left( \frac{T_j(r\omega)}{P_j(\omega)} - \frac{H(r(M')^{-1}\omega)T_j(r(M')^{-1}\omega)}{P_j((M')^{-1}\omega)} \right)$$

$$= \sum_{\deg P_j = s_0} \left( \frac{T_j(r\omega) - \lambda_j H(r(M')^{-1}\omega)T_j(r(M')^{-1}\omega)}{P_j(\omega)} \right) \to 0$$

as $r \to +\infty$ a.e. $\omega \in \mathbb{S}^{n-1}$. Write

$$T_j(\omega) - \lambda_j H((M')^{-1}\omega)T_j((M')^{-1}\omega) = \sum_k c_{j,k} \exp(iy_{j,k})\omega$$

and denote

$$D_k(\omega) = \sum_j \frac{c_{j,k}}{P_j(\omega)}$$
Thus, (3) holds. And from (1), there exists a trigonometric polynomial \( H(\omega) \), such that
\[
\sum_{\deg P_j = s_0} \left( \frac{T_j(r\omega)}{P_j(\omega)} - \frac{H(r(M')^{-1}\omega)T_j(r(M')^{-1}\omega)}{P_j((M')^{-1}\omega)} \right) = 0
\]

Recursively, for degree of \( P_j = s_0 + 1, s_0 + 2, \ldots, T_j(\omega) = \lambda_j H((M')^{-1}\omega)T_j((M')^{-1}\omega) \), so that for all \( T_j \) we have \( T_j(\omega) = \lambda_j H((M')^{-1}\omega)T_j((M')^{-1}\omega) \). By using lemma 2.5 and proposition 2.6, there exists a trigonometric polynomial \( \hat{T}(\omega) \) such that
\[
T_j(\omega) = c_j \exp(il((M')^{-1}I)^{-1}\omega)\hat{T}(\omega) = c_j \exp(il((M')^{-1}I)^{-1}\omega)\hat{T}(\omega),
\]
which implies
\[
\dot{\phi}(\omega) = \sum_{s \geq s_0} \sum_{\deg P_j = s} \frac{c_j}{P_j(\omega)} \exp(il((M')^{-1}I)^{-1}\omega)\hat{T}(\omega).
\]

Let \( P(\omega)/Q(\omega) = \sum_{s \geq s_0} \sum_{\deg P_j = s} c_j/s_j(\omega)\), where \( Q(\omega) \) is a principle homogeneous polynomial, and \( P(\omega) \) and \( Q(\omega) \) do not have common factors. Hence
\[
Q(\omega)\dot{\phi}(\omega) = \exp(il((M')^{-1}I)^{-1}\omega)\hat{T}(\omega)P(\omega)
\]

Let \( Q(\omega) = \prod_{j=1}^{N} a_j'\omega, 0 \neq a_j \in \mathbb{R}^n \). Because \( |\exp(il((M')^{-1}I)^{-1}\omega)| = 1 \), and there is no common factors for \( P \) and \( Q \), we obtain \( \hat{T}(\omega) = 0 \) on the hyperplanes of \( a_j'\omega = 0, j = 1, 2, \ldots, N \). From lemma 2.7 we know that there exists a trigonometric polynomial \( R(\omega) \) and \( \tilde{a}_j = \alpha_j a_j \in \mathbb{Z}^n \) such that \( \hat{T}(\omega) = R(\omega)\prod_{j=1}^{N} (\exp(i\tilde{a}_j\omega) - 1) \). Hence we have
\[
\dot{\phi}(\omega) = \exp(il((M')^{-1}I)^{-1}\omega)P(\omega)R(\omega)\prod_{j=1}^{N} (\exp(i\tilde{a}_j\omega) - 1)/i\tilde{a}_j\omega
\]

Thus, (3) holds. And from (1), there exists a trigonometric polynomial \( H(\omega) \), such that
\[
H(\omega) = \frac{\dot{\phi}((M')^{-1}\omega)}{\phi(\omega)}
\]
\[
= \frac{P((M')^{-1}\omega)R(\exp(i(M')\omega))}{P(\omega)R(\exp(i\omega))} \exp(i((M')^{-1}I)^{-1}(M' - I)\omega)
\]
\[
\times \prod_{j=1}^{N} \left( \frac{\exp(i\tilde{a}_j(M')\omega) - 1}{\exp(i\tilde{a}_j\omega) - 1} \frac{a_j'\omega}{a_j'(M')\omega} \right)
\]

Let \( \tilde{l} = (M - I)((M')^{-1}I)^{-1}l, \tilde{R}(\omega) = R(\exp(i\omega))\prod_{j=1}^{N} (\exp(i\tilde{a}_j\omega) - 1) \). By using lemma 3 of [2], for \( \tilde{l} \in \mathbb{Z}^n \)
\[
P(M'\omega)\prod_{j=1}^{N} a_j'\omega = CP(\omega)\prod_{j=1}^{N} a_j'M'\omega
\]
\[
(21)
\]

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\[ \tilde{R}(M'\omega) = C^{-1} H(\omega) \exp(i\tilde{\ell} \omega) \tilde{R}(\omega) \]

From (21) and noting that \( Q(\omega) = \prod_{j=1}^{N} a'_j \omega \) and \( P(\omega) \) do not have common factor, we obtain

\[ P(M'\omega) = \tilde{c}_1 P(\omega) \prod_{j=1}^{N} a'_j \omega = \tilde{c}_2 \prod_{j=1}^{N} \tilde{a}'_j M'\omega \]

Sufficiency of the second part: The sufficiency holds from (3) and lemma 3 of [3].

Necessity of the second part: First we prove that if there is a nonzero constant term in \( P(\omega) \), then \( P(\omega) \) must be degree of zero. Let \( E = (e_1, e_2, \cdots, e_n) \) be a nonsingular matrix consisting of the eigenvector of \( M' \), namely, \( M' e_j = \rho_j e_j \). Write \( \omega = E\alpha \), then

\[ P(M'\omega) = c P(\omega) = P(M'E\alpha) = P(E(\rho \cdot \alpha)) = c P(E\alpha) \]

Suppose \( P(\alpha) = \sum_j \beta_j \alpha^j \), then we have \( c \beta_j = \beta_j \rho^j \) for all \( j \). Since \( \beta_0 \neq 0 \), we have \( c = 1 \). And from \( |\rho_1| > 1 \) we have \( \beta_j = 0 \) for \( j \neq 0 \).

Because \( \sum_j B(x - j) \) is a constant, we know \( \sum_j \phi(x - j) \) is also a constant. Thus, from [6, theorem 5.1](or [7, theorem 1.1]) the shifts of \( \phi \) can not form a Riesz basis when there isn’t constant term in \( P(D) \). So \( P(D) \) must be a polynomial of degree zero.

Similar to the proof of theorem 2 in [3], one can prove \( A \) is unimodular and \( R(z) \) does not have any root on \( T^n = \{ z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid |z_1| = |z_2| = \cdots = |z_n| = 1 \} \). Using the same argument of the proof of proposition 9 in [3], it can be proved that all prime factors of \( \prod_{j=1}^{s} (z^{a_j} - 1) \) have roots on \( T^n \), which implies that \( R(\exp(i\omega)) \) have no any prime factor of \( \prod_{j=1}^{s} (z^{a_j} - 1) \). From this fact and noting \( R(z) \prod_{j=1}^{s} (z^{a_j} - 1) \) is \( M' \)-closed, we know \( R(\exp(i\omega)) \) is also \( M' \)-closed. Furthermore, similar to the proof of proposition 6 and 7 in [3], we have learnt that \( R(z) \) is a monomial.

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References


