ON THE CONSTRUCTION OF NUMBER SEQUENCE IDENTITIES

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Abstract. to construct a class of identities for number sequences generated by linear recurrence relations. An alternative method based on the generating functions of the sequences is given. The equivalence between two methods for linear recurring sequences are also shown. However, the second method is not limited to the linear recurring sequences, which can be used for a wide class of sequences possessing rational generating functions. As examples, Many new and known identities of Stirling numbers of the second kind, Pell numbers, Jacobsthal numbers, etc., are constructed by using our approach. Finally, we discuss the hyperbolic expression of the identities of linear recurring sequences.

1. Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}_{n\geq 0}$ is called sequence of order r if it satisfies a linear recurrence relation of order r

$$a_n = \sum_{j=1}^{r} p_j a_{n-j}, \quad n \geq r, \tag{1.1}$$

for some constants p_j $(j = 1, 2, \ldots, r)$, $p_r \neq 0$, and initial conditions a_j $(j = 0, 1, \ldots, r-1)$. Linear recurrence relations with constant coefficients are important in subjects including pseudo-random number generation, circuit design, and cryptography, and they have been studied extensively. To construct an explicit formula of the general term of a number sequence of order r, one may use generating function, characteristic equation, or a matrix method (See Comtet [6], Hsu [12], Strang [16], Wilf [17], etc.). In [10], He and Shiue presented a method for the sequences of order 2 using the reduction of order, which can be considered as a class of how to make difficult an easy thing. In next section, the method shown in [10] will be modified to give a unified approach to construct a class of identities of linear recurring sequences with any orders. An alternative method will be given in Section 3 by using the generating functions of the recursive sequences discussed in Section 2. The equivalence between these two methods for linear recurring sequences will be shown. However, the second method can be applied for all the sequences with rational generating functions. Inspired by Askey's and Ismail's works shown in [1], [4], and [13], respectively, we discuss the hyperbolic expression of the identities constructed by using our approach, which and another extension will be presented in Section 4.

This work is partially supported by the National Science Council in Taiwan under the grant number 100-2115-M-001-013. The author Wun-Seng Chou would like to thank the Department of Mathematics, Illinois Wesleyan University, for its hospitality.

2. Identities of linear recurring sequences

We now modify the method applied in [10] and extend it to the higher order setting. More precisely, we will give a unify approach to derive identities of linear recurring sequences of arbitrary order r. The key idea can be described in the following theorem.

Theorem 2.1. *Let sequence $\{a_n\}_{n\geq 0}$ be defined by the linear recurrence relation (1.1) of order r, and let its characteristic polynomial $P_r(t) = t^r - \sum_{j=1}^{r} p_j t^{r-j}$ have r roots α_j ($j = 1, 2, \ldots, r$), where the root set may be multiset. Denote $a_n^{(j)} := a_n^{(j-1)} - \alpha_{j-1} a_{n-1}^{(j-1)}$ ($2 \leq j \leq r$) and $a_n^{(1)} := a_n$. Then*

$$a_n^{(r)} = \alpha_r^{n-r+1} a_{r-1}^{(r)}, \tag{2.1}$$

where

$$\begin{aligned} a_n^{(r)} \quad &= a_n - a_{n-1}\sum_{i=1}^{r-1}\alpha_i + a_{n-2}\sum_{1\leq i<j\leq r-1}\alpha_i\alpha_j \\ &-a_{n-3}\sum_{1\leq i<j<k\leq r-1}\alpha_i\alpha_j\alpha_k + \cdots + (-1)^{r-1}a_{n-r+1}\Pi_{i=1}^{r-1}\alpha_i \end{aligned} \tag{2.2}$$

for $n \geq r-1$.

Remark 2.1 $a_n^{(r)}$ shown in (2.2) can be written as

$$a_n^{(r)} = \sum_{i=0}^{r-1}(-1)^n a_{n-i}\sum_{1\leq k_1<\cdots<k_i\leq r-1}\alpha_{k_1}\cdots\alpha_{k_i}.$$

Proof. Denote $a_n^{(2)} := a_n - \alpha_1 a_{n-1}$. Then the recurrence relation (1.1) can be reduced to

$$\begin{aligned} a_n^{(2)} &= a_{n-1}^{(2)}\sum_{k=2}^{r}\alpha_k - a_{n-2}^{(2)}\sum_{2\leq i<j\leq r}\alpha_i\alpha_j + a_{n-3}^{(2)}\sum_{2\leq i<j<k\leq r}\alpha_i\alpha_j\alpha_k - \cdots \\ &+(-1)^r a_{n-r+1}^{(2)}\Pi_{k=2}^{r}\alpha_k, \end{aligned} \tag{2.3}$$

a linear recurrence relation of order $r-1$ for sequence $\{a_n^{(2)}\}_{n\geq 0}$. Similarly, we denote $a_n^{(3)} := a_n^{(2)} - \alpha_2 a_n^{(2)}$. Hence, from (2.3), we obtain

$$\begin{aligned} a_n^{(3)} &= a_{n-1}^{(3)}\sum_{k=3}^{r}\alpha_k - a_{n-2}^{(3)}\sum_{3\leq i<j\leq r}\alpha_i\alpha_j + a_{n-3}^{(3)}\sum_{3\leq i<j<k\leq r}\alpha_i\alpha_j\alpha_k - \cdots \\ &+(-1)^{r-1} a_{n-r+2}^{(3)}\Pi_{k=3}^{r}\alpha_k. \end{aligned}$$

The above expression is a linear recurrence relation of order $r-2$ for sequence $\{a_n^{(3)}\}_{n\geq 0}$. Repeating this process and denoting $a_n^{(r)} := a_n^{(r-1)} - \alpha_{r-1} a_{n-1}^{(r-1)}$, we finally obtain

$$a_n^{(r)} = \alpha_r a_{n-1}^{(r)}, \tag{2.4}$$

which implies (2.1). In (2.1), for $n \geq r-1$,

$$\begin{aligned} a_n^{(r)} &= a_n^{(r-1)} - \alpha_{r-1} a_{n-1}^{(r-1)} \\ &= a_n^{(r-2)} - (\alpha_{r-1} + \alpha_{r-2}) a_{n-1}^{(r-2)} + \alpha_{r-1}\alpha_{r-2} a_{n-2}^{(r-2)} \\ &= a_n^{(r-3)} - a_{n-1}^{(r-3)} \sum_{i=r-3}^{r-1} \alpha_i + a_{n-2}^{(r-3)} \sum_{r-3\leq i<j\leq r-1} \alpha_i\alpha_j - a_{n-3}^{(r-3)} \alpha_{r-3}\alpha_{r-2}\alpha_{r-1}, \end{aligned}$$

which yields (2.2) by using mathematical induction. □

As an example, for $r=2$, if the characteristic polynomial $P_2(t) = t^2 - p_1 t - p_2$ of (1.1) has roots α_1 and α_2, then we denote $a_n^{(2)} := a_n - \alpha_1 a_{n-1}$ and obtain

$$a_n^{(2)} := a_n - \alpha_1 a_{n-1} = \alpha_2(a_{n-1} - \alpha_1 a_{n-2}) = \alpha_2 a_{n-1}^{(2)} = \alpha_2^{n-1} a_1^{(n)}.$$

Similarly, for $r=3$, we denote the roots of the characteristic polynomial $P_3(t) = t^3 - p_1 t^2 - p_2 t - p_3$ of (1.1) by α_j ($j=1,2,3$). Then,

$$\begin{aligned} & a_n^{(2)} := a_n - \alpha_1 a_{n-1} = (\alpha_2 + \alpha_3) a_{n-1} - \alpha_1(\alpha_2+\alpha_3) a_{n-2} - \alpha_2\alpha_3(a_{n-2} - \alpha_1 a_{n-3}) \\ = \; & (\alpha_2+\alpha_3) a_{n-1}^{(2)} - \alpha_2\alpha_3 a_{n-2}^{(2)}, \end{aligned}$$

which implies

$$a_n^{(3)} := a_n^{(2)} - \alpha_2 a_{n-1}^{(2)} = \alpha_3(a_{n-1}^{(2)} - \alpha_2 a_{n-2}^{(2)}) = \alpha_3 a_{n-1}^{(3)} = \alpha_3^{n-2} a_2^{(3)}.$$

Remark 2.2 If $\alpha_r = 1$, then (2.1) becomes

$$a_n^{(r)} = a_{r-1}^{(r)}, \tag{2.5}$$

where $a_n^{(r)}$ is shown in (2.2). In particular, for $r=2$, we have

$$a_n^{(2)} = a_1^{(2)},$$

or equivalently,

$$a_n = \alpha_1 a_{n-1} + a_1 - \alpha_1 a_0.$$

Thus, we have shown that the last non-homogenous recurrence relation of order 1 is equivalent to the homogeneous recurrence relation of order 2, $a_n = (\alpha_1+1)a_{n-1} - \alpha_1 a_{n-2}$, for the same sequence $\{a_n\}_{n\geq 0}$. Similarly, we have the equivalence of the homogenous recurrence relation of order 3, $a_n = (p+1)a_{n-1} - (p-q)a_{n-2} + q a_{n-3}$, and the non-homogenous recurrence relation of order 2, $a_n = p a_{n-1} + q a_{n-2} + d$ for uniquely determined constant $d = a_2 - p a_1 - q a_0$.

We now consider three special cases $r = 2$, 3, and 4 for some particular cases of Theorem 2.1.

Corollary 2.2. *Let $\{a_n\}_{n\geq 0}$ be a sequence satisfying the linear recurrence relation of order 2:*

$$a_n = p_1 a_{n-1} + p_2 a_{n-2}, \quad n \geq 2,$$

with initial conditions a_0 and a_1, and let the characteristic polynomial $P_2(t) = t^2 - p_1 t - p_2$ have roots α and β. Then the sequence $\{a_n\}_{n\geq 0}$ satisfies the identity

$$a_n^{(2)} = \beta^{n-1} a_1^{(2)},$$

where $a_n^{(2)} = a_n - \alpha a_{n-1}$ *for* $n \geq 1$

As an example, we consider *Pell number* sequence $\{P_n\}_{n\geq 0}$ generated by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}$$

with initial conditions $P_0 = 0$ and $P_1 = 1$. The roots of the characteristic polynomial $t^2 - 2t - 1$ are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Hence, Corollary 2.2 gives identity for $n \geq 1$:

$$P_n - (1+\sqrt{2})P_{n-1} = (1-\sqrt{2})^{n-1},$$

or equivalently,

$$(1-\sqrt{2})P_n + P_{n-1} = (1-\sqrt{2})^n.$$

Similarly, we have

$$(1+\sqrt{2})P_n + P_{n-1} = (1+\sqrt{2})^n$$

for $n \geq 1$.

Jacobsthal number sequence $\{J_n\}_{n\geq 0}$ is generated by

$$J_n = J_{n-1} + 2J_{n-2}$$

with initial conditions $J_0 = 0$ and $J_1 = 1$. The characteristic polynomial $t^2 - t - 2$ has two roots $\alpha = 2$ and $\beta = -1$. Hence, from Corollary 2.2, we obtain

$$J_n - 2J_{n-1} = (-1)^{n-1}$$

and

$$J_n + J_{n-1} = 2^{n-1}.$$

For Fibonacci number sequence $\{F_n\}_{n\geq 0}$ and Lucas number sequence $\{L_n\}_{n\geq 0}$, we may use the same argument shown above to construct the well-known identities as follows (see also [14] and [10]) .

$$\frac{1-\sqrt{5}}{2}F_n + F_{n-1} = \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

$$\frac{1+\sqrt{5}}{2}F_n + F_{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

$$\frac{\sqrt{5}-1}{2}L_n - L_{n-1} = \sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

$$-\frac{\sqrt{5}+1}{2}L_n - L_{n-1} = -\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Corollary 2.3. *Let* $\{a_n\}_{n\geq 0}$ *be a sequence satisfying the linear recurrence relation of order* 3*:*

$$a_n = p_1 a_{n-1} + p_2 a_{n-2} + p_3 a_{n-3}, \quad n \geq 3,$$

with initial conditions a_0 a_1*, and* a_2*, and let the characteristic polynomial* $P_3(t) = t^3 - p_1 t^2 - p_2 t - p_3$ *have roots* α*,* β*, and* γ*. Then the sequence* $\{a_n\}_{n\geq 0}$ *satisfies the identity*

$$a_n^{(3)} = \gamma^{n-2} a_2^{(3)},$$

where $a_n^{(3)} = a_n - (\alpha+\beta)a_{n-1} + \alpha\beta a_{n-2}$ *for* $n \geq 2$.

Corollary 2.4. *Let $\{a_n\}_{n\geq 0}$ be a sequence satisfying the linear recurrence relation of order 4:*

$$a_n = p_1 a_{n-1} + p_2 a_{n-2} + p_3 a_{n-3} + p_4 a_{n-4}, \quad n \geq 4,$$

with initial conditions a_0 a_1, a_2, and a_3, and let the characteristic polynomial $P_4(t) = t^4 - p_1 t^3 - p_2 t^2 - p_3 t - p_4$ have roots α, β, γ, and δ. Then the sequence $\{a_n\}_{n\geq 0}$ satisfies the identity

$$a_n^{(4)} = \delta^{n-3} a_2^{(4)},$$

where $a_n^{(4)} = a_n - (\alpha+\beta+\gamma)a_{n-1} + (\alpha\beta+\alpha\gamma+\beta\gamma)a_{n-2} + \alpha\beta\gamma a_{n-3}$ for $n \geq 3$.

Examples related to some famous linear recurring sequences in combinatorics are presented below for the applications of Corollaries 2.2, 2.3, and 2.4.

Example 1. We now construct identities for sequences shown in Table 6 of [9] (see also in http://www.research.att.com/ njas/sequences/). Sequence *A001047*, $a_n = 3^n - 2^n = 2\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\}$ satisfies recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denote Stirling numbers of the second kind. Thus, from Corollary 2.2, we have

$$a_n = 2a_{n-1} + 3^{n-1}, \quad \textit{and} \quad a_n = 3a_{n-1} + 2^{n-1},$$

which implies the following identities of Stirling numbers of the second kind:

$$2\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} - 4\left\{ \begin{array}{c} n \\ 3 \end{array} \right\} - 2\left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 3^{n-1},$$
$$2\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} - 6\left\{ \begin{array}{c} n \\ 3 \end{array} \right\} - 3\left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 2^{n-1}.$$

The above identities imply $2\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} = 3^n - 2^n$, but the converse implication is not obvious.

Similarly, for Sequence *A003462*, $a_n = (3^n - 1)/2$ satisfying $a_n = 4a_{n-1} - 3a_{n-2}$, there hold

$$\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} - \left\{ \begin{array}{c} n \\ 3 \end{array} \right\} - \left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 3^{n-1},$$
$$\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} - 3\left\{ \begin{array}{c} n \\ 3 \end{array} \right\} - 3\left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 1.$$

Mersenne number sequence $a_n = 2^n - 1$ (*A000225*) satisfying $a_n = 3a_{n-1} - 2a_{n-2}$ generates

$$a_n = a_{n-1} + 2^{n-1}, \quad \textit{and} \quad a_n = 2a_{n-1} + 1,$$

or equivalently,

$$\left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} - \left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 2^{n-1},$$
$$\left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\} - 2\left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 1$$

due to $a_n = \left\{ \begin{array}{c} n+1 \\ 2 \end{array} \right\}$.

We now consider examples of sequences of order 3. If the sequence $\{a_n\}_{n\geq 0}$ satisfies a linear recurrence relation of order 3:

$$a_n = 3ka_{n-1} - (3k^2-1)a_{n-2} + k(k^2-1)a_{n-3}$$

for some positive integer k, then solutions of the equation $t^3 - 3kt^2 + (3k^2-1)t - k(k^2-1) = 0$ are $k \pm 1$ and k. Thus, Corollary 2.3 shows that the sequence satisfies identities

$$a_n - 2ka_{n-1} + (k^2-1)a_{n-2} = k^{n-2}(a_2 - 2ka_1 + (k^2-1)a_0), \ (k>0), \tag{2.6}$$

$$\begin{aligned} a_n &- (2k+1)a_{n-1} + k(k+1)a_{n-2} \\ &= (k-1)^{n-2}(a_2 - (2k+1)a_1 + k(k+1)a_0), \ (k>1), \end{aligned} \tag{2.7}$$

$$\begin{aligned} a_n &- (2k-1)a_{n-1} + k(k-1)a_{n-2} \\ &= (k+1)^{n-2}(a_2 - (2k-1)a_1 + k(k-1)a_0), \ (k>1). \end{aligned} \tag{2.8}$$

In particular, if $a_0 = a_1 = 1$ and $a_2 = 2$, then (2.6)-(2.8) can be written as

$$a_n - 2ka_{n-1} + (k^2-1)a_{n-2} = (k-1)^2k^{n-2}, \ (k>0), \tag{2.9}$$

$$a_n - (2k+1)a_{n-1} + k(k+1)a_{n-2} = (k^2-k+1)(k-1)^{n-2}, \ (k>1), \tag{2.10}$$

$$a_n - (2k-1)a_{n-1} + k(k-1)a_{n-2} = (k^2-3k+3)(k+1)^{n-2}, \ (k>1), \tag{2.11}$$

respectively.

Example 2. Sequence *A129652*, $\{a_n\}_{n\geq 0} = \{1, 1, 2, 7, 26, 91, \ldots\}$, is defined by the linear recurrence relation of order 3:

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with initial conditions $a_0 = a_1 = 1$ and $a_2 = 2$. It is easy to see the three roots of the characteristic polynomial equation $t^3 - 6t^2 + 11t - 6 = 0$ are 1, 2, and 3. Thus, using (2.9)-(2.11) for $k = 2$, we obtain identities

$$\begin{aligned} a_n - 4a_{n-1} + 3a_{n-2} &= 2^{n-2}, \\ a_n - 5a_{n-1} + 6a_{n-2} &= 3, \\ a_n - 3a_{n-1} + 2a_{n-2} &= 3^{n-2}, \end{aligned}$$

respectively. Let P(A) be the power set of an n-element set A. Then a_{n-1} is the number of pairs of elements x,y of P(A) for which either (1) x and y are disjoint and for which x is not a subset of y and y is not a subset of x, or (2) x and y are intersecting and for which either x is a proper subset of y or y is a proper subset of x, or (3) x = y. Hence, the general term of $\{a_n\}_{n\geq 0} = \{1, 1, 2, 7, 26, 91, \ldots\}$ is $a_n = \left\{ \begin{matrix} n+1 \\ 3 \end{matrix} \right\} + 1$ (See Haye [9]). From the above identities, we obtain identities of Stirling numbers of the second kind as follows:

$$\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} - 4\left\{ \begin{array}{c} n \\ 3 \end{array} \right\} + 3\left\{ \begin{array}{c} n-1 \\ 3 \end{array} \right\} = 2^{n-2},$$
$$\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} - 5\left\{ \begin{array}{c} n \\ 3 \end{array} \right\} + 6\left\{ \begin{array}{c} n-1 \\ 3 \end{array} \right\} = 1,$$
$$\left\{ \begin{array}{c} n+1 \\ 3 \end{array} \right\} - 3\left\{ \begin{array}{c} n \\ 3 \end{array} \right\} + 2\left\{ \begin{array}{c} n-1 \\ 3 \end{array} \right\} = 3^{n-2}.$$

Sequence *A162723*, $\{a_n\}_{n\geq 0} = \{1, 1, 2, 16, 116, 676, \ldots\}$ is defined by $a_n = 9a_{n-1} - 26a_{n-2} + 24a_{n-3}$ with initial conditions $a_0 = a_1 = 1$ and $a_2 = 2$. Its characteristic polynomial $p(t) = t^3 + 9t^2 - 26t + 24$ has roots 2, 3, and 4. Thus, we apply (2.9)-(2.11) for $k = 3$ to the sequence and obtain

$$a_n - 6a_{n-1} + 8a_{n-2} = 4 \cdot 3^{n-2},$$
$$a_n - 7a_{n-1} + 12a_{n-2} = 7 \cdot 2^{n-2},$$
$$a_n - 5a_{n-1} + 6a_{n-2} = 3 \cdot 4^{n-2},$$

respectively.

If a sequence $\{a_n\}_{n\geq 0}$ satisfies the linear recurrence relation of order 3:

$$a_n = 2(k+1)a_{n-1} - (k^2 + 3k + 1)a_{n-2} + k(k+1)a_{n-3}$$

for some positive integer $k \geq 1$, then roots of the characteristic polynomial $P_3(t) = t^3 - 2(k+1)t^2 + (k^2+3k+1)t - k(k+1)$ are 1, k and $k+1$. Thus, Corollary 2.3 shows that the sequence satisfies identities

$$a_n - (2k+1)a_{n-1} + k(k+1)a_{n-2} = a_2 - (2k+1)a_1 + k(k+1)a_0, \tag{2.12}$$
$$a_n - (k+2)a_{n-1} + (k+1)a_{n-2} = k^{n-2}(a_2 - (k+2)a_1 + (k+1)a_0), \tag{2.13}$$
$$a_n - (k+1)a_{n-1} + ka_{n-2} = (k+1)^{n-2}(a_2 - (k+1)a_1 + ka_0). \tag{2.14}$$

In particular, if $a_0 = a_1 = 1$ and $a_2 = 2$, then (2.12)-(2.14) can be written as

$$a_n - (2k+1)a_{n-1} + k(k+1)a_{n-2} = k^2 - k + 1, \tag{2.15}$$
$$a_n - (k+2)a_{n-1} + (k+1)a_{n-2} = k^{n-2}, \tag{2.16}$$
$$a_n - (k+1)a_{n-1} + ka_{n-2} = (k+1)^{n-2}, \tag{2.17}$$

respectively.

Example 3. Consider Sequence *A000325*, $\{a_n\}_{n\geq 0} = \{1, 1, 2, 5, 12, 27, 58, \ldots\}$, which is defined by $a_n = 2^n - n$ and satisfies the recurrence relation

$$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}, \quad n \geq 3,$$

with initial conditions $a_0 = a_1 = 1$ and $a_2 = 2$. DeSario and Wenstrom [8] have shown that a_n is the number of different permutations of a deck of n cards that can be produced by a single shuffle. From Lascoux and Schutzenberger [15], one may see that a_n is also the number of permutations of degree n with at most one fall, called Grassmannian permutations. Since the corresponding characteristic polynomial equation $t^3 - 4t^3 + 5t^2 - 2 = 0$ has solutions 1, 1, and 2, we may use (2.16)-(2.17) (Note (2.15) and (2.16) are identical for $k = 1$) and obtain identities

$$a_n - 3a_{n-1} + 2a_{n-2} = 1,$$
$$a_n - 2a_{n-1} + a_{n-2} = 2^{n-2},$$

respectively.

For $k = 2$, we obtain Sequence *A129652*, $\{a_n\}_{n\geq 0} = \{1, 1, 2, 7, 26, 91, \ldots\}$ from (2.15)-(2.17). This sequence and its three identities have been presented in Example 2. Similarly, if $k = 3$, we get Sequence *A162725*, $\{a_n\}_{n\geq 0} = \{1, 1, 2, 9, 46, 221, \ldots\}$, which is defined by

$$a_n = 8a_{n-1} - 19a_{n-2} + 12a_{n-2}.$$

Hence, there hold

$$a_n - 7a_{n-1} + 12a_{n-2} = 7,$$
$$a_n - 5a_{n-1} + 4a_{n-2} = 3^{n-2},$$
$$a_n - 4a_{n-1} + 3a_{n-2} = 4^{n-2}.$$

3. An alternative method using the generating functions

Let $\{a_n\}_{n\geq 0}$ be the linear recurring sequence defined by (1.1). Then its generating function $P(t)$ can be written as

$$P(t) = \{a_0 + \sum_{n=1}^{r-1} \left(a_n - \sum_{j=1}^{n} p_j a_{n-j} \right) t^n\}/\{1 - \sum_{j=1}^{r} p_j t^j\}. \tag{3.1}$$

Hence, we have the following result.

Proposition 3.1. *Let the characteristic polynomial of the linear recurring sequence $\{a_n\}_{n\geq 0}$ defined by (1.1) be $p(t) = \Pi_{i=1}^r (t - \alpha_i)$. Then the denominator of the generating function $P(t)$ of $\{a_n\}_{n\geq 0}$ equals $\Pi_{i=1}^r (1 - \alpha_i t)$.*

The proof is straightforward and omitted. Based on this fact, we may give the following method, which is an alternative method of that presented in Section 2. The equivalence of two methods will be shown later.

Proposition 3.2. *Let $\{a_n\}_{n\geq 0}$ be a sequence with the generating function $P(t) = A(t)/B(t)$, in which $A(t)$ can be expressed as a formal power series and $B(t)$ is a non-null polynomial. Suppose that $B(t)$ can be factored as $B(t) = q_1(t)q_2(t)$ with*

$$q_1(t) = g_0 + g_1 t + g_2 t^2 + \cdots + g_r t^r,$$

then

$$[t^n]P(t) = [t^n]\frac{A(t)}{q_1(t)q_2(t)}$$

implies

$$[t^n]q_1(t)P(t) = [t^n]\frac{A(t)}{q_2(t)},$$

or equivalently,

$$g_0a_n + g_1a_{n-1} + \cdots + g_ra_{n-r} = [t^n]\frac{A(t)}{q_2(t)}. \tag{3.2}$$

Observe that no formal proof is necessary, since everything depends on the "rules of the generating function" and the "coefficient of operators", which are immediate. If we are able to extract the coefficient of t^n from $A(t)/q_2(t)$ then we have obtained a (non homogeneous) recurrence relation of order r by using Proposition 3.2.

First, let $\{a_n\}_{n\geq 0}$ be the linear recurring sequence defined by (1.1) and let $P(t)$ be the generating function. From equation (3.1) and Proposition 3.1, we have

$$P(t) = \frac{a_0 + \sum_{i=1}^{r-1}\left(a_i - \sum_{j=1}^{i} p_j a_{i-j}\right)t^i}{\Pi_{i=1}^{r}(1-\alpha_i t)},$$

and so

$$P(t)\Pi_{i=1}^{r-1}(1-\alpha_i t) = \frac{a_0 + \sum_{i=1}^{r-1}\left(a_i - \sum_{j=1}^{i} p_j a_{i-j}\right)t^i}{1-\alpha_r t}. \tag{3.3}$$

Multiplying out the left hand side of (3.3), we have

$$[t^n]P(t)\Pi_{i=1}^{r-1}(1-\alpha_i t) = a_n - a_{n-1}\sum_{i=1}^{r-1}\alpha_i + \cdots + (-1)^{r-1}a_{n-r+1}\Pi_{i=1}^{r-1}\alpha_i = a_n^{(r)} \tag{3.4}$$

for all $n \geq r-1$. Multiplying out the right hand side of (3.3), we have

$$\begin{aligned}
& a_0\alpha_r^n + \sum_{i=1}^{r-1}\left(a_i - \sum_{j=1}^{i} p_j a_{i-j}\right)\alpha_r^{n-i} \\
= \; & \sum_{i=0}^{r-1} a_i\alpha_r^{n-i} - \sum_{i=1}^{r-1}\sum_{j=1}^{i} p_j a_{i-j}\alpha_r^{n-i} \\
= \; & \sum_{i=0}^{r-1} a_i\alpha_r^{n-i} - \sum_{i=0}^{r-1} a_i \sum_{j=1}^{r-1-i} p_j\alpha_r^{n-i-j} \\
= \; & \alpha_r^{n-r+1}\sum_{i=0}^{r-1} a_i\left(\alpha_r^{r-1-i} + \sum_{j=1}^{r-1-i}(-1)^j\left(\sum_{1\leq k_1<\cdots<k_j\leq r}\alpha_{k_1}\cdots\alpha_{k_j}\right)\alpha_r^{r-1-i-j}\right) \\
= \; & \alpha_r^{n-r+1}\sum_{i=0}^{r-1} a_i\left((-1)^{r-1-i}\sum_{1\leq k_1<\cdots<k_{r-1-i}\leq r-1}\alpha_{k_1}\cdots\alpha_{k_{r-1-i}}\right) \\
= \; & \alpha_r^{n-r+1}a_{r-1}^{(r)}
\end{aligned} \tag{3.5}$$

with the convention that $(-1)^{r-1-i}\sum_{1\leq k_1<\cdots<k_{r-1-i}<r}\alpha_{k_1}\cdots\alpha_{k_{r-1-i}} = 1$ for $i = r-1$. From (3.3), (3.4) and (**??**), we have $a_n^{(r)} = \alpha_r^{n-r+1}a_{r-1}^{(r)}$ for all $n \geq r$, which is the same as the result in Theorem 2.1. So, the method in Proposition 3.2 is indeed an alternative method of that presented in Section 2 for linear recurring sequences.

Let $P(t) = A(t)/B(t)$ be the generating function of sequence $\{a_n\}_{n\geq 0}$, in which $B(t)$ is a non-null polynomial of degree r. Then, $B(t)$ can be factoring into a product of r linear factors defined in $\mathbb{C}$, which implies that 2^r identities can be constructed by using the method shown in Proposition 3.2. In other words, there are 2^r identities in $\mathbb{C}$ can be found from the linear recurring sequences defined by the homogeneous recurrence relation (1.1) of order r, because the generating function of $\{a_n\}_{n\geq 0}$ is $A(t)/B(t)$ with $B(t)=1-\sum_{j=1}^{r} p_j t^j$. The more important is that this method can be applied to any sequence possessing a rational generating function, or even the numerator is a formal power series. The latest case is the most interesting case in this section. We demonstrate it with some examples as follows.

Example 4. For each nonnegative integer n, the *Fine number* f_n is considered to be the number of rooted trees of order n with root of even degree. In [7], the generating function of Fine number sequence $\{f_n\}_{n\geq 0}$ is presented as

$$F(t)=\frac{1+2t-\sqrt{1-4t}}{2t(t+2)}. \tag{3.6}$$

Using *Faà di Bruno's formula*, Chou, Hsu and Shiue [5] give the expressions

$$F(t)=\frac{4t+2t^2+\sum_{n\geq 3}\frac{2}{n}\binom{2(n-1)}{n-1}t^n}{2t(t+2)} \tag{3.7}$$

and

$$f_n=\frac{1}{2}\sum_{k=2}^{n}\frac{(-1)^{n-k}}{(k+1)2^{n-k}}\binom{2k}{k}, \quad n\geq 2,$$

with $f_0=1$ and $f_1=0$. Using Preposition 3.2 and equations (3.6) and (3.7), we may obtain an identity of Fine number f_n for $n\geq 1$ as follows

$$f_{n-1}+2f_n=\frac{1}{n+1}\binom{2n}{n}=C_n, \tag{3.8}$$

where C_n is the nth Catalan number.

For any non-negative integer n, the *Riordan number* r_n can be viewed as the number of tall bushes with $n+1$ edges (see Bernhart [3]). Let $R(t)=\sum_{n=0}^{\infty} r_n t^n$ be the generating function of Riordan numbers. As shown in [5],

$$R(t)=\frac{1+t-\sqrt{1-2t-3t^2}}{2t(1+t)} \tag{3.9}$$

with

$$\sqrt{1-2t-3t^2}=1-\sum_{n=1}^{\infty}\frac{t^n}{2^{n-1}}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2n-2k-2)!3^k}{(n-k-1)!(n-2k)!k!}. \tag{3.10}$$

[3] also gives that

$$r_n=\frac{1}{n+1}\sum_{m=0}^{n}(-1)^m\binom{n+1}{m}\binom{2n-2m}{n-m}.$$

Using Preposition 3.2 and equations (3.9) and (3.10), we obtain an identity of Riordan number r_n for $n \geq 1$ as follows

$$r_n + r_{n-1} = \frac{1}{2^n} \sum_{k=0}^{[\frac{n+1}{2}]} \frac{(2n-2k)!3^k}{(n-k)!(n-2k+1)!k!}. \tag{3.11}$$

For any non-negative integer n, the *central Delannoy number* d_n is defined by the number of lattice paths on the plane from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$, and $(1,1)$. Let $D(t) = \sum_{n=0}^{\infty} d_n t^n$ be the generating function of central Delannoy numbers. Then we have (see Banderier and Schwer [2])

$$D(t) = \frac{1}{\sqrt{1-6t+t^2}} = \frac{\sqrt{1-6t+t^2}}{1-6t+t^2} \tag{3.12}$$

and

$$d_n = \sum_{i=0}^{n} \binom{n}{i}^2 2^i.$$

Using the same method in [5], we have

$$\sqrt{1-6t+t^2} = 1 + \sum_{n=1}^{\infty} \frac{3^n t^n}{2^{n-1}} \sum_{i=0}^{[\frac{n}{2}]} \frac{(-1)^i}{3^{2i}(n-i-1)!(n-2i)!i!}. \tag{3.13}$$

From Preposition 3.2 and equations (3.12) and (3.13), we obtain an identity of the central Delannoy number d_n for $n \geq 2$ as

$$d_n - 6d_{n-1} + d_{n-2} = \frac{3^n}{2^{n-1}} \sum_{i=0}^{[\frac{n}{2}]} \frac{(-1)^i}{3^{2i}(n-i-1)!(n-2i)!i!}. \tag{3.14}$$

Note that the roots of $1-6t+t^2$ are $3 \pm \sqrt{2}$. So, one can obtain other two identities for the central Delannoy numbers using Preposition 3.2 and equations (3.12) and (3.13) similarly.

4. Extensions

In the last section, we will apply the following two techniques to derive more identities or to give hyperbolic expressions of identities from the results obtained in Sections 2 and 3.

Proposition 4.1. *Let sequence $\{a_n\}_{n\geq 0}$ be defined by the linear recurrence relation (1.1) of order r and let the characteristic polynomial $P(t) = t^r - \sum_{j=1}^{r} p_j t^{r-j}$ have r roots α_j $(j = 1, 2, \ldots, r)$, where the root set may be multiset. Denote*

$$a_n^{(j)} := a_n^{(j-1)} - \alpha_{j-1} a_n^{(j-1)} \quad (2 \leq j \leq r)$$

and $a_n^{(1)} := a_n$. Then there hold identities

$$a_n^{(r)} \pm \left(a_{n+k} - \sum_{j=1}^{r} p_j a_{n+k-j} \right) = \alpha_r^{n-r+1} a_{r-1}^{(r)} \tag{4.1}$$

for any integer k satisfying $n+k \geq r$, where

$$
\begin{aligned}
a_n^{(r)} &= a_n - a_{n-1}\sum_{i=1}^{r-1}\alpha_i + a_{n-2}\sum_{1\le i<j\le r-1}\alpha_i\alpha_j - a_{n-3}\sum_{1\le i<j<k\le r-1}\alpha_i\alpha_j\alpha_k \\
&+\cdots+(-1)^{r-1}a_{n-r+1}\Pi_{i=1}^{r-1}\alpha_i
\end{aligned}
\tag{4.2}
$$

for $n\ge r-1$.

Example 5. In Section 2 (see the paragraphs after Corollary 2.2), we obtain two identities for Pell number sequence $\{P_n\}_{n\ge 0}$, which has the characteristic polynomial t^2-2t-1. From Proposition 4.1, for $k=1$, we immediately have identities:

$$
\begin{aligned}
&P_{n+1}-P_n-(2+\sqrt{2})P_{n-1}=(1-\sqrt{2})^{n-1},\\
&P_{n+1}-P_n-(2-\sqrt{2})P_{n-1}=(1+\sqrt{2})^{n-1}.
\end{aligned}
$$

Similarly, for Jacobsthal number sequence $\{J_n\}_{n\ge 0}$ and Fibonacci number sequence $\{F_n\}_{n\ge 0}$, there hold identities:

$$
\begin{aligned}
&J_{n+1}-4J_{n-1}=(-1)^{n-1},\\
&J_{n+1}-J_{n-1}=2^{n-1},\\
&F_{n+1}-\frac{3+\sqrt{5}}{2}F_{n-1}=\left(\frac{1-\sqrt{5}}{2}\right)^{n-1},\\
&F_{n+1}-\frac{3-\sqrt{5}}{2}F_{n-1}=\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}.
\end{aligned}
$$

From [10], let a $\{a_n\}_{n\ge 0}$ be linear recurring sequence of order 2 satisfying the linear recurrence relation,

$$
a_n = pa_{n-1}+qa_{n-2}. \tag{4.3}
$$

and denote by α and β the two roots of the characteristic polynomial $p(t)=t^2-pt-q$, then

$$
a_n=\begin{cases}\left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n-\left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & if\ \alpha\ne\beta;\\ na_1\alpha^{n-1}-(n-1)a_0\alpha^n, & if\ \alpha=\beta.\end{cases} \tag{4.4}
$$

Inspired by [1, 4, 11, 13], denote

$$
\alpha(\theta)=\sqrt{q}e^{\theta},\quad \beta(\theta)=-\sqrt{q}e^{-\theta} \tag{4.5}
$$

for some real or complex number θ, where $q>0$. For the case of $q<0$, we denote

$$
(\alpha(\theta),\beta(\theta))=\begin{cases}(\sqrt{-b}e^{\theta},\sqrt{-b}e^{-\theta}) & for\ p>0,\\ (-\sqrt{-b}e^{\theta},-\sqrt{-b}e^{-\theta}) & for\ p<0,\end{cases}
$$

for some real or complex number θ, and the remaining process is similar, which we leave for the interested readers. From (4.5) we may have

$$
p(\theta)=2\sqrt{q}\sinh(\theta) \tag{4.6}
$$

and define a parametric expression of $\{a_n\}_{n\ge 0}$ as

$$a_n(\theta) = 2\sqrt{q}\sinh(\theta)a_{n-1}(\theta) + qa_{n-2}(\theta),\ a_0(\theta) = a_0,\ a_1(\theta) = \frac{2a_1\sqrt{q}}{p}\sinh\theta. \tag{4.7}$$

Obviously, if

$$\theta = \sinh^{-1}\left(\frac{p}{2\sqrt{q}}\right), \tag{4.8}$$

$\{a_n(\theta)\}_{n\geq 0}$ is reduced to $\{a_n\}_{n\geq 0}$.

Substituting expressions (4.5) into (4.4), we obtain

$$\begin{aligned} & a_n(\theta) \\ = \quad & q^{(n-1)/2}\frac{a_1(e^{n\theta} - (-1)^n e^{-n\theta}) + \sqrt{q}a_0(e^{(n-1)\theta} + (-1)^n e^{-(n-1)\theta})}{e^\theta + e^{-\theta}} \\ = \quad & \begin{cases} \frac{q^{(n-1)/2}}{\cosh\theta}\left(a_1\sinh n\theta + \sqrt{q}a_0\cosh(n-1)\theta\right), & if\, n\, is\, even; \\ \frac{q^{(n-1)/2}}{\cosh\theta}\left(a_1\cosh n\theta + \sqrt{q}a_0\sinh(n-1)\theta\right), & if\, n\, is\, odd. \end{cases} \end{aligned} \tag{4.9}$$

Some properties and extensions of $\{a_n(\theta)\}_{n\geq 0}$ can be derived from (4.9). For instance, from the first equation of (4.9) and using $r = -e^{-2\theta}$, we have

$$a_n(\theta) = q^{(n-1)/2}\left(a_1 e^{(n-1)\theta}\frac{1-r^n}{1-r} + \sqrt{q}a_0 e^{(n-2)\theta}\frac{1-r^{n-1}}{1-r}\right),$$

which enables us to extend the definition of $a_n(\theta)$ to nonpositive values of n.

Since $\alpha(\theta)$ and $\beta(\theta)$ shown in (4.5) are two roots of the characteristic polynomial equation $x^2 - p(\theta)x - q = 0$, we may write (4.3) as

$$a_n(\theta) = (\alpha(\theta) + \beta(\theta))a_{n-1}(\theta) - \alpha(\theta)\beta(\theta)a_{n-2}(\theta), \tag{4.10}$$

where $\alpha(\theta)$ and $\beta(\theta)$ satisfy $\alpha(\theta) + \beta(\theta) = p(\theta)$ and $\alpha(\theta)\beta(\theta) = -q$. Therefore, from (4.10), we have

$$a_n(\theta) - \alpha(\theta)a_{n-1}(\theta) = \beta(\theta)(a_{n-1}(\theta) - \alpha(\theta)a_{n-2}(\theta)), \tag{4.11}$$

which implies

Proposition 4.2. *A sequence $\{a_n(\theta)\}_{n\geq 0}$ of order 2 satisfies the linear recurrence relation (4.3) if and only if it satisfies the non-homogeneous linear recurrence relation of order 1 with the form*

$$a_n(\theta) = \alpha(\theta)a_{n-1}(\theta) + d(\theta)\beta^{n-1}(\theta), \tag{4.12}$$

where $d(\theta)$ is uniquely determined.

Proof. The necessity is clearly from (4.11). We now prove sufficiency. If the sequence $\{a_n(\theta)\}_{n\geq 0}$ satisfies the non-homogeneous recurrence relation of order 1 shown in (4.12), then by substituting $n = 1$ into the above equation we obtain $d(\theta) = a_1(\theta) - \alpha(\theta)a_0$. Thus, (4.12) can be written as

$$a_n(\theta) - \alpha(\theta)a_{n-1}(\theta) = (a_1(\theta) - \alpha(\theta)a_0(\theta))\beta^{n-1}(\theta), \tag{4.13}$$

which yields

$$a_{n-1}(\theta) - \alpha(\theta)a_{n-2}(\theta) = (a_1(\theta) - \alpha(\theta)a_0(\theta))\beta^{n-2}(\theta). \tag{4.14}$$

Multiplying the both sides of (4.14) by $\beta(\theta)$ and evaluating the difference of the resulting equation and (4.13), one immediately knows that $\{a_n(\theta)\}_{n\geq 0}$ satisfies the linear recurrence relation of order 2: $a_n(\theta) = p(\theta)a_{n-1}(\theta)+qa_{n-2}(\theta)$ with $p(\theta) = \alpha(\theta)+\beta(\theta)$ and $q = -\alpha(\theta)\beta(\theta)$. □

Example 6. As an example, we may consider the parametric Fibonacci numbers defined by

$$F_n(\theta) = 2\sinh(\theta)F_{n-1}(\theta) + F_{n-2}(\theta), \quad F_0 = 0, F_1 = 2\sinh(\theta).$$

Here $\alpha(\theta) = e^{\theta}$ and $\beta(\theta) = -e^{-\theta}$. From (4.12) there holds an identity for the parametric Fibonacci numbers

$$F_n(\theta) = e^{\theta}F_{n-1}(\theta) + 2(-1)^{n-1}\sinh(\theta)e^{-(n-1)\theta},$$

or equivalently,

$$-e^{-\theta}F_n(\theta) + F_{n-1}(\theta) = 2(-1)^n e^{-n\theta}\sinh(\theta).$$

Similarly, we have

$$e^{\theta}F_n(\theta) + F_{n-1}(\theta) = 2e^{n\theta}\sinh(\theta).$$

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MSC2010: 05A15, 33C45, 65B10, 39A70, 41A80.

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