Multivariate Expansion Associated with Sheffer-type Polynomials and Operators

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Abstract

With the aid of multivariate Sheffer-type polynomials and differential operators, this paper provides two kinds of general expansion formulas, called respectively the first expansion formula and the second expansion formula, that yield a constructive solution to the problem of the expansion of $A(\hat{t})f(\widehat{g(t)})$ (a composition of any given formal power series) and the expansion of the multivariate entire functions in terms of multivariate Sheffer-type polynomials, which may be considered an application of the first expansion formula and the Sheffer-type operators. The results are applicable to combinatorics and special function theory.

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differential operators, multivariate weighted Stirling numbers, multivariate Riordan array pair, multivariate exponential polynomials.

1 Introduction

The purpose of this paper is to study the following expansion problem.

Problem 1. Let $\hat{t} = (t_1, t_2, \ldots, t_r)$, $A(\hat{t})$, $\widehat{g(t)} = (g_1(t_1), g_2(t_2), \cdots, g_r(t_r))$ and $f(\hat{t})$ be any given formal power series over the complex number field $\mathbb{C}^r$ with $A(\hat{0}) = 1$, $g_i(0) = 0$ and $g_i'(0) \neq 0$ $(i = 1, 2, \cdots, r)$. We wish to find the power series expansion in $\hat{t}$ of the composite function $A(\hat{t})f(\widehat{g(t)})$.

For this problem, there is a significant body of relevant work in terms of the choices of univariate functions $A(t)$ and $g(t)$ (see, for example, Comtet [5]). Certainly, such the problem is of fundamental importance in combinatorial analysis as well as in special function theory, inasmuch as various generating functions (GF) often used or required of the form $A(\hat{t})f(\widehat{g(t)})$. In the case of $r = 1$, it is known that [5] has dealt with various explicit expansions of $f(g(t))$ using either Faa di Bruno formula or Bell polynomials. In addition, for $r = 1$, such a problem also gives a general extension of the Riordan array sum. Here, the Riordan array is an infinite lower triangular matrix $(a_{n,k})_{n,k\in\mathbb{N}}$ with $a_{n,k} = [t^n]A(t)(g(t))^k$, and the matrix is denoted by $(A(t), g(t))$. Indeed, if $f(t) = t^k$, then $A(t)f(g(t))$ is the kth column sum of the array; i.e., $A(t)f(g(t))$ yields the GF of the kth column of $(A(t), g(t))$. Hence, the row sum can be also used to derive the Sheffer-type polynomials from the Riordan array (*cf.* our recent work [7]). In this paper we will show that a power series expansion of $A(\hat{t})f(\widehat{g(t)})$ could quite readily be obtained via the use of Sheffer-type differential operators. Also it will be shown that some generalized weighted Stirling numbers would be naturally entering into the coefficients of the general expansion formula developed.

We now give the definitions of the Sheffer-type polynomials, an extension of the Appell polynomials (*cf.* Barrucand [1] and Sheffer [14]), and the Sheffer-type differential operators (see Section 2).

Definition 1.1 *Let $A(\hat{t})$ and $\widehat{g(t)}$ be defined as in Problem 1. Then the polynomials $p_{\hat{n}}(\hat{x})$ ($\hat{n} \in \mathbb{N}^r \cup \{\hat{0}\}$) as defined by the GF*

$$A(\hat{t})e^{\hat{x}\cdot\widehat{g(t)}} = \sum_{\hat{n}\geq\hat{0}} p_{\hat{n}}(\hat{x})t^{\hat{n}} \tag{1.1}$$

are called the Sheffer-type polynomials, where $p_{\hat{0}}(\hat{x}) = 1$. *Accordingly,* $p_{\hat{n}}(\hat{D})$ *with* $\hat{D} \equiv (D_1, D_2 \cdots, D_r)$ *is called Sheffer-type differential operator of degree* $\hat{n}$ *associated with* $A(\hat{t})$ *and* $\widehat{g(t)}$. *In particular,* $p_{\hat{0}}(\hat{D}) \equiv I$ *is the identity operator.*

Note that for $r = 1$, $\{p_n(x)\}$ is also called the sequence of Sheffer A-type zero, which has been treated thoroughly by Roman [11] and Roman-Rota [12] using umbral calculus (*cf.* also Broder [3] and Hsu-Shiue [9]).

For the formal power series $f(\hat{t})$, the coefficient of $t^\lambda = (t_1^{\lambda_1}, t_2^{\lambda_2}, \cdots, t_r^{\lambda_r})$ is usually denoted by $[t^\lambda]f(\hat{t})$. Accordingly, (1.1) is equivalent to the expression $p_\lambda(\hat{x}) = [t^\lambda]A(\hat{t})e^{\hat{x}\cdot\widehat{g(t)}}$. Also, we shall frequently use the notation

$$p_\lambda(\hat{D})f(\hat{0}) := [p_\lambda(\hat{D})f(\hat{t})]_{\hat{t}=\hat{0}}. \tag{1.2}$$

In the next section, we shall give the expansion theorem for our problem. As an application of our results, in Section 3 we shall find the series expansion of the multivariate entire function $f(\hat{z})$ defined on $\mathbb{C}^r$ in terms of multivariate Sheffer-type polynomials. As for this expansion, we would like to mention the related work in Boas-Buck [2], in which the univariate case for entire functions was thoroughly discussed, and the expansion coefficients were described by using contour integrals. In this paper, we will show that the expansion of the multivariate entire function $f(\hat{z})$ possesses a much simpler form in terms of multivariate Sheffer-type polynomials. All of those results including other applications will be presented in the following sections, which will show how Sheffer-type polynomials and operators could make the entire thing both simplified and generalized.

2 The First expansion theorem and its consequences

In what follows we shall adopt the multi-index notational system. Denote

$$
\begin{aligned}
&\hat{t} \equiv (t_1, \cdots, t_r),\ \hat{x} \equiv (x_1, \cdots, x_r),\\
&\hat{t} + \hat{x} \equiv (t_1 + x_1, \cdots, t_r + x_r),\\
&\hat{0} \equiv (0, \cdots, 0),\ \hat{D} \equiv (D_1, \cdots, D_r),\\
&\hat{D}^{\hat{k}} = D_1^{k_1} D_2^{k_2} \cdots D_r^{k_r},\ (\hat{k} \geq \hat{0})\\
&\widehat{g(t)} \equiv (g_1(t_1), \cdots, g_r(t_r)),\\
&\hat{x} \cdot \hat{D} \equiv \sum_{i=1}^{r} x_i D_i,\ \hat{x} \cdot \widehat{g(t)} \equiv \sum_{i=1}^{r} x_i g_i(t_i).
\end{aligned}
$$

Here, we define $D_i \equiv \partial/\partial t_i$ as the partial differentiation with respect to t_i. Also, E_i means the shift operator acting on t_i, namely for $1 \leq i \leq r$,

$$
\begin{aligned}
E_i f(\cdots, t_i, \cdots) &= f(\cdots, t_i + 1, \ldots),\\
E_i^{x_i} f(\cdots, t_i, \cdots) &= f(\cdots, t_i + x_i, \cdots).
\end{aligned}
$$

Formally we may denote $E_i = e^{D_i} = \exp\,(\partial/\partial t_i)$. Moreover, we write $t^\lambda \equiv t_1^{\lambda_1} \cdots t_r^{\lambda_r}$ with $\lambda \equiv (\lambda_1, \cdots, \lambda_r), r$ being non-negative integers. Also, $\lambda \geq \hat{0}$ means $\lambda_i \geq 0$ $(i = 1, \cdots, r)$, and $\lambda \geq \mu$ means $\lambda_i \geq \mu_i$ for all $i = 1, \cdots, r$.

Let $g_i(t) (i = 1, \cdots, r)$ be the formal power series in t over the complex number field C, with $g_i(0) = 0, g_i'(0) \neq 0$. Let $A(\hat{t})$ be a multiple formal power series in $\hat{t}$ with $A(\hat{0}) = 1$. Then a kind of r-dimensional Sheffer-type polynomial $p_\lambda(\hat{x}) \equiv p_{\lambda_1, \cdots, \lambda_r}(x_1, \cdots, x_r)$ of degree λ with highest degree term $x_1^{\lambda_1} \cdots x_r^{\lambda_r}$ can be defined via the multiple formal power series expansion

$$
A(\hat{t}) e^{\hat{x} \cdot \widehat{g(t)}} = \sum_{\lambda \geq \hat{0}} p_\lambda(\hat{x}) t^\lambda. \tag{2.1}
$$

For the formal power series $f(\hat{t})$, the coefficient of $t^{\hat{k}}$ is usually denoted by $[t^{\hat{k}}] f(\hat{t})$. Accordingly, (2.1) is equivalent to the expression $p_\lambda(\hat{x}) = [t^{\hat{k}}] A(\hat{t}) e^{\hat{x} \cdot \widehat{g(t)}}$. Throughout this section all series expansions are formal, so that the symbolic calculus with formal differentiation operator $\hat{D} = (D_1, \cdots, D_r)$ and shift operator $E = (E_1, \cdots, E_r)$ can be applied to all formal series, where $E^{\hat{x}}$ $(\hat{x} \in \mathbb{C}^r)$ is defined by

$$
E^{\hat{x}} f(\hat{t}) := E_1^{x_1} \cdots E_r^{x_r} f(\hat{t}) = f(\hat{t} + \hat{x}) \quad (\hat{x} \in \mathbb{C}^r),
$$

and satisfies the formal relations

$$E^{\hat{x}} = e^{\hat{x}\cdot\hat{D}} \quad (\hat{x} \in \mathbb{C}^r) \tag{2.2}$$

because of the following formal process:

$$E^{\hat{x}} f(\hat{0}) = f(\hat{x}) = \sum_{i=0}^{\infty} \frac{1}{i!}(\hat{x}\cdot\hat{D})^i f(\hat{0}) = e^{\hat{x}\cdot\hat{D}} f(\hat{0}).$$

As may be observed, the following theorem contains a constructive solution to Problem 1 mentioned in §1.

Theorem 2.1 *(First Expansion Theorem) Let $A(\hat{t})$, $f(\hat{t})$, and components of $\widehat{g(t)}$ each be a formal power series over $\mathbf{C}^r$, which satisfy $A(\hat{0}) = 1, g_i(0) = 0$ and $g_i'(0) \neq 0$ $(i = 1, \cdots, r)$. Then there holds an expansion formula of the form*

$$A(\hat{t}) f(\widehat{g(t)}) = \sum_{\lambda \geq \hat{0}} t^{\lambda} p_{\lambda}(\hat{D}) f(\hat{0}), \tag{2.3}$$

where $p_\lambda(\hat{D})$ are called Sheffer-type r-variate differential operators of degree λ associated with $A(\hat{t})$ and $\widehat{g(t)}$, and

$$p_\lambda(\hat{D}) f(\hat{0}) = p_\lambda(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_r}) f(t_1, \cdots, t_r)|_{\hat{t}=\hat{0}}.$$

Proof. Clearly, the conditions imposed on $\widehat{g(t)}$ ensure that the method of formal power series applies to the composite formal power series $A(\hat{t}) f(\widehat{g(t)})$. Thus, using symbolic calculus with operators $\hat{D}$ and E, we find via (2.2)

$$A(\hat{t}) f(\widehat{g(t)}) = A(\hat{t}) E^{\widehat{g(t)}} f(\hat{0}) = A(\hat{t}) e^{\widehat{g(t)}\cdot\hat{D}} f(\hat{0}) = \sum_{\lambda \geq \hat{0}} t^{\lambda} p_{\lambda}(\hat{D}) f(\hat{0}).$$

This is the desired expression give by (2.3).

■

Remark 2.1 For $r = 1$ $p_k(D)(k = 0, 1, 2, \cdots)$ satisfy the recurrence relations

$$(k+1)p_{k+1}(D) = \sum_{j=0}^{k}(\alpha_j + \beta j D)p_{k-j}(D) \tag{2.4}$$

with $p_0(D) = I$ and α_j, β_j being given by

$$\alpha_j = (j+1)[t^{j+1}] \log A(t),\ \beta_j = (j+1)[t^{j+1}]g(t). \tag{2.5}$$

Accordingly we have

$$(k+1)p_{k+1}(x) = \sum_{j=0}^{k}\lambda_{j+1}(x)p_{k-j}(x) \tag{2.6}$$

where $\lambda_{j+1}(x)$ are given by

$$\lambda_{j+1}(x) = (j+1)[t^{j+1}] \log(A(t)e^{xg(t)}). \tag{2.7}$$

Thus, from (2.6)-(2.7) we may infer that the differential operators $p_k(D)$'s satisfy the relations (2.4) with α_j, β_j being defined by (2.5).
Remark 2.2 If $f(x) = \sum_{n=0}^{\infty} b_n x^n$ be a formal power series and $a_{n,k} = [t^n]A(t)\ (g(t))^k$, then $A(t)f(g(t))$ yields the GF of the column of

$$\begin{pmatrix} a_{0,0} & 0 & 0 & 0 & \cdots \\ a_{1.0} & a_{1,1} & 0 & 0 & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}$$

(*cf.* Theorem 1.1 of Shapiro-Getu-Woan-Woodson [13] and Sprugnoli [15]). From the description on the Riordan array shown as in the introduction, $A(t)f(g(t))$ can be considered as an infinite linear combination of column sums of the array $(A(t), g(t))$.

Noting that formula (2.3) is equivalent to the computational rule:

$$p_{\hat{k}}(\hat{x}) := [t^k]A(\hat{t})e^{\hat{x}\cdot\widehat{g(t)}} \tag{2.8}$$

$$\Longrightarrow \quad p_{\hat{k}}(\hat{D})f(\hat{0}) := [t^k]A(\hat{t})f(\widehat{g(t)}). \tag{2.9}$$

Of course the number-sequence $\{p_{\hat{k}}(\hat{D})f(\hat{0})\}_{\hat{k}\geq\hat{0}}$ has the GF as $A(\hat{t})f(\widehat{g(t)})$.

Several immediate consequences of Theorem 2.1 may be stated as examples as follows.

Example 2.1 For $A(\hat{t}) \equiv 1$ and $\widehat{g(t)} \equiv \hat{t}$ we see that (2.3) yields the Maclaurin expansion $f(\hat{t}) = \sum_{\lambda \geq \hat{0}} t^\lambda D^\lambda f(\hat{0})/\lambda!$, where $\lambda! = (\lambda_1)! \cdots (\lambda_r)!$.

Example 2.2 If $A(\hat{z}), \widehat{g(z)}$ and $f(\hat{z})$ are entire functions with $A(\hat{0}) = 1, g_i(0) = 0, g_i'(0) \neq 0$ for $i = 1, 2, \cdots, r$, then the equality (2.3) holds for the entire function $A(\hat{z})f(\widehat{g(z)})$ (with $\hat{t} = \hat{z}$).

Example 2.3 As an example of (2.1), we set $A(\hat{t}) \equiv 1$ and $exp(\hat{x} \cdot \widehat{g(t)}) = exp(x_1(e^{t_1} - 1) + x_2(e^{t_2} - 1) + \cdots + x_r(e^{t_r} - 1))$ in (2.1) and obtain

$$e^{\hat{x} \cdot \widehat{g(t)}} = \sum_{\lambda \geq \hat{0}} \hat{\tau}_\lambda(\hat{x}) t^\lambda, \tag{2.10}$$

where

$$\hat{\tau}_\lambda(\hat{x}) = \Pi_{j=1}^r \tau_{\lambda_j}(x_j)$$

and $\tau_u(s)$ is the Touchard polynomial of degree u. Hence we may call $\hat{\tau}_\lambda(\hat{x})$ the higher dimensional Touchard polynomial of order λ.

Example 2.4 Sheffer-type expansion (2.1) also includes the following two special cases shown as in Liu [10]. Let $A(\hat{t}) = 2^m/(exp\sum_{i=1}^r t_i + 1)^m$, $\widehat{g(t)} = (t_1, \cdots, t_r)$, and $exp\left(\hat{x} \cdot \widehat{g(t)}\right) = exp\left(\sum_{i=1}^r x_i t_i\right)$, then the corresponding Sheffer-type expansion of (2.1), shown as in [10], have the form

$$A(\hat{t}) e^{\hat{x} \cdot \widehat{g(t)}} = \sum_{\lambda \geq \hat{0}} \frac{E_\lambda^{(m)}(\hat{x})}{\lambda!} t^\lambda,$$

where $E_\lambda^{(m)}(\hat{x})$ $(\lambda \geq \hat{0})$ is defined as the mth order r-variable Euler's polynomial in [10].

Similarly, substituting $A(\hat{t}) = \left(\sum_{i=1}^r t_i\right)^m / (exp\sum_{i=1}^r t_i - 1)^m$, $\widehat{g(t)} = (t_1, \cdots, t_r)$, and $exp\left(\hat{x} \cdot \widehat{g(t)}\right) = exp\left(\sum_{i=1}^r x_i t_i\right)$ into (2.1) yields

$$A(\hat{t}) e^{\hat{x} \cdot \widehat{g(t)}} = \sum_{\lambda \geq \hat{0}} \frac{B_\lambda^{(m)}(\hat{x})}{\lambda!} t^\lambda,$$

where $B_\lambda^{(m)}(\hat{x})$ $(\lambda \geq \hat{0})$ is called in [10] the mth order r-variable Bernoulli polynomial. Some basic properties of $E_\lambda^{(m)}(\hat{x})$ and $B_\lambda^{(m)}(\hat{x})$ were studied in [10].

Example 2.5 For the case $A(\hat{t}) \equiv 1$, the expansion (2.3) is essentially equivalent to the Faa di Bruno formula. Indeed, if $\widehat{g(t)} = \sum_{\hat{m}\geq(1,\cdots,1)} a_{\hat{m}}$ $t^{\hat{m}}/(\hat{m})!$, where $a_{\hat{m}} = a^{(1)}{}_{m_1}\cdots a^{(r)}{}_{m_r}$, it follows that $e^{\hat{x}\cdot\widehat{g(t)}}$ may be written in the form

$$\begin{aligned} e^{\hat{x}\cdot\widehat{g(t)}} &= \Pi_{\ell=1}^{r} \exp\{x_\ell \sum_{m_\ell\geq 1} a^{(\ell)}{}_{m_\ell} \frac{t_\ell{}^{m_\ell}}{m_\ell!}\} \\ &= \Pi_{\ell=1}^{r}\left(1+\sum_{k_\ell\geq 1}\frac{t_\ell{}^{k_\ell}}{k_\ell!}\{\sum_{j_\ell=1}^{k_\ell} x_\ell{}^{j_\ell} B_{k_\ell j_\ell}(a^{(\ell)}{}_1, a^{(\ell)}{}_2,\cdots)\}\right) \end{aligned} \quad (2.11)$$

so that

$$p_\lambda(\hat{x}) = [t^\lambda]e^{\hat{x}\cdot\widehat{g(t)}} = \Pi_{\ell=1}^{r}\frac{1}{\lambda_\ell!}\sum_{j_\ell=1}^{\lambda_\ell} x_\ell{}^{j_\ell} B_{\lambda_\ell j_\ell}(a^{(\ell)}{}_1, a^{(\ell)}{}_2,\cdots).$$

Consequently we have

$$[t^\lambda]f(\widehat{g(t)}) = \Pi_{\ell=1}^{r}\frac{1}{\lambda_\ell!}\sum_{j_\ell=1}^{\lambda_\ell} B_{\lambda_\ell j_\ell}(a^{(\ell)}{}_1, a^{(\ell)}{}_2,\cdots) D_\ell{}^{j_\ell} f(\hat{0}).$$

This is precisely the multivariate extension of the univariate Faa di Bruno formula (*cf.* Constantine [6] for another type extension).

$$[(d/dt)^k f(g(t))]_{t=0} = \sum_{j=1}^{k} B_{kj}(g'(0), g''(0),\cdots) f^{(j)}(0). \quad (2.12)$$

Note that $B_{\lambda_\ell j_\ell}(a^{(\ell)}{}_1, a^{(\ell)}{}_2,\cdots)$ is the so-called incomplete Bell polynomial whose explicit expression can easily be derived from the relation (2.11) (*cf.* [5] for the setting of $r = 1$), namely

$$\begin{aligned} & B_{\lambda_\ell j_\ell}(a^{(\ell)}{}_1, a^{(\ell)}{}_2,\cdots, a^{(\ell)}{}_{\lambda_\ell - j_\ell + 1}) \\ = & \sum_{(c)} \frac{\lambda_\ell!}{c_1! c_2!\cdots}\left(\frac{a^{(\ell)}{}_1}{1!}\right) c_1 \left(\frac{a^{(\ell)}{}_2}{2!}\right) c_2 \cdots \end{aligned} \quad (2.13)$$

where the summation extends over all integers $c_1, c_2, \cdots \geq 0$, such that $c_1 + 2c_2 + 3c_3 + \cdots = k$ and $c_1 + c_2 + \cdots = j$.

Example 2.6 For $j = 0, 1, \cdots, m$, let $f_j(t)$ be a formal power series satisfying the conditions $f_j(0) = 0$ and $f_j'(0) \neq 0$. Denote $(f_j \circ f_{j-1})(t) = f_j(f_{j-1}(t))(j \geq 1)$, then the power series expansion of $(f_m \circ f_{m-1} \circ \cdots \circ f_0)(t)(m \geq 2)$ can be obtained recursively via the implicative relations

$$\begin{aligned} & p_{jk}(x) = [t^k]e^{x \cdot (f_j \circ \cdots \circ f_0)(t)} \\ \Longrightarrow \quad & p_{jk}(D)f_{j+1}(0) = [t^k](f_{j+1} \circ f_j \circ \cdots f_0)(t), \end{aligned} \tag{2.14}$$

where $1 \leq j \leq m-1$, and $p_{jk}(D)$ are Sheffer-type operators.

Remark 2.3 For any power series f with $f(0) = 0$ and $f'(0) \neq 0$, the compositional inverse of f will be denoted by $f^{\langle -1 \rangle}$ so that $\big(f^{\langle -1 \rangle} \circ f(t)\big) = \big(f \circ f^{\langle -1 \rangle}(t)\big) = t$. Now suppose that $f_1, \cdots$, and f_m are given as in Example 2.6 and that

$$g_m(t) = (f_m \circ f_{m-1} \circ \cdots \circ f_0)(t)$$

is a known series, but $f_0(t)$ is an unknown series to be determined. Certainly one may get $f_0(t)$ via computing

$$(f_1^{\langle -1 \rangle} \circ f_2^{\langle -1 \rangle} \circ \cdots f_m^{\langle -1 \rangle} \circ g_m)(t) = f_0(t). \tag{2.15}$$

Here it may be worth mentioning that the processes (2.14) and (2.15) suggest a kind of compositional power-series-techniques that could be used to devise a certain procedure for the modification of a sequence represented by the coefficient sequence of $f_0(t)$.

Example 2.7 For the case of $r = 1$, let $B_n(x), \hat{C}_n^{(\alpha)}(x)$ and $T_n^{(p)}(x)$ be Bernouilli, Charlier and Touchard polynomials, respectively. Then, for any given formal power series $f(t)$ over $\mathbf{C}$ we have three weighted expansion formulas as follows

$$\frac{t}{e^t - 1} f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(D) f(0) \tag{2.16}$$

$$e^{-\alpha t} f(\log(1+t)) = \sum_{n=0}^{\infty} t^n \hat{C}_n^{(\alpha)}(D) f(0) \qquad (\alpha \neq 0) \tag{2.17}$$

$$(1-t)^p f(e^t - 1) = \sum_{n=0}^{\infty} t^n T_n^{(p)}(D) f(0) \qquad (p > 0) \tag{2.18}$$

Actually, (2.16)-(2.18) are just three instances drawn from the following table of special Sheffer-type polynomials, $B_n(D), \hat{C}_n^{(\alpha)}(D)$ and $T_n^{(p)}(D)$ may be called Bernoulli's, Charlier's and Touchard's differential operators, respectively.

$A(t)$	$g(t)$	$p_n(x)$	*Name of polynomials*
$t/(e^t-1)$	t	$\frac{1}{n!}B_n(x)$	Bernoulli
$2/(e^t+1)$	t	$\frac{1}{n!}E_n(x)$	Euler
e^t	$\log(1+t)$	$(PC)_n(x)$	Poisson-Charlier
$e^{-\alpha t}(\alpha \neq 0)$	$\log(1+t)$	$\hat{C}_n^{(\alpha)}(x)$	Charlier
1	$\log((1+t)\ /\ (1-t))$	$(ML)_n(x)$	Mittag-Leffler
$(1-t)^{-1}$	$\log((1+t)\ /\ (1-t))$	$p_n(x)$	Pidduck
$(1-t)^{-p}(p>0)$	$t/(t-1)$	$L_n^{(p-1)}(x)$	Laguerre
$e^{\lambda t}(\lambda \neq 0)$	$1-e^t$	$(Tos)_n^{(\lambda)}(x)$	Toscano
1	e^t-1	$\tau_n(x)$	Touchard
$1/(1+t)$	$t/(t-1)$	$A_n(x)$	Angelescu
$(1-t)/(1+t)^2$	$t/(t-1)$	$(De)_n(x)$	Denisyuk
$(1-t)^{-p}(p>0)$	e^t-1	$T_n^{(p)}(x)$	Weighted-Touchard
$(1-e^{\lambda t})^p$	e^t-1	$\frac{1}{n!}\beta_n(x)$	Boole

The higher dimensional Touchard polynomials, Euler's polynomials, and Bernoulli polynomials are shown in Examples 2.3 and 2.4. Hence, the corresponding expansions (2.3) associated with the higher dimensional polynomials can be immediately drawn. Interest readers may extend the remaining Sheffer-type polynomials shown in the above table to the setting of r-dimension, thus establishing the corresponding expansions (2.3).

Remark 2.4 The general expansion formula (2.3) can also be employed conversely. For instance, when $r=1$, suppose that we are concerned with a summation problem of power series $\sum_0^\infty \alpha(k)t^k$, where $\{\alpha(k)\}$ is a given sequence of complex numbers. If there can be found three power series $A(t), g(t)$ and $f(t)$ (with $A(0) = 1, g(0) = 0$, $[t]g(t) \neq 0$) such that $\alpha(k) = p_k(D)f(0)$, where $p_k(D)$ are Sheffer-type operators associated with $A(t)$ and $g(t)$, then the series $\sum_0^\infty \alpha(k)t^k$ can be represented by $A(t)f(g(t))$. In this case $\sum_{k=0}^\infty \alpha(k)t^k$ is said to be $Af(g)$-representable. Surely, the class of $Af(g)$-representable series is a useful concept in the Computational Theory of Formal Series as well in the Computational Combinatorics (see [7] for different approach for $\sum_{k\geq 0} \alpha(k)t^k$).

Let us discuss the convergence of the formal expansion (2.3). We now consider the setting of $r = 1$ and the higher dimensional case can be discussed similarly.

Definition 2.2 *For any real or complex series $\sum_{k=0}^\infty a_k$, the Cauchy root is defined by $\rho = \overline{\lim}_{k\to\infty}|a_k|^{1/k}$. Clearly, $\sum_{k=0}^\infty a_k$ converges absolutely whenever $\rho < 1$.*

With the aid of Cauchy root, we have the following convergence result.

Theorem 2.3 *Let $\{p_k(D)f(0)\}$ be a given sequence of numbers (in $\mathbb{R}$ or $\mathbb{C}$), and let $\theta = \overline{\lim_{k\to\infty}}|p_k(D)f(0)|^{1/k}$. Then for any given t with $t \neq 0$ we have the convergent expressions (2.3), provided that $\theta < 1/|t|$.*

Proof. Suppose that the condition $\theta < 1/|t|$ $(t \neq 0)$ is fulfilled, so that $\theta|t| < 1$. Hence the convergence of the series on the right-hand side of (2.3) is obviously in accordance with Cauchy's root test.

∎

3 Generalized Stirling Numbers and the expansion of entire functions

We now introduce a kind of extended weighted Stirling-type pair in the higher dimensional setting.

Definition 3.1 *Let $A(\hat{t})$ and $\widehat{g(t)}$ be a formal power series defined on $\mathbb{C}^r$, with $A(\hat{0}) = 1, g_i(0) = 0$ and $g_i'(0) \neq 0$ $(i = 1, 2, \cdots, r)$. Then* *we have a multivariate weighted Stirling-type pair* $\{\sigma(\hat{n},\hat{k}),\sigma^*(\hat{n},\hat{k})\}$ *as defined by*

$$\frac{1}{\hat{k}!}A(\hat{t})\Pi_{i=1}^{r}\left(g_i(t_i)\right)^{k_i}=\sum_{\hat{n}\geq\hat{k}}\sigma(\hat{n},\hat{k})\frac{t^{\hat{n}}}{\hat{n}!} \tag{3.1}$$

$$\frac{1}{\hat{k}!}A(\widehat{g^*(t)})^{-1}\Pi_{i=1}^{r}\left(g_i^*(t_i)\right)^{k_i}=\sum_{\hat{n}\geq\hat{k}}\sigma^*(\hat{n},\hat{k})\frac{t^{\hat{n}}}{\hat{n}!}, \tag{3.2}$$

where $A(\cdot)^{-1}$ *is the reciprocal of* $A(\cdot)$, $\widehat{g^*(t)}=(g_1^*(t_1),g_2^*(t_2),\cdots,g_r^*(t_r))$, *and* $g_i^*=g_i{}^{\langle -1\rangle}$ *is the compositional inverse of* g_i $(i=1,2,\cdots,r)$ *with* $g_i^*(0)=0$, $[t_i]g_i^*(t_i)\neq 0$, *and* $\sigma(\hat{0},\hat{0})=\sigma^*(\hat{0},\hat{0})=1$. *We call* $\sigma(\hat{n},\hat{k})$ *the dual of* $\sigma^*(\hat{n},\hat{k})$ *and vice versa. We will also call* $\sigma(\hat{n},\hat{k})$, $\sigma^*(\hat{n},\hat{k})$ *the multivariate Riordan arrays and denote them by* $\left(A(\hat{t}),\widehat{g(t)}\right)$ *and* $\left(1/A(\widehat{g^*(t)}),\widehat{g^*(t)}\right)$, *respectively. The multivariate weighted Stirling-type pair* $\{\sigma(\hat{n},\hat{k}),\sigma^*(\hat{n},\hat{k})\}$ *can also be called the Riordan array pair with respect to* $A(\hat{t})$ *and* $\widehat{g(t)}$.

Theorem 3.2 *The equations (3.1) and (3.2) imply the biorthogonality relations*

$$\sum_{\hat{m}\geq\hat{n}\geq\hat{k}}\sigma(\hat{m},\hat{n})\sigma^*(\hat{n},\hat{k})=\sum_{\hat{m}\geq\hat{n}\geq\hat{k}}\sigma^*(\hat{m},\hat{n})\sigma(\hat{n},\hat{k})=\delta_{\hat{m}\hat{k}} \tag{3.3}$$

with $\delta_{\hat{m}\hat{k}}$ *denoting the Kronecker delta, i.e.,* $\delta_{\hat{m}\hat{k}}=1$ *if* $\hat{m}=\hat{k}$ *and* 0 *otherwise. It then follows that there hold the inverse relations*

$$f_{\hat{n}}=\sum_{\hat{n}\geq\hat{k}\geq\hat{0}}\sigma(\hat{n},\hat{k})g_{\hat{k}}\Longleftrightarrow g_{\hat{n}}=\sum_{\hat{n}\geq\hat{k}\geq\hat{0}}\sigma^*(\hat{n},\hat{k})f_{\hat{k}}. \tag{3.4}$$

Proof. Transforming t_i by $g_i^*(t_i)$ in (3.1) and multiplying $A\left(\widehat{g^*(t)}\right)^{-1}(\hat{k}!)$ on both sides of the resulting equation yields

$$t^{\hat{k}}=\sum_{\hat{n}\geq\hat{k}}\sigma(\hat{n},\hat{k})\frac{\hat{k}!}{\hat{n}!}A(\widehat{g^*(t)})^{-1}\Pi_{i=1}^{r}\left(g_i^*(t_i)\right)^{n_i}. \tag{3.5}$$

By substituting (3.2) into the above equation, we obtain

$$\begin{aligned} t^{\hat{k}} &= \sum_{\hat{n}\geq\hat{k}} \sigma(\hat{n},\hat{k}) \sum_{\hat{m}\geq\hat{n}} \sigma^*(\hat{m},\hat{n}) \frac{\hat{k}!}{\hat{m}!} t^{\hat{m}} \\ &= \sum_{\hat{m}\geq\hat{k}} \frac{\hat{k}!}{\hat{m}!} t^{\hat{m}} \sum_{\hat{m}\geq\hat{n}\geq\hat{k}} \sigma^*(\hat{m},\hat{n})\sigma(\hat{n},\hat{k}). \end{aligned}$$

Equating the coefficients of the terms $t^{\hat{m}}$ on the leftmost side and the rightmost side of the above equation leads (3.3), and (3.4) is followed immediately. This completes the proof.

■

From (3.3), we can see that the $2r$ dimensional infinite matrices $\left(\sigma(\hat{n},\hat{k})\right)$ and $\left(\sigma^*(\hat{n},\hat{k})\right)$ are inverse for each other, i.e., their product is the identity matrix $\left(\delta_{\hat{n},\hat{k}}\right)_{\hat{n}\geq\hat{k}\geq\hat{0}}$.

As an application of Theorem 3.2, we now turn to the problem for finding an expansion of a multivariate entire function f in terms of a sequence of higher Sheffer-type polynomials $\{p_\lambda\}$. For this purpose, we establish the following expansion theorem.

Theorem 3.3 *(Second Expansion Theorem) The Sheffer-type operator $p_{\hat{n}}(\hat{D})$ defined in (2.3) has an expression of the form*

$$p_{\hat{n}}(\hat{D}) = \frac{1}{\hat{n}!} \sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma(\hat{n},\hat{k}) \hat{D}^{\hat{k}}. \tag{3.6}$$

Proof. Using the multivariate Taylor's formula and (3.1) we have formally

$$\begin{aligned} A(\hat{t}) f(\widehat{g(t)}) &= A(\hat{t}) \sum_{\hat{k}\geq\hat{0}} \frac{1}{\hat{k}!} \Pi_{i=1}^r g_i^{k_i}(t_i) \hat{D}^{\hat{k}} f(\hat{0}) \\ &= \sum_{\hat{k}\geq 0} \left(\sum_{\hat{n}\geq\hat{k}} \sigma(\hat{n},\hat{k}) \frac{t^{\hat{n}}}{\hat{n}!} \right) \hat{D}^{\hat{k}} f(\hat{0}) \\ &= \sum_{\hat{n}\geq\hat{0}} \left(\sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma(\hat{n},\hat{k}) \hat{D}^{\hat{k}} f(\hat{0}) \right) \frac{t^{\hat{n}}}{\hat{n}!}. \end{aligned}$$

Thus the rightmost expression, comparing with (2.3), leads to (3.6).

∎

We have the following corollaries from Theorem 3.3.

Corollary 3.4 *The formula (2.3) may be rewritten in the form*

$$A(\hat{t})f(\widehat{g(t)}) = \sum_{\hat{n}\geq\hat{0}} \frac{t^{\hat{n}}}{\hat{n}!}\left(\sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma(\hat{n},\hat{k})\hat{D}^{\hat{k}}f(\hat{0})\right), \tag{3.7}$$

where $\sigma(\hat{n},\hat{k})$'s are defined by (3.1).

Corollary 3.5 *The multivariate generalized exponential polynomials related to the generalized Stirling numbers $\sigma(\hat{n},\hat{k})$ and $\sigma^*(\hat{n},\hat{k})$ are given, respectively by the following*

$$\hat{n}!p_{\hat{n}}(\hat{x}) = \sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma(\hat{n},\hat{k})\hat{x}^{\hat{k}} \tag{3.8}$$

and

$$\hat{n}!p^*_{\hat{n}}(\hat{x}) = \sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma^*(\hat{n},\hat{k})\hat{x}^{\hat{k}}, \tag{3.9}$$

where $p_{\hat{n}}(\hat{x})$ and $p^*_{\hat{n}}(\hat{x})$ are multivariate Sheffer-type polynomials associated with $\{A(\hat{t}),\Pi_{i=1}^r(g_i(t_i))^{k_i}\}$ and $\{A(\widehat{g^*(t)})^{-1},\Pi_{i=1}^r(g_i^*(t_i))^{k_i}\}$, respectively.

Polynomials defined by (3.8) and (3.9) can be considered as the higher dimensional exponential polynomials, and the corresponding numbers when $\hat{x} = (1,\cdots,1)$ can be called the higher dimensional Bell numbers.

Applying the inverse relations (3.4) to (3.8) and (3.9) we get

Corollary 3.6 *There hold the relations*

$$\sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma^*(\hat{n},\hat{k})\hat{k}!p_{\hat{k}}(\hat{x}) = \hat{x}^{\hat{n}} \tag{3.10}$$

and

$$\sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma(\hat{n},\hat{k})\hat{k}!p^*_{\hat{k}}(\hat{x}) = \hat{x}^{\hat{n}}. \tag{3.11}$$

These may be used as recurrence relations for $p_{\hat{n}}(\hat{x})$ *and* $p^*_{\hat{n}}(\hat{x})$ *respectively with the initial conditions* $p_{\hat{0}}(\hat{x}) = p^*_{\hat{0}}(\hat{x}) \equiv 1$.

Evidently (3.6) and (3.7) imply a higher derivative formula for $A(\hat{t})f(\widehat{g(t)})$ at $\hat{t} = \hat{0}$, namely

$$\hat{D}^{\hat{n}}(A(\hat{t})f(\widehat{g(t)}))\Big|_{\hat{t}=\hat{0}} = \sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma(\hat{n},\hat{k})\hat{D}^{\hat{k}}f(\hat{0}) = \hat{n}!p_{\hat{n}}(\hat{D})f(\hat{0}).$$

We now establish the following theorem.

Theorem 3.7 *If* $f(\hat{z})$ *is a multivariate entire function defined on* $\mathbb{C}^r$, *then we have the formal expansion of* f *in terms of a sequence of multivariate Sheffer-type polynomials* $\{p_{\hat{k}}\}$ *as*

$$f(\hat{z}) = \sum_{\hat{k}\geq\hat{0}} \alpha_{\hat{k}} p_{\hat{k}}(\hat{z}), \tag{3.12}$$

where

$$\alpha_{\hat{k}} = \sum_{\hat{n}\geq\hat{k}} \frac{\hat{k}!}{\hat{n}!}\sigma^*(\hat{n},\hat{k})\hat{D}^{\hat{n}}f(\hat{0}) \tag{3.13}$$

or

$$\alpha_{\hat{k}} = \Lambda_{\hat{k}}(\hat{D})f(\hat{0}) \tag{3.14}$$

with

$$\Lambda_{\hat{k}}(\hat{D}) = \sum_{\hat{n}\geq\hat{k}} \frac{\hat{k}!}{\hat{n}!}\sigma^*(\hat{n},\hat{k})\hat{D}^{\hat{n}}. \tag{3.15}$$

Proof. Let $f(\hat{z})$ be a multivariate entire function defined on $\mathbb{C}^r$, then, we can write its Taylor's series expansion as

$$\begin{aligned} f(\hat{z}) = \sum_{\hat{n}\geq\hat{0}} \frac{\hat{D}^{\hat{n}} f(\hat{0})}{\hat{n}!}\hat{z}^{\hat{n}} &= \sum_{\hat{n}\geq\hat{0}} \frac{\hat{D}^{\hat{n}} f(\hat{0})}{\hat{n}!} \sum_{\hat{n}\geq\hat{k}\geq\hat{0}} \sigma^*(\hat{n},\hat{k})\hat{k}! p_{\hat{k}}(\hat{z}) \\ = \sum_{\hat{k}\geq\hat{0}} p_{\hat{k}}(\hat{z})\hat{k}! \sum_{\hat{n}\geq\hat{k}} \frac{1}{\hat{n}!}\sigma^*(\hat{n},\hat{k})\hat{D}^{\hat{n}} f(\hat{0}) &= \sum_{\hat{k}\geq\hat{0}} \alpha_{\hat{k}} p_{\hat{k}}(\hat{z}), \end{aligned}$$

where $\alpha_{\hat{k}}$ can be written as the forms of (3.13) or (3.14)-(3.15), which completes the proof of the theorem.

■

Remark 3.1. From (3.13), it is easy to derive Boas-Buck formulas (7.3) and (7.4) (*cf.* [2]) of the coefficients of the series expansion of an entire function in terms of polynomial $p_k(z)$. Indeed, for the fixed k, using (3.13), (3.2), and Cauchy's residue theorem yields

$$\begin{aligned} \alpha_k &= \sum_{n=k}^{\infty} \frac{k!}{n!}\sigma^*(n,k) f^{(n)}(0) = \sum_{n=k}^{\infty} [t^n]\left(A(g^*(t))^{-1}(g^*(t))^k\right) f^{(n)}(0) \\ &= \frac{1}{2\pi i}\oint_\Gamma \sum_{n=k}^{\infty} \frac{(g^*(t))^k}{A(g^*(t))t^{n+1}} f^{(n)}(0)dt \\ &= \frac{1}{2\pi i}\oint_\Gamma \frac{(g^*(t))^k}{A(g^*(t))}\left(\sum_{n=k}^{\infty}\frac{f^{(n)}(0)}{t^{n+1}}\right)dt \\ &= \frac{1}{2\pi i}\oint_\Gamma \frac{(g^*(t))^k}{A(g^*(t))}\left(\sum_{n=k}^{\infty}\frac{n! f_n}{t^{n+1}}\right)dt = \frac{1}{2\pi i}\oint_\Gamma \frac{(g^*(\zeta))^k}{A(g^*(\zeta))} F(\zeta)d\zeta, \end{aligned}$$

where $F(\zeta)$ is the Borel's transform of $\{f_n = f^{(n)}(0)/n!\}$.

We now give two algorithms to derive the series expansion (3.12) in terms of a Sheffer-type polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$.

Algorithm 3.1

Step 1 For given Sheffer-type polynomial $\{p_n(x)\}_{n\in\mathbb{N}}$, we determine its GF pair $(A(t), g(t))$ and the compositional inverse $g^*(t)$ of $g(t)$.

Step 2 Use (3.2) to evaluate set $\{\sigma^*(n,k)\}_{n\geq k}$ and substitute it into (3.13) to find α_k $(k \geq 0)$.

Algorithm 3.2

Step 1 For given Sheffer-type polynomial $\{p_n(x)\}_{n\in\mathbb{N}}$, apply (3.8) to obtain set $\{\sigma(n,k)\}_{n\geq k\geq 0}$.

Step 2 Use (3.3) to solve for set $\{\sigma^*(n,k)\}_{n\geq k}$ and substitute it into (3.13) to find α_k $(k \geq 0)$.

It is easy to see the equivalence of the two algorithms. However, the first algorithm is more readily applied than the second one.

Example 3.1 If $p_n(x) = x^n/n!$, then the corresponding GF pair is $(A(t), g(t)) = (1, t)$. Hence, noting $g^*(t) = t$ and $A(g^*(t))^{-1} = 1$, from (3.2) we have

$$\sigma^*(\hat{n}, \hat{k}) = \delta_{n,k},$$

the Kronecker delta. Therefore

$$\alpha_k = \sum_{n=k}^{\infty} \frac{k!}{n!}\sigma^*(n,k) f^{(n)}(0) = f^{(k)}(0)$$

and the expansion of f is its Maclaurin expansion.

Example 3.2 We now use Algorithm 3.1 to find the expansion of an entire function in terms of Bernoulli polynomials.

Since the GF of the Bernoulli polynomials is $A(t)exp(xg(t))$ with $A(t) = t/(e^t - 1)$ and $g(t) = t$, we have the compositional inverse of $g(t)$ as $g^*(t) = t$ and

$$A(g^*(t))^{-1} = (e^t - 1)/t.$$

Hence from (3.2) we can present

$$\begin{aligned} \frac{1}{k!}A(g^*(t))^{-1}(g^*(t))^k &= \frac{1}{k!}(e^t - 1)t^{k-1} = \frac{1}{k!}\sum_{n=1}^{\infty}\frac{t^{n+k-1}}{n!} \\ = \quad \sum_{n=k}^{\infty}\frac{1}{k!(n-k+1)!}t^n &= \sum_{n=k}^{\infty}\frac{1}{n+1}\binom{n+1}{k}\frac{t^n}{n!}. \end{aligned}$$

Hence $\sigma^*(\hat{n}, \hat{k}) = \binom{n+1}{k}/(n+1)$. Substituting this expression into (3.13) yields

$$\alpha_k = \sum_{n=k}^{\infty} \frac{k!}{n!} \sigma^*(\hat{n}, \hat{k}) f^{(n)}(0) = \sum_{n=k}^{\infty} \frac{k!}{(n+1)!} \binom{n+1}{k} f^{(n)}(0).$$

Noting $f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) t^n/n!$ formally, for $k=0$, we can write α_0 as

$$\alpha_0 = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f^{(n)}(0) = \int_0^1 f(t)dt$$

and for $k > 0$ we have

$$\begin{aligned} \alpha_k &= \sum_{n=k}^{\infty} \frac{1}{(n-k+1)!} f^{(n)}(0) = \sum_{n=k-1}^{\infty} \frac{1}{(n-k+1)!} f^{(n)}(0) - f^{(k-1)}(0) \\ &= \sum_{n=k-1}^{\infty} \frac{f^{(n)}(0)}{n!} \, D_x^{k-1} x^n \big|_{x=1} - f^{(k-1)}(0) \\ &= f^{(k-1)}(1) - f^{(k-1)}(0), \end{aligned}$$

which are exactly the expressions of the expansion coefficients obtained on page 29 of [2], which were derived by using contour integrals.

Example 3.3 Let $p_n(x)$ be the Laguerre polynomial with its GF pair $(A(t), g(t)) = ((1-t)^{-p}, t/(t-1))$, $p > 0$, then $g^*(t) = t/(t-1)$ and $A(g^*(t))^{-1} = (1-t)^{-p}$. Thus, using a similar argument of Example 3.2, we obtain

$$\sigma^*(n,k) = (-1)^k \frac{n!}{k!} \binom{n+p-1}{n-k}.$$

Hence, the coefficients of the corresponding expansion (3.12) can be written as

$$\alpha_k^{(p)} = \sum_{n=k}^{\infty} \frac{k!}{n!} \sigma^*(n,k) f^{(n)}(0) = \sum_{n=k}^{\infty} (-1)^k \binom{n+p-1}{n-k} f^{(n)}(0)$$

for $k = 0, 1, \cdots$.

Example 3.4 If expansion basis polynomials are Angelescu polynomial, $A_n(x)$ $(n \in \mathbb{N})$, then their GF pair is $(A(t), g(t)) = (1/(1+t), t/(t-1))$ and the dual $\sigma^*(n,k)$ can be found as follows:

$$\sigma^*(n,k) = (-1)^{k+1} n! \left[2\binom{n-1}{k} - \binom{n}{k} \right].$$

Substituting the above expression of $\sigma^*(n,k)$ into (3.13) yields the coefficients of expansion (3.12) as

$$\alpha_k = \sum_{n=k}^{\infty} (-1)^{k+1} k! \left[2\binom{n-1}{k} - \binom{n}{k} \right] f^{(n)}(0)$$

for $k = 0, 1, \cdots$.

Remark 3.2 The partial sum in (3.12) can be used to approximate the entire function $f(\hat{z})$. However, the corresponding remainder and error bound remain to be further investigated.

4 More Applications and Selected Examples

Here we shall mention several examples in order to indicate some applications of what has been developed in §2 − §3. In order to compare some of them with the well-known results, we only consider the applications in the univariate setting, $r = 1$.

Example 4.1 Taking $A(t) \equiv 1, g(t) = log(1+t)$, so that $g^*(t) = e^t - 1$, we see that (3.8)-(3.9) of Corollary 3.5 yield

$$n!\binom{x}{n} = (x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \tag{4.1}$$

$$n! \tau_n(x) = \sum_{k=0}^{n} S_2(n,k) x^k, \tag{4.2}$$

where $\tau_n(x)$ is the Touchard polynomials mentioned in Example 2.3. Here (4.1) is a familiar expression defining Stirling numbers of the first kind. (4.2) shows that the GF of Stirling numbers of the second kind is just the Touchard polynomial apart from a constant factor $n!$.

Example 4.2 Taking $A(t) \equiv 1, g(t) = e^t - 1$ and $f(t) = e^t$, we find that Corollary 3.4 gives

$$e^{e^{t-1}} = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} S_2(n,k))\frac{t^n}{n!}, \tag{4.3}$$

where the inner sum contained in the RHS of (4.3) represents Bell numbers W_n. Thus (4.3) is just the well-known formula

$$e^{e^{t-1}} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!}. \tag{4.4}$$

Example 4.3 The inverse expression of (4.2) is given by

$$\sum_{k=0}^{n} S_1(n,k)k!\tau_k(x) = x^n. \tag{4.5}$$

Actually this follows easily from (3.11). Note that $n!\tau_n(1) = W_n$. Thus (4.5) implies

$$\sum_{k=0}^{n} S_1(n,k)W_k = 1. \tag{4.6}$$

This seems to be a "strange identity", not easily found in the combinatorial literature.

Example 4.4 There are two kinds of weighted Stirling numbers, $S_1^{(\alpha)}(n,k)$ and $S_2^{(\alpha)}(n,k)$, defined by (*cf.* Carlitz [4] and Howard [8])

$$\frac{1}{k!}e^{-\alpha t}(log(1+t))^k = \sum_{n=k}^{\infty}\frac{t^n}{n!}S_1^{(\alpha)}(n,k) \tag{4.7}$$

$$\frac{1}{k!}e^{\alpha t}(e^t-1)^k = \sum_{n=k}^{\infty}\frac{t^n}{n!}S_2^{(\alpha)}(n,k), \tag{4.8}$$

where $\alpha \neq 0$. Comparing with (3.1) and (3.2) we have here

$$A(t) = e^{-\alpha t}, g(t) = log(1+t), g^*(t) = e^t - 1.$$

Note that $S_1^{(\alpha)}(n,k)$ and $S_2^{(\alpha)}(n,k)$ do not form a weighted Stirling-type pair as defined by (3.1)-(3.2), inasmuch as $A(g^*(t))^{-1} \neq e^{\alpha t}$. However, for $\alpha = 0$, pair of (4.7) and (4.8) is obviously a special case of pair (3.1) and (3.2). Making use of (3.8) and the table for Sheffer-type polynomials, we can obtain

$$n!\hat{C}_n^{(\alpha)}(x) = \sum_{k=0}^{n} S_1^{(\alpha)}(n,k)x^k \tag{4.9}$$

and

$$n!(Tos)_n^{(\alpha)}(x) = \sum_{k=0}^{n} S_2^{(\alpha)}(n,k)x^k. \tag{4.10}$$

Accordingly,

$$n!\hat{C}_n^{(\alpha)}(1) = \sum_{k=0}^{n} S_1^{(\alpha)}(n,k) \tag{4.11}$$

$$n!(Tos)_n^{(\alpha)}(-1) = \sum_{k=0}^{n} S_2^{(\alpha)}(n,k), \tag{4.12}$$

where numbers given by (4.12) are the generalized Bell numbers.

Example 4.5 The well-known tangent numbers $T(n,k)$ and arctangent numbers $T^*(n,k)$ are defined by

$$\frac{1}{k!}(\tan\, t)^k = \sum_{n=k}^{\infty} \frac{t^n}{n!}T(n,k) \tag{4.13}$$

$$\frac{1}{k!}(\arctan\, t)^k = \sum_{n=k}^{\infty} \frac{t^n}{n!}T^*(n,k). \tag{4.14}$$

Evidently (3.7) implies the following expansions

$$f(\tan\, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\sum_{k=0}^{n} T(n,k)f^{(k)}(0)) \tag{4.15}$$

$$f(\arctan\, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\sum_{k=0}^{n} T^*(n,k)f^{(k)}(0)). \tag{4.16}$$

Moreover, we have a pair of exponential polynomials and inverse relations as follows:

$$n!g_n(x) = \sum_{k=0}^{n} T(n,k)x^k \tag{4.17}$$

$$n!g_n^*(x) = \sum_{k=0}^{n} T^*(n,k)x^k \tag{4.18}$$

$$\sum_{k=0}^{n} T^*(n,k)k!g_k(x) = x^n \tag{4.19}$$

$$\sum_{k=0}^{n} T(n,k)k!g_k^*(x) = x^n, \tag{4.20}$$

where $g_n(x) = [t^n]\ \exp(x\ \tan\ t)$ and $g_n^*(x) = [t^n]\ \exp\ (x\ \arctan\ t)$.

Remark 4.1 Obviously, a weighted Stirling-type pair $\{\sigma(n,k), \sigma^*(n,k)\}$ defined in (3.1)-(3.2) satisfies the biorthogonal relation

$$\sum_{k\le n\le m} \sigma(m,n)\sigma^*(n,k) = \sum_{k\le n\le m} \sigma^*(m,n)\sigma(n,k) = \delta_{mk}$$

with δ_{mk} denoting the Kronecker delta, while the weighted Stirling number pair $\{S_1^{(\alpha)}(n,k), S_2^{(\alpha)}(n,k)\}$ does not satisfy the biorthogonal relation . Hence, from each element of the latter pair we can construct the corresponding weighted Stirling-type biorthogonal pair. In Section 3, we have seen that weighted Stirling-type pair $\{\sigma(n,k), \sigma^*(n,k)\}$ is equivalent to a Riordan array pair.

Remark 4.2 There is an interesting combinatorial interpretation of the relation (4.12). It is known that Broder's second kind of ℓ-Stirling number, denoted by $\left\{ {n \atop k} \right\}_\ell$ (*cf.* [3]), has its denotation and combinatorial meaning as follows. $\left\{ {n \atop k} \right\}_\ell =$ the number of partitions of the set $\{1, 2, \cdots, n\}$ into k non-empty disjoint subsets, such that the integers $1, 2, \cdots, \ell$ are in distinct subsets.

Of course, we may define $\left\{ {n \atop k} \right\}_\ell = 0$ whenever $k > n$ or $\ell > k$. As may be observed from [3], the numbers $S_2^{(\ell)}(n,k)$ as defined by (4.8) with $a = \ell$ are just equivalent to $\left\{ {n+\ell \atop k+\ell} \right\}_\ell$. Thus the equality (4.12) precisely means that $n!(Tos)_n^{(\ell)}(-1)$ gives the number of partitions of the set $\{1, 2, 3, \cdots, n+\ell\}$ into at least ℓ disjoint non-empty subsets such that the integers $1, 2, \cdots$, and ℓ are in distinct subsets.

Evidently, $S_2^{(0)}(n,k) = \left\{ {n \atop k} \right\}_0 = \left\{ {n \atop k} \right\}_1 = S_2^{(1)}(n,k)$, so that for $\ell = 0$ we have the particular case

$$n!(Tos)_n^{(0)}(-1) = W_n.$$

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