On the Generalized Möbius Inversion Formulas

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Abstract

We provide a wide class of Möbius inversion formulas in terms of the generalized Möbius functions and its application to the setting of the Selberg multiplicative functions.

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1 Introduction

This paper is concerned with the problem of construction for a general type of Möbius inversion formulas for the set of the generalized Möbius

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function μ_α. For other types of generalized Möbius inversion formulas can be seen in the recent article by Sándor and Bege [7]. In addition, an application of the Möbius inversion formulas in the physics can be found in [3].

For any $z \in \mathbb{C}$, a generalized Fleck-type Möbius function (*cf.* [4]) is defined by

$$\mu_z(t) := \Pi_p(-1)^{e_p([|t|])}\binom{z}{e_p([|t|])} \tag{1.1}$$

for any $t \in \mathbb{C}$, where $[|t|]$ is the integer part of $|t|$, p runs through all the prime divisors of $[|t|]$, and $e_p([|t|]) = ord_p([|t|])$ denotes the highest power k of p such that p^k divides $[|t|]$. Obviously, $\mu_1(n) = \mu(n)$, $n \in \mathbb{N}$, is the classical Möbius function.

[1] shows that μ_z $(z \in \mathbb{C})$ is a multiplicative function, i.e., if m and n are relatively prime, then $\mu_z(mn) = \mu_z(m)\mu_z(n)$. In addition, $\mu_0(n) = \delta_{1n}$ $(n \in \mathbb{N})$, the Kronecker symbol, i.e., $\mu_0(n)$ equals to 1 if $n = 1$ and 0 otherwise. It can be easily verified that, for each complex number z, μ_z is not a completely multiplicative function except for μ_0, which is completely multiplicative.

Let E denote an arbitrary arithmetical semigroup. The set of all complex-valued arithmetical functions on E will be denoted by $A(E)$. Note that the definition of an arithmetical semigroup E implies that E is countable. For the sake of definiteness, one could if desired express E in the form $E = \{a_1 = 1, a_2, a_3, \ldots\}$, where $|a_i| \le |a_{i+1}|$.

Let $h \in A(E)$ be a nonzero completely multiplicative function. We now give generalized Möbius inversion formulas for the set $M_h = \{\mu_{z,h} = \mu_z h : z \in \mathbb{C}\}$ with respect to the the generalized Dirichlet convolution $*$ defined as follows.

Definition 1.1 *Let $h \in A(E)$ be a nonzero completely multiplicative function. Given two functions $f, g \in A(E)$, the generalized Dirichlet convolution, denoted by $(f * g)_h$, is also in $A(E)$ which is defined by $(f * g)_h = (fh) * (gh)$, where $*$ is the Dirichlet convolution; i.e.,*

$$\begin{aligned}(f * g)_h(n) &:= ((fh) * (gh))(n) \\ &= \sum_{d|n}(fh)(d)(gh)\left(\frac{n}{d}\right) = \sum_{d|n}(fh)\left(\frac{n}{d}\right)(gh)(d)\end{aligned} \tag{1.2}$$

for $n = [|t|]$ with $t \in E$.

Next we recall a fairly more general notation of multiplicativity, introduced first by Selberg [8], which apparently did not prevail in the literature.

Definition 1.2 *A number-theoretical function F is said to be Selberg-multiplicative if, for each prime p, there exists $f_p : \mathbb{N}_0 \mapsto \mathbb{C}$ with $f_p(0) = 1$ for all but finitely many p such that*

$$F(n) = \Pi_p f_p(e_p(n)) \tag{1.3}$$

holds for every $n \in \mathbb{N}$. The class of all Selberg-multiplicative functions is denoted by $\mathcal{G}$.

Definition 1.3 *A Dirichlet formal series $\hat{f}(s)$ $(s \in \mathbb{C})$ of the sequence $\{F(n)\}$ is defined by*

$$\hat{f}(s) = \sum_{n=1}^{\infty} F(n) n^{-s} \quad (s \neq 0).$$

One of the main advantages of this more general notation of multiplicativity is that it can be used without change to define multiplicative functions of several variables.

In next section, we will show that $(M_h, *)$ is a group with the identity element $\mu_{0,h} = \mu_0 h$ $(\mu_0(n) = \delta_{n1})$ and establish a kind of Möbius type inversion. We shall also apply these results to the setting of Selberg-multiplicative functions class $(\mathcal{G}_h, *) = \{Fh : F \in \mathcal{G}\}$ in Section §3, where $h \in A(E)$ is a nonzero completely multiplicative function with $h(0) = 1$. Hence, $h(n) = \Pi_p h_p(e_p(n))$ with $h_p(e_p(n)) := h(p^{e_p(n)})$ and $h_p(0) = 1$. The application is based on the following lemmas.

Lemma 1.4 *Let $\mathcal{G}$ be the Selberg class of multiplicative functions defined in Definition 1.2. If $F \in \mathcal{G}$, then for $s \in \mathbb{C}$ we have the fomal identity*

$$\sum_{n=1}^{\infty} F(n) n^{-s} \equiv \sum_{n=1}^{\infty} n^{-s} \Pi_p f_p(e_p(n)) = \Pi_p \sum_{r=0}^{\infty} f_p(r) p^{-rs}, \tag{1.4}$$

in which the product on the rightmost extends over all prime numbers.

Proof. We can denote $n = \Pi_p p^{e_p(n)}$. Clearly we have

$$LHS\ of\ (1.4) = \sum_{n=1}^{\infty} \frac{\Pi_p f_p(e_p(n))}{\Pi_p p^{se_p(n)}} = \sum_{n=1}^{\infty} \Pi_p \frac{f_p(e_p(n))}{p^{se_p(n)}}$$

and

$$RHS\ of\ (1.4) = \Pi_p \left\{ \frac{f_p(0)}{p^{0\cdot s}} + \frac{f_p(1)}{p^{1\cdot s}} + \frac{f_p(2)}{p^{2\cdot s}} + \frac{f_p(3)}{p^{3\cdot s}} + \cdots \right\}.$$

In addition, on the LHS of (1.4), for every $n = p_1^{r_1} p_2^{r_2} p_3^{r_3} \cdots$, where $r_j \geq 0$ and $2 \leq p_1 < p_2 < \cdots$ are primes, the term

$$\frac{f_{p_1}(r_1) f_{p_2}(r_2) f_{p_3}(r_3) \cdots}{p_1^{sr_1} p_2^{sr_2} p_3^{sr_3} \cdots} \tag{1.5}$$

occurs exactly once in the expansion of the RHS of (1.4).

On the other hand, in the RHS expansion, every term with the form (1.5), corresponds to a term with $n = p_1^{r_1} p_2^{r_2} p_3^{r_3} \cdots$. Hence, the lemma holds.

■

Lemma 1.4 has its own importance, which can be seen from its special case shown as in Theorem 2.6.1 in [9].

From Definition 1.3 and product of series, we immediately have the following lemma.

Lemma 1.5 *Given formal series* $\hat{f}(s) = \sum_{n=1}^{\infty} F(n) n^{-s}$ *and* $\hat{g}(s) = \sum_{n=1}^{\infty} G(n) n^{-s}$ $(s \neq 0)$. *Then* $\hat{f}(s) \cdot \hat{g}(s)$ *generates the sequence* $\{\sum_{d|n} F\left(\frac{n}{d}\right) G(d)\}$.

2 A Generalized Möbius Inversion Formula

Theorem 2.1 *The generalized Dirichlet convolution is commutative and satisfies the associative law. In addition,* $(f * g)_h$ *is also a multiplicative function whenever* f *and* g *are multiplicative functions.*

Proof. Evidently, the generalized Dirichlet convolution is commutative. The associativity can be established from the following statement.

$$
\begin{aligned}
((f*g)_h * u)_h(n) &= \sum_{d|n} (f*g)_h(d)(uh)\left(\frac{n}{d}\right) \\
&= \sum_{d|n}\sum_{c|d}(fh)(c)(gh)\left(\frac{d}{c}\right)(uh)\left(\frac{n}{d}\right) \\
&= \sum_{c|n}\sum_{\ell|\frac{n}{c}}(fh)(c)(gh)(\ell)(uh)\left(\frac{n}{c\ell}\right) \quad (\ell = \frac{d}{c}) \\
&= \sum_{c|n}(fh)(c)\sum_{\ell|\frac{n}{c}}(gh)(\ell)(uh)\left(\frac{n}{c\ell}\right) \\
&= \sum_{c|n}(fh)(c)(g*u)_h\left(\frac{n}{c}\right) \\
&= (f*(g*u)_h)_h(n).
\end{aligned}
$$

We now prove that $(f*g)_h$ is also multiplicative whenever f and g are multiplicative functions. If $(a,b)=1$, then

$$
(f*g)_h(ab) = \sum_{d|ab}(fh)(d)(gh)\left(\frac{ab}{d}\right).
$$

Let $u=(a,d)$ and $v=(b,d)$. Then $uv=d$, $(u,v)=1$, and

$$
\begin{aligned}
(f*g)_h(ab) &= \sum_{u|a}\sum_{v|b}(fh)(uv)(gh)\left(\frac{ab}{uv}\right) \\
&= \sum_{u|a}(fh)(u)(gh)\left(\frac{a}{u}\right)\sum_{v|b}(fh)(v)(gh)\left(\frac{b}{v}\right) \\
&= (f*g)_h(a)(f*g)_h(b).
\end{aligned}
$$

This completes the proof of the theorem.

■

Theorem 2.2 *Let $h \in A(E)$ be a nonzero completely multiplicative function, and let $*$ be the Dirichlet convolution. Then $(M_h, *)$ is an Abelian group with the identity element $\mu_{0,h} = \mu_0 h$.*

Proof. First, the set M_h is closed with respect to $*$: for all $\mu_{\alpha,h}, \mu_{\beta,h} \in M_h$,

$$
\begin{aligned}
(\mu_{\alpha,h} * \mu_{\beta,h})(n) &= \sum_{d|n} \mu_\alpha(d) h(d) \mu_\beta\left(\frac{n}{d}\right) h\left(\frac{n}{d}\right) \\
&= h(n) \sum_{d|n} \mu_\alpha(d) \mu_\beta\left(\frac{n}{d}\right) \\
&= \mu_{\alpha+\beta,h}(n), \qquad (2.1)
\end{aligned}
$$

where the last step is due to Lemma 2 in [1].

Secondly, the convolution $*$ is commutative and associative in M_h from Theorem 2.1. Finally, from (2.1) the identity element of $(M_h, *)$ is $\mu_{0,h} = \mu_0 h$ and the inverse of $\mu_{\alpha,h}$ is $\mu_{-\alpha,h}$. This completes the proof of Theorem 2.2.

■

By using Theorem 2.2, we immediately obtain a generalized Möbius inversion formula as follows.

Theorem 2.3 *Let $h \in A(E)$ be any nonzero completely multiplicative function with $h(-1) = 1$, and let U, V denote complex-valued functions of a positive real variable t. Then*

$$V(t) = \sum_{k \in E; |k| \le t} \mu_z(k) U(t/|k|) h(k) \qquad (2.2)$$

for all $t > 0$, if and only if

$$U(t) = \sum_{k \in E; |k| \le t} \mu_{-z}(k) V(t/|k|) h(k) \qquad (2.3)$$

for all $t > 0$.

Proof. (2.2) and (2.3) are equivalent to the statemen

$$V = U * \mu_{z,h} \Longleftrightarrow U = V * \mu_{-z,h},$$

which follows from

$$V = U * \mu_{z,h} \Longleftrightarrow V * \mu_{-z,h} = (U * \mu_{z,h}) * \mu_{-z,h} = U * \mu_{0,h} = U.$$

Remark 1. Various consequences including some classical inversion formulas can be deduced from Theorem 2.3 by special choices of E, z, and $h(\cdot)$. In particular, the classical Möbius inversion pair is a particular case of (2.2) and (2.3) shown in Theorem 2.3 with $E = \mathbb{N}$, $z = 1$, and $h(\cdot) \equiv 1$ (*cf.* [6] for a different approach).

Corollary 2.4 *Let $t_0 \geq 1$, and let $h(k)$ be a nonzero completely multiplicative function. Then for all t, $1 \leq t \leq t_0$, and complex-valued functions F and G,*

$$G(t) = \sum_{1 \leq k \leq t} \mu_z(k) F(t/k) h(k) \tag{2.4}$$

if and only if

$$F(t) = \sum_{1 \leq k \leq t} \mu_{-z}(k) G(t/k) h(k). \tag{2.5}$$

In Corollary 2.4, we may take

$$F(t) = \begin{cases} f(t) & \text{if } t \in \mathbb{Z}, \\ 0 & \text{if } t \notin \mathbb{Z}, \end{cases} \qquad G(t) = \begin{cases} g(t) & \text{if } t \in \mathbb{Z}, \\ 0 & \text{if } t \notin \mathbb{Z}. \end{cases}$$

Hence, for $n \in \mathbb{N}$ we write (2.4) as

$$\begin{aligned} g(n) &= G(n) = \sum_{1 \leq k \leq n} \mu_z(k) F(n/k) h(k) \\ &= \sum_{k|n} \mu_z(k) f(n/k) h(k) = \sum_{d|n} \mu_z(n/d) f(d) h(n/d). \end{aligned}$$

Similarly, from (2.5),

$$\begin{aligned} f(n) &= F(n) = \sum_{k|n} \mu_{-z}(k) g(n/k) h(k) \\ &= \sum_{d|n} \mu_{-z}(n/d) g(d) h(n/d). \end{aligned}$$

Therefore, we obtain the following corollary.

Corollary 2.5 *Let h be any nonzero completely multiplicative function, and let f and g be complex-valued functions. Then for all $n \in \mathbb{N}$,*

$$g(n) = \sum_{k|n} \mu_z(k) f(n/k) h(k) \tag{2.6}$$

if and only if

$$f(n) = \sum_{k|n} \mu_{-z}(k) g(n/k) h(k). \tag{2.7}$$

Remark 2. Corollary 2.5 can be derived by using a similar argument as that in [1] with the consideration of the Abelian group $M_h = \{\mu_{z,h} = \mu_z h : z \in \mathbb{C}\}$ with respect to the addition $*$ defined by

$$(\mu_{\alpha,h} * \mu_{\beta,h})(n) := \sum_{k|n} \mu_\alpha(k) h(k) \mu_\beta(n/k) h(n/k) = \mu_{\alpha+\beta,h}(n),$$

where h is a nonzero completely multiplicative function.

In Theorem 2.3, after substituting $F(t) = f(1/t)$ and $G(t) = g(1/t)$ into (2.2) and (2.3) and replacing $1/t$ by t afterwards, we can obtain the following results.

Theorem 2.6 *Let $h \in A(E)$ be any nonzero completely multiplicative function, and let f and g be complex-valued functions of positive real variable $t < 1$,*

$$g(t) = \sum_{k \in E; |k| \le 1/t} \mu_{-z}(k) f(|k|t) h(k) \tag{2.8}$$

for all $0 < t < 1$, if and only if

$$f(t) = \sum_{k \in E; |k| \le 1/t} \mu_z(k) g(|k|t) h(k) \tag{2.9}$$

for all $0 < t < 1$.

Remark 3. (2.8) and (2.9) can be proved directly as follows.

Proof. If (2.8) holds, then

$$\begin{aligned} &\sum_{|k|\le 1/t} \mu_z(k) g(|k|t) h(k) \\ = &\sum_{|k|\le 1/t} \mu_z(k) h(k) \sum_{|m|\le 1/(|k|t)} \mu_{-z}(m) f(|mk|t) h(m). \end{aligned}$$

By setting $mk = r$ on the right-hand side of the above equation and noting that $\mu_z * \mu_{-z} := \sum_{mk=r} \mu_z(m)\mu_{-z}(k) = \mu_0(r)$, we can write it as follows.

$$\begin{aligned} &\sum_{|mk|\le 1/t} \mu_z(k)\mu_{-z}(m) h(mk) f(|mk|t) \\ = &\sum_{|r|\le 1/t} h(mk) f(|mk|t) \sum_{mk=r} \mu_z(k)\mu_{-z}(m) \\ = &\ h(1)f(t) = f(t). \end{aligned}$$

Similarly, from (2.9) we can derive (2.8).

If we choose $E = \mathbb{N}$, then we have

Corollary 2.7 *Let $0 < n_0 \le n_1$, and let $h(k)$ be any nonzero completely multiplicative function. For every $n_0 \le t \le n_1$ and complex-valued functions f and g,*

$$g(t) = \sum_{1\le k\le n_1/t} \mu_{-z}(k) f(kt) h(k) \tag{2.10}$$

if and only if

$$f(t) = \sum_{1\le k\le n_1/t} \mu_z(k) g(kt) h(k). \tag{2.11}$$

Remark 4. A special case of $h \equiv 1$ and $z = 1$ can be found in Theorems 268 and 270 of Section §16.5 in [5].

3 A Möbius Inversion Formula for the Setting of the Selberg Multiplicative Functions

We now give an application of Theorem 2.2 to the setting of Selberg-multiplicative functions. Similar to Theorems 2.1 and 2.2, for a nonzero completely multiplicative function h with $h(0) = 1$, $A(E)_h := \{fh : f \in A(E) - \{0\}\}$ forms an Abelian group with respect to the Dirichlet convolution defined in Definition 1.1. We will show that $(\mathcal{G}_h, *)$ is a subgroup of $(A(E)_h, *)$.

Let $F, G \in \mathcal{G}$; i.e., for any $n \in \mathbb{N}$, $F(n) = \Pi_p f_p(e_p(n))$ and $G(n) = \Pi_p g_p(e_p(n))$, where for each prime p $f_p, g_p : \mathbb{N}_0 = \mathbb{N} \cup \{0\} \mapsto \mathbb{C}$ with $f_p(0) = g_p(0) = 1$. For each prime p and a fixed nonzero completely multiplicative function h_p with $h_p(0) = 1$, we define

$$q_p(r) := h_p(r) \sum_{\rho=0}^{r} f_p(\rho) g_p(r-\rho) \ (r \in \mathbb{N}_0). \tag{3.1}$$

Thus $q_p(0) = 1$. Furthermore, since $Q := (F * G)_h \in A(E)_h$, where $h(n) = \Pi_p h_p(e_p(n))$ and $h_p(e_p(n)) := h(p^{e_p(n)})$, we have

$$\begin{aligned} Q(n) &= \Pi_p Q(p^{e_p(n)}) = \Pi_p \sum_{\rho=0}^{e_p(n)} F(p^\rho) G(p^{e_p(n)-\rho}) h(p^{e_p(n)}) \\ &= \Pi_p \sum_{\rho=0}^{e_p(n)} f_p(\rho) g_p(e_p(n)-\rho) h_p(e_p(n)) = \Pi_p q_p(e_p(n)), \end{aligned}$$

where the last two equalities are derived from the expressions of F, G, and H and (3.1). Consequently, $(F * G)_h \in \mathcal{G}$ if $Fh, Gh \in \mathcal{G}_h$.

To prove that $Fh \in \mathcal{G}_h$ implies its inverse $F^{-1}h \in \mathcal{G}_h$, for each prime p we define

$$g_p(0) = 1, \ \sum_{\rho=0}^{r} f_p(\rho) g_p(r-\rho) = 0 \tag{3.2}$$

for $r = 1, 2, \cdots$. Since $f_p(0) = 1$ this is uniquely possible. From $(F * F^{-1})_h = \epsilon$, the identity of $(A(E)_h, *)$, we have

$$0 = \epsilon(p^r) = h(p^r) \sum_{\rho=0}^{r} F(p^\rho) F^{-1}(p^{r-\rho}) = h(p^r) \sum_{\rho=0}^{r} f(\rho) F^{-1}(p^{r-\rho})$$

for each prime p and $r \in \mathbb{N}$. Then from (3.2), we immediately obtain $F^{-1}(1) = 1 = g_p(0)$ and $F^{-1}(p^r) = g_p(r)$ for each prime p and $r \in \mathbb{N}$. Consequently,

$$F^{-1}(n) = \Pi_p F^{-1}(p^{e_p(n)}) = \Pi_p g_p(e_p(n)), \tag{3.3}$$

where $F^{-1} \in \mathcal{G}$. Hence, constructed $F^{-1}h$ is the inverse of Fh in $\mathcal{G}_h$.

(3.2) also give an algorithm to find F^{-1}.

We now use Lemmas 1.4 and 1.5 to derive the Möbius inversion formulas over $(\mathcal{G}_h, *)$ as follows.

Theorem 3.1 *Let either sequence of* $\{\alpha(n)\}$ *and* $\{\beta(n)\}$ *be given arbitrarily. For* $F \in \mathcal{G}$*, its inverse* $G \in (\mathcal{G}_h, *)$ *exists so that the general type of Möbius inversion formulas*

$$\beta(n) = \sum_{d|n} F(d)\alpha\left(\frac{n}{d}\right) h(d) \Longleftrightarrow \alpha(n) = \sum_{d|n} G(d)\beta\left(\frac{n}{d}\right) h(d) \tag{3.4}$$

hold for any nonzero completely multiplicative function h *with* $h(0) = 1$ *and for* $\alpha, \beta : \mathbb{N} \mapsto \mathbb{C}$.

Proof. For given $F \in \mathcal{G}$, $F(n) = \Pi_p f_p(e_p(n))$, from Lemma 1.4 we have

$$\hat{f}_h(s) := \sum_{n=1}^{\infty} F(n)h(n)n^{-s} = \Pi_p \sum_{p}^{\infty} f_p(e_p(n))h_p(e_p(n)), \tag{3.5}$$

where $\hat{f}_h(s)$ is the Dirichlet formal series of the sequence $\{F(n)h(n)\}$ (see Definition 1.3). Solving system (3.2) or its equivalent form

$$\left(\sum_{r=1}^{\infty} f_p(r)h_p(r)x^r\right)^{-1} = \sum_{r=1}^{\infty} g_p(r)h_p(r)x^r \tag{3.6}$$

for each prime p and $r = 1, 2, \cdots$. From (3.3), the inverse of $F(n)h(n)$ in $\mathcal{G}_h, *)$ can be written as $G(n)h(n)$, where

$$G(n) = \Pi_p g_p(e_p(n)) \ (g_p(0) = 1). \tag{3.7}$$

Denote by $\hat{g}_h(s)$ the Dirichlet formal series of the sequence $\{G(n)h(n)\}$. From Lemma 1.4, we obtain

$$\hat{g}_h(s) := \sum_{n=1}^{\infty} G(n)h(n)n^{-s} = \Pi_p \sum_{p}^{\infty} g_p(e_p(n))h_p(e_p(n)). \tag{3.8}$$

Since (3.6) implies

$$\hat{g}_h(s) = 1/\hat{f}_h(s), \tag{3.9}$$

for the Dirichlet formal series, $\hat{\alpha}(s) : \mathbb{N} \mapsto \mathbb{C}$, of any given sequence $\{\alpha(n)\}$ we have $\hat{\beta}(s) := \hat{f}_h(s)\hat{\alpha}(s)$ implies $\hat{\alpha}(s) = \hat{g}_h(s)\hat{\beta}(s)$, and the last expression implies the previous one if $\hat{\beta}(s)$ is indicated, namely

$$\hat{\beta}(s) = \hat{f}_h(s)\hat{\alpha}(s) \Longleftrightarrow \hat{\alpha}(s) = \hat{g}_h(s)\hat{\beta}(s). \tag{3.10}$$

Obviously, from Lemma 1.5, inversion relation (3.10) is equivalent to (3.4). This completes the proof of the theorem.

■

Remark 5. A special case of (3.4) for $h \equiv 1$ and a subgroup of $\mathcal{G} \equiv \mathcal{G}_1$ can be found in [2].

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