Generalized Zeta Functions

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Abstract

We provide a wide class of generalized zeta function in terms of the generalized Möbius functions and its properties.

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1 Introduction

For any integer $z \in \mathbb{C}$, a Fleck-type generalized Möbius function (*cf.* [3]) of order z can be defined by

$$\mu_z(n) := \Pi_p (-1)^{e_p(n)} \binom{z}{e_p(n)} \tag{1.1}$$

for any $n \in \mathbb{N}$, where p runs through all the prime divisors of n, and $e_p(n) = ord_p(n)$ denotes the highest power k of p such that p^k divides n. Obviously, $\mu_1(n) = \mu(n)$, $n \in \mathbb{N}$, is the classical Möbius function: $\mu(1) = 1$; if n is not square free then $\mu(n) = 0$; if n is square free and if q is the number of distinct primes dividing n, then $\mu(n) = (-1)^q$. In addition,

$$\mu_0 = \Pi_{p|n} (-1)^{e_p(n)} \binom{0}{e_p(n)} = \left\{ \begin{array}{ll} 1 & n = 1, \\ 0 & n > 1, \end{array} \right.$$

and

$$\mu_{-1} = \Pi_p(-1)^{e_p(n)} \binom{-1}{e_p(n)} = \Pi_{p|n} \frac{(e_p(n))!}{(e_p(n))!} = 1.$$

It is easy to verify that for each complex number α, μ_α is a multiplicative function, but is not complete multiplicative except μ_0, which is complete multiplicative.

The generalized zeta function, denoted by ξ_z, is defined accordingly by

$$\xi_z(s) = 1/\sum_{n\geq 1} \frac{\mu_z(n)}{n^s}, \tag{1.2}$$

where $z \in \mathbb{C}$. Hence, $\xi_1 = \xi$ the classical zeta function. And $\xi_0 = 1$.
Remark 1 $\xi_z(s)$ defined in (1.2) can be extended to $\mathbb{C}$. However, throughout this note, we do not consider the zero points of $\xi_1(s)$ in its domain, i.e., the points at which $\sum_{n\geq 1} \frac{\mu_1(n)}{n^s}$ diverges to infinite.

In this note, we will show that the set of functions ξ_α ($\alpha \in \mathbb{C}$) forms an Abelian group with the Dirichlet series multiplication followed by a number of applications.

2 Generalized zeta function group

We now recall the definition of the Dirichlet product (or convolution) of two arithmetic functions f and g (*cf* [1], [2]).

Definition 2.1 *Given two arithmetic functions f and g, the Dirichlet (convolution) product $f * g$ is again an arithmetic function which is defined by*

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d), \tag{2.1}$$

where the summations are taken over all positive divisors d of n.

Definition 2.2 *Denote*

$$M := \{\mu_z : z \in \mathbb{C}\},$$

where $\mathbb{C}$ denotes the set of complex numbers. We call M is called the set of generalized Möbius functions of complex order. The set, denoted by

N, of the corresponding nonzero generalized zeta functions of complex order is defined by

$$N := \{\xi_z : z \in \mathbb{C}\},$$

where ξ_z are presented in (1.2).

From [2, 5], $(M, *)$ forms an Abelian group with identity element μ_0 under the operation $* : M \times M \mapsto M$,

$$\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta},$$

where $\alpha, \beta \in \mathbb{C}$.

Lemma 2.3 *For any given $\alpha, \beta \in \mathbb{C}$, we define $\cdot : N \times N \mapsto N$ by*

$$\xi_\alpha \cdot \xi_\beta = \xi_\gamma$$

for some $\gamma \in \mathbb{C}$ if

$$\frac{1}{\xi_\alpha} * \frac{1}{\xi_\beta} = \frac{1}{\xi_\gamma},$$

where $*$ is the regular Dirichlet product of Dirichlet series (cf., for example, [8]). Thus, we have

$$\xi_\alpha \cdot \xi_\beta = \xi_{\alpha+\beta}.$$

Proof. By writing

$$1/\xi_\alpha(s) = \sum_{n\geq 1} \frac{\mu_\alpha(n)}{n^s}, \text{ and } 1/\xi_\beta(s) = \sum_{n\geq 1} \frac{\mu_\beta(n)}{n^s},$$

we obtain

$$\begin{aligned}
1/(\xi_\alpha\xi_\beta)(s) &= \sum_{n\geq 1} \left(\sum_{d|n} \mu_\alpha(d)\mu_\beta\left(\frac{n}{d}\right)\right) /n^s \\
&= \sum_{n\geq 1} \left(\mu_\alpha * \mu_\beta\right)/n^s \\
&= \sum_{n\geq 1} \frac{\mu_{\alpha+\beta}}{n^s} = 1/\xi_{\alpha+\beta}(s).
\end{aligned}$$

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We now ready to show $(N,\cdot)$ is an Abelian group.

Theorem 2.4 *Let $\cdot$ be the operation define in Lemma 2.3. Then $(N,\cdot)$ is an Abelian group with identity element $\xi_0 = 1$.*

Proof. From Lemma 2.3, we see that N is closed respect to the operation $\cdot$. Moreover, for any α and $\beta, \in \mathbb{C}$, we have

$$\xi_\alpha \cdot \xi_\beta = \xi_{\alpha+\beta} = \xi_\beta \cdot \xi_\alpha.$$

And for any α, β, and $\gamma, \in \mathbb{C}$,

$$\begin{aligned}
(\xi_\alpha \cdot \xi_\beta) \cdot \xi_\gamma &= \xi_{\alpha+\beta} \cdot \xi_\gamma \\
&= 1/\sum_{n\geq 1}\left(\sum_{d|n} \mu_{\alpha+\beta}(d)\mu_\gamma\left(\frac{n}{d}\right)\right)/n^s \\
&= 1/\sum_{n\geq 1}\frac{\mu_{\alpha+\beta+\gamma}}{n^s} = \xi_{\alpha+\beta+\gamma}(s).
\end{aligned}$$

Similarly, $\xi_\alpha \cdot (\xi_\beta \cdot \xi_\gamma) = \xi_{\alpha+\beta+\gamma}$. Thus,

$$(\xi_\alpha \cdot \xi_\beta) \cdot \xi_\gamma = \xi_\alpha \cdot (\xi_\beta \cdot \xi_\gamma).$$

It is also easy to check $\xi_\alpha \cdot 1 = 1 \cdot \xi_\alpha = \xi_\alpha$ and

$$\xi_\alpha \cdot \xi_{-\alpha} = \xi_{-\alpha} \cdot \xi_\alpha = 1.$$

Therefore, the theorem is proved.

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From Theorem 2.4 and Equation (1.2) we have

Corollary 2.5 *For all $\alpha \in \mathbb{Z}$*

$$\xi_\alpha(s) = (\xi(s))^\alpha,$$

where $(\xi(s))^\alpha := \xi(s)(\xi(s))^{\alpha-1}$.

Theorem 2.6 *Group $(M,*)$ and $(N,\cdot)$ are isomorphic.*

Proof. Mapping $\phi : M \mapsto N$ is defined by

$$\phi(\mu_\alpha) := \xi_\alpha = 1/\sum_{n\geq 1} \frac{\mu_\alpha(n)}{n^s},$$

where $\alpha \in \mathbb{C}$. It is easy to verify that the mapping is one-to-one and onto [[a subspace of M in which $\sum_{n\geq 1} \mu_\alpha(n)/n^s$ is not divergent]]. In addition, for any $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} \phi(\mu_\alpha * \mu_\beta) &= \phi(\mu_{\alpha+\beta}) = \xi_{\alpha+\beta} \\ &= \xi_\alpha \cdot \xi_\beta = \phi(\mu_\alpha) \cdot \phi(\mu_\beta). \end{aligned}$$

This completes the proof.

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3 Some results from generalized zeta function group

A series $\sum_{n\geq 1} a_n n^{-s}$ is called an arithmetic Dirichlet series if all of its coefficients $a_n = a(n)$ are arithmetic functions.

Theorem 3.1 *(Generalized zeta inversion formulae) For all $\alpha \in \mathbb{C}$ and Dirichlet series f and g,*

$$f = \xi_\alpha g \Leftrightarrow g = \xi_{-\alpha} f. \tag{3.1}$$

Moreover, if both $f = \sum_{n\geq 1} f_n n^{-s}$ and $g = \sum_{n\geq 1} g_n n^{-s}$ are arithmetic Dirichlet series, then for any $n \in \mathbb{N}$

$$f(n) = \sum_{d|n} \mu_\alpha\left(\frac{n}{d}\right) g(d) \Leftrightarrow g(n) = \sum_{d|n} \mu_{-\alpha}\left(\frac{n}{d}\right) f(d). \tag{3.2}$$

Proposition 3.2 *For all $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$,*

$$\sum_{d|n} \mu_{\alpha-1}(d) = \mu_\alpha(n). \tag{3.3}$$

Proof. From Corollary 2.5

$$\begin{aligned}\sum_{n\geq 1}\frac{\mu_\alpha(n)}{n^s} &= \frac{1}{\xi_\alpha(s)} = \frac{1}{\xi_{\alpha-1}(s)}\cdot\sum_{n\geq 1}\frac{\mu_{-1}(n)}{n^s}\\ &= \sum_{n\geq 1}\frac{\mu_{\alpha-1}(n)}{n^s}\cdot\sum_{n\geq 1}\frac{\mu_{-1}(n)}{n^s}\\ &= \sum_{n\geq 1}\sum_{d|n}\mu_{\alpha-1}(d)\mu_{-1}\left(\frac{n}{d}\right)/n^s,\end{aligned}$$

which leads (3.3) by applying the Dirichlet series multiplication and noting that $\mu_{-1}\equiv 1$.

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Proposition 3.3 *Let* $f = \sum_{n\geq 1} c_n n^{-s}$*, and let all* c_n *are completely multiplicative functions. For any fixed positive integer* α*,*

$$f^{\alpha-1}\sum_{n\geq 1}\frac{\mu_\alpha(n)c_n}{n^{-s}} = \sum_{n\geq 1}\frac{\mu_0(n)c_n}{n^{-s}}. \tag{3.4}$$

Proof. This follows easily from Proposition 3.2 and mathematical induction on α. In fact, first we have

$$\begin{aligned} f\sum_{n\geq 1}\frac{\mu_\alpha(n)c_n}{n^{-s}} &= \sum_{n\geq 1}\frac{((\mu_\alpha c)*c)(n)}{n^{-s}}\\ &= \sum_{n\geq 1}\frac{c(n)\sum_{d|n}\mu_\alpha(d)}{n^{-s}}\\ &= \sum_{n\geq 1}\frac{c(n)\mu_{\alpha-1}(n)}{n^{-s}}.\end{aligned}$$

Secondly, using mathematical induction on α we obtain (3.4).

■

It is known (*cf.*, for examples, [4] and [6]) that for any fixed integer $\alpha\geq 1$

$$\xi_1^\alpha(s) = \left(\sum_{n\geq 1} \mu(n) n^{-s}\right)^\alpha = \sum_{n\geq 1} r_\alpha(n) n^{-s}, \tag{3.5}$$

where

$$r_\alpha(n) = \sum_{n_1 n_2 \cdots n_\alpha = n} 1$$

is the number of ways that n can be written as a product of α fixed factors, so that $r_\alpha(n)$ is clearly a multiplicative function of n. In particular, $r_2(n)$ denotes the number of positive divisors of n.

Theorem 3.4 *(Characteristic of generalized Möbius functions) For any fixed integer $\alpha \geq 1$, the inverse of μ_α in the group $(M, *)$ is r_α; or equivalently, $r_\alpha = \mu_{-\alpha}$, i.e., for all integers $n \geq 1$*

$$(r_\alpha * \mu_\alpha)(n) = \sum_{d|n} r_\alpha(d) \mu_\alpha(n/d) = \delta_{n,1}, \tag{3.6}$$

where $\delta_{n,1} = 1$ if $n = 1$ and 0 otherwise.

Proof. Multiplying $\xi_1^\alpha(s)$ shown as in (3.5) with

$$\frac{1}{\xi_\alpha(s)} = \frac{1}{\xi_1^\alpha(s)} = \sum_{n\geq 1} \frac{\mu_\alpha(n)}{n^s}$$

yields

$$\sum_{n\geq 1} \frac{(r_\alpha * \mu_\alpha)(n)}{n^s} = 1,$$

which leads $r_\alpha(1)\mu_\alpha(1) = 1$ and

$$(r_\alpha * \mu_\alpha)(n) = \sum_{d|n} r_\alpha(d) \mu_\alpha(n/d) = 0$$

for $n \geq 2$, completing the proof.

■

Denote

$$F_\alpha(s) = \sum_{n\geq 1} r_\alpha(n) n^{-s}. \tag{3.7}$$

Then we obtain

$$\begin{aligned} F_\alpha(s)\xi_{1-\alpha}(s) &= \sum_{n\geq 1}\sum_{d|n} r_\alpha(d)\mu_{1-\alpha}\left(\frac{n}{d}\right)/n^s = \xi(s) \ and \\ F_\alpha(s)\xi_{-\alpha}(s) &= (\xi_1)^\alpha(\xi_1)^{-\alpha} = \xi_1^0 = \xi_0(s) = 1, \end{aligned}$$

i.e., identities (3.6) and

$$\sum_{d|n} r_\alpha(d)\mu_{1-\alpha}\left(\frac{n}{d}\right) = 1.$$

In particular, for $r_2 = \sum_{d|n} 1$, the number of positive divisors of n, from $\mu_{-2}(n) = r_2(n)$ we obtain

$$F_2(s)\xi_{-1}(s) = 1 \quad and \quad F_2(s)\xi_{-2}(s) = \xi_0(s) = 1,$$

i.e.,

$$\sum_{d|n} r_2(d)\mu\left(\frac{n}{d}\right) = 1 \quad and \quad \sum_{d|n} r_2(d)\mu_2\left(\frac{n}{d}\right) = \delta_{n,1}.$$

References

[**1**] Apostol, T., Möbius functions of rank k, Pacific J. Math., 32(1970), 21-27.

[**2**] Brown, T.C., Hsu, L.C., Wang, J., and Shiue, P.J.S., On a certain kind of generalized number-theoretical Möbius function, Math. Sci., **25**(2000), 72-77.

[**3**] Fleck, A., Über gewisse allgemeine zahlentheoretische Funktionen, insbesondere eine der Funktion $\mu(n)$ verwandte Funktion $\mu_k(m)$, S.-B. Berlin. Math. Ges 15, 1916, 3-8.

[**4**] Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, fifth edition, Oxford Science Publications, 1979.

[5] He, T. X., Hsu, L. C., and Shiue, P. J.-S., On Generalised Möbius Inversion Formulas, Bulletin AMS, (2006),

[6] Knopfmacher, J., *Abstract Analytic Number Theory*, North-Holland Publishing Co., Amsterdam, 1975.

[7] Sándor, J. and Bege, A., The Möbius Function: Generalizations and Extensions, *Adv. Stud. Contemp. Math. (Kyungshang)*, **6**(2003), No.2, 77-128.

[8] H. S. Wilf, Generatingfunctionology, Acad. Press, New York, 1990.