A Symbolic Operator Approach to Power Series Transformation-Expansion Formulas

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Abstract
Here expounded is a kind of symbolic operator method by making use of the defined Sheffer-type polynomial sequences and their generalization, which can be used to construct many power series transformation and expansion formulas. The convergence of the expansions are also discussed.

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(Concerned with sequences A000262, A052844, A084358, A129652, A129653, and A133289.)

1 Introduction
The closed form representation of series has been studied extensively. See, for examples, Comtet [1], Ch. Jordan [11], Egorechev [2], Roman-Rota [15], Sofo [17], Wilf [18], Petkovšek-Wilf-Zeilberger’s book [13],
"A=B," and the author’s recent work with Hsu, Shiue, and Toney [4].
This paper is a sequel to the work [4] and the paper with Hsu and Shiue [5], in which the main results are special cases of Theorem 2.1 shown below. The object of this paper is to make use of the following defined generalized Sheffer-type polynomial sequences and the classical operators $\Delta$ (difference), $E$ (shift), and $D$ (derivative) to construct a method for the summation of power series expansions that appears to have a certain wide scope of applications.

As an important tool using in the Calculus of Finite Differences and in Combinatorial Analysis, the operators $E$, $\Delta$, $D$ are defined by the following relations.

$$
Ef(t) = f(t+1), \quad \Delta f(t) = f(t+1) - f(t), \quad Df(t) = \frac{d}{dt}f(t).
$$

Powers of these operators are defined in the usual way. In particular for any real numbers $x$, one may define $E^x f(t) = f(t + x)$. Also, the number 1 may be used as an identity operator, viz. $1f(t) \equiv f(t)$. Then it is easy to verify that these operators satisfy the formal relations (cf. [11])

$$
E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1 + \Delta).
$$

Note that $E^k f(0) = \left[E^k f(t)\right]_{t=0} = f(k)$, so that $(xE)^k f(0) = f(k)x^k$. This means that $(xE)^k$ with $x$ as a parameter may be used to generate a general term of the series $\sum_{k=0}^{\infty} f(k)x^k$.

**Definition 1.1** Let $A(t)$, $B(t)$, and $g(t)$ be any formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $A(0) = 1$, $B(0) = 1$, $g(0) = 0$, and $g'(0) \neq 0$. Then the polynomials $p_n(x)$ ($n = 0, 1, 2, \cdots$) defined by the generating function (GF)

$$
A(t)B(xg(t)) = \sum_{n \geq 0} p_n(x)t^n
$$

(1.1)

are called generalized Sheffer-type polynomials associated with $(A(t), B(t), g(t))$. Accordingly, $p_n(D)$ with $D \equiv d/dt$ is called Sheffer-type differential operator of degree $n$ associated with $A(t)$, $B(t)$, and $g(t)$. In particular, $p_0(D) \equiv I$ is the identity operator due to $p_0(x) = 1$. 

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In Definition 1.1, if \( B(t) = e^t \), then the defined \( \{ p_n(x) \} \) is a classical Sheffer-type polynomial sequence associated with \((A(t), g(t)) \equiv (A(t), \exp(t), g(t))\). As examples, classical Sheffer-type polynomials include Bernoulli polynomials, Euler polynomials, and Laguerre polynomials generalized by \((A(t), g(t)) = (\frac{t}{e^t - 1}, t), (\frac{2}{e^t + 1}, t), \) and \((\frac{(1 - t)^{-p}}{t}, \frac{t}{t - 1}) \) \((p > 0)\), respectively.

We call the infinite matrix \( [d_{n,k}]_{n,k \geq 0} \) with real entries or complex entries a generalized Riordan matrix (The originally defined Riordan matrices need \( g'(0) = 1 \)) if its \( k \)th column satisfies

\[
\sum_{n \geq 0} d_{n,k} t^n = A(t)(g(t))^k;
\]

that is,

\[
d_{n,k} = [t^n]A(t)(g(t))^k,
\]

the \( n \)th term of the expansion of \( A(t)(g(t))^k \). The Riordan matrix is denoted by \((A(t), g(t))\) or \([d_{n,k}]\) described in (1.2). Then the generalized Sheffer-type polynomial sequence associated with \((A(t), B(t), g(t))\) is the result of the following matrix multiplication

\[
\begin{bmatrix}
1 \\
b_1 x \\
b_2 x^2 \\
 \vdots \\
b_n x^n \\
 \vdots 
\end{bmatrix}
\]

If \([d_{n,k}] = [t^n]A(t)(g(t))^k\) and \([c_{n,k}] = [t^n]C(t)(f(t))^k\) are two Riordan matrices, and \( \{p_n(x)\} \) and \( \{q_n(x)\} \) are two corresponding generalized Sheffer-type polynomial sequences associated with \((A(t), B(t), g(t))\) and \((C(t), B(t), f(t))\), respectively, then we can define a umbral composition (cf. its special case of \( B(t) = e^t \) is given in [14] and [15]) between \( \{p_n(x)\} \) and \( \{q_n(x)\} \), denoted by \( \{p_n(x)\} \# \{q_n(x)\} \). The resulting sequence is the generalized Sheffer-type polynomial sequence associated with \((A(t)C(g(t)), B(t), f(g(t)))\). Clearly, the sequence \( \{p_n(x)\} \# \{q_n(x)\} \) is the result of the following matrix multiplication.
A power series $B(t) = 1 + \sum_{k=1}^{\infty} b_k t^k$ is said to be regular if $b_k \neq 0$ for all $k \geq 1$. Under the composition operator $\#$, it can be proved that all generalized Sheffer-type polynomial sequences associated with a regular $B(t)$ form a group, called the generalized Sheffer group associated with $B(t)$. Its verification is analogous to the classical Sheffer group associated with $e^t$ established in [14] (see also in [6]).

In [6], the author established the isomorphism between the classic Sheffer group and the Riordan group based on the following bijective mapping: $\theta : [d_{n,k}] \mapsto \{p_n(x)\}$ or $\theta : (A(t), g(t)) \mapsto \{p_n(x)\}$, i.e.,

$$\theta([d_{n,k}]_{n \geq k \geq 0}) := \sum_{j=0}^{n} d_{n,j} x^j / j! = [d_{n,k}]_{n \geq k \geq 0} X, \quad (1.3)$$

for fixed $n$, where $X = (1, x, x^2/2!, \ldots)^T$, or equivalently,

$$\theta((A(t), g(t))) := [t^n] A(t) e^{xg(t)} \quad (1.4)$$

It is clear that $(1, t)$, the identity Riordan array, maps to the identity Sheffer-type polynomial sequence $\{p_n(x) \equiv x^n/(n!)\}_{n \geq 0}$. From the definitions (1.2) we immediately know that

$$p_n(x) = [t^n] A(t) e^{xg(t)} \text{ if and only if } d_{n,k} = [t^n] A(t) (g(t))^k. \quad (1.5)$$

Therefore, the Riordan matrices from the sequences shown in the On-Line Encyclopedia of Integer Sequences (OLEIS) also present the coefficients of the corresponding classic Sheffer-type polynomials. For example, sequence A129652

$$1, 1, 1, 3, 2, 1, 13, 9, 3, 1, 73, 52, 18, 4, 1, \ldots$$

presents both Riordan array $(e^{x/(1-x)}, x)$ and the corresponding Sheffer-type polynomial sequence:
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\[ p_0(x) = 1 \]
\[ p_1(x) = 1 + x \]
\[ p_2(x) = 3 + 2x + x^2/2! \]
\[ p_3(x) = 13 + 9x + 3x^2/2! + x^3/3! \ldots \]

The row sums of Riordan array \( (e^{x/(1-x)}, x) \), i.e., the polynomial values at \( x = 1 \), are \( A_{052844} \), while diagonal sums of the array are \( A_{129653} \) (cf. [16]). Other examples including \( A_{000262} \), \( A_{084358} \), \( A_{133289} \), etc. can be also found in [16].

Suppose that \( \Phi(t) \) is an analytic function of \( t \) or a formal power series in \( t \), say

\[ \Phi(t) = \sum_{k=0}^{\infty} c_k t^k, \quad c_k = [t^k] \Phi(t), \quad (1.6) \]

where \( c_k \) can be real or complex numbers. Then, formally we have a sum of general form

\[ \Phi(xE)f(0) = \sum_{k=0}^{\infty} c_k f(k)x^k. \quad (1.7) \]

In certain cases, \( \Phi(\alpha + \beta) \) or \( \Phi(\alpha \beta) \) can be decomposed into something having a power series in \( \beta \) as a part. Accordingly the operator \( \Phi(xE) = \Phi(x + x\Delta) = \Phi(xe^D) \) can be expressed as some power series involving operators \( \Delta^k \) or \( D^k \)'s. Then it may be possible to compute the right-hand side of (1.7) by means of operator-series in \( \Delta^k \) or \( D^k \)'s.

This idea could be readily applied to various elementary functions \( \Phi(t) \). Therefore, we can obtain various transformation formulas as well as series expansion formulas for the series of the form (1.7).

It is well-known that the Eulerian fraction is a powerful tool to study the Eulerian polynomial, Euler function and its generalization, Jordan function (cf. [1]).

The classical Eulerian fraction can be expressed in the form

\[ \alpha_m(x) = \frac{A_m(x)}{(1-x)^{m+1}} \quad (x \neq 1), \quad (1.8) \]

where \( A_m(x) \) is the \( m \)th degree Eulerian polynomial of the form
\[ A_m(x) = \sum_{j=0}^{m} j! S(m, j) x^j (1 - x)^{m-j}, \quad (1.9) \]

\( S(m, j) \) being Stirling numbers of the second kind, i.e., \( j! S(m, j) = [\Delta^j t^m]_{t=0} \), which is also denoted by \( \left\{ \begin{array}{c} m \\ j \end{array} \right\} \). Evidently \( \alpha_m(x) \) can be written in the form (cf. [4])

\[ \alpha_m(x) = \sum_{j=0}^{m} \frac{j! S(m, j) x^j}{(1 - x)^{j+1}}. \quad (1.10) \]

In order to express some new formulas for certain general types of power series, we need to introduce the following extension of Euler fraction, denoted by \( \alpha_n(x, A(x), B(x), g(x)) \), using the generalized Sheffer-type polynomials associated with analytic functions or power series \( A(t) \), \( B(t) \), and \( g(t) \), which satisfy conditions in Definition 1.1.

\[ \alpha_n(x, A(x), B(x), g(x)) := \sum_{j=0}^{n} \sum_{\ell=0}^{\infty} j! \binom{\ell}{j} p_{\ell}(x) S(n, j), \quad (1.11) \]

where \( p_{\ell}(x) \) is the generalized Sheffer-type polynomial of degree \( \ell \) defined in Definition 1.1. In particular, if \( A(x) = 1 \) and \( g(x) = x \), then \( p_{\ell}(x) = B^{(\ell)}(0) x^\ell / \ell! \), and the generalized Euler fraction is hence

\[ \alpha_n(x, 1, B(x), x) = \sum_{j=0}^{n} S(n, j) B^{(j)}(x) x^j \quad (1.12) \]

because

\[ \sum_{\ell=0}^{\infty} j! \binom{\ell}{j} p_{\ell}(x) = \sum_{\ell=0}^{\infty} j! \binom{\ell}{j} B^{(\ell)}(0) x^\ell / \ell! = B^{(j)}(x) x^j. \]

Obviously, \( \alpha_m(x) \) defined by (1.8) can be presented as

\[ \alpha_m(x) = A_m(x, 1, (1 - x)^{-1}, x). \]

From (1.11), two kinds of generalized Eulerian fractions in terms of \( A(x) = 1 \), \( g(x) = x \), \( B(x) = (1 + x)^a \) and \( B(x) = (1 - x)^{-a-1} \), respectively, with real number \( a \) as a parameter, can be introduced respectively, namely
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\[ A_n(x, 1, (1 + x)^a, x) = \sum_{j=0}^{n} \binom{a+j}{j} \frac{j! S(n,j)x^j}{(1 + x)^{j-a}} \]  \hspace{1cm} (1.13)

for \( x \neq -1 \), and

\[ A_n(x, 1, (1 - x)^{-a-1}, x) = \sum_{j=0}^{n} \binom{a+j}{j} \frac{j! S(n,j)x^j}{(1 - x)^{a+j+1}} \]  \hspace{1cm} (1.14)

for \( x \neq 1 \). These two generalized Eulerian fractions were given in [5]. Another extension of the classical Eulerian fraction is presented in [10].

Two major transformation and expansion formulas and their applications will be displayed in next section, and the convergence of the series in the formulas is presented in 3.

2 Series transformation-expansion formulas

\textbf{Theorem 2.1} Let \( \{ f(k) \} \) be a sequence of numbers (in \( \mathbb{R} \) or \( \mathbb{C} \)), and let \( h(t) \) be infinitely differentiable. Assume \( A(t), B(t), \) and \( g(t) \) are analytic functions in a disk centered at the origin or power series defined as in Definition 1.1 and \( \{ p_n(x) \} \) is the generalized Sheffer-type polynomial sequence associated with \( A(x), B(x), \) and \( g(x) \), we have formally

\[ \sum_{n=0}^{\infty} f(n)p_n(x) = \sum_{n=0}^{\infty} \Delta^n f(0) \left( \sum_{\ell=n}^{\infty} \binom{\ell}{n} p_\ell(x) \right) \] \hspace{1cm} (2.1)

\[ \sum_{n=0}^{\infty} h(n)p_n(x) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0) \alpha_n(x, A(x), B(x), g(x)), \] \hspace{1cm} (2.2)

where \( S(n, j) \) is the Stirling numbers of the second kind, and \( \alpha_n(x, A(x), B(x), g(x)) = \sum_{j=0}^{n} \sum_{\ell=j}^{\infty} j! \binom{j}{\ell} p_\ell(x) S(n, j) \) is the generalized Eulerian fraction defined as in (1.11).

In particular, if \( g(t) = t \), then the transformation and expansion formulas (2.1) becomes to
\[
\sum_{n=0}^{\infty} \frac{f(n)}{n!} \sum_{\ell=0}^{n} \binom{n}{\ell} A^{(n-\ell)}(0) B^{(\ell)}(0) x^\ell
= \sum_{n=0}^{\infty} \frac{\Delta^n f(0)}{n!} \left( \sum_{\ell=0}^{n} \binom{n}{\ell} A^{(n-\ell)}(1) B^{(\ell)}(x) x^\ell \right). \tag{2.3}
\]

**Proof.** Applying the operator \(A(E)B(xg(E))\) to \(f(t)\) at \(t = 0\), where \(E\) is the shift operator, we obtain the left-hand side of (2.1).

On the other hand, we have

\[
A(E)B(xg(E))f(t)\big|_{t=0} = A(1 + \Delta)B(xg(1 + \Delta))f(t)\big|_{t=0}
= \sum_{\ell=0}^{\infty} p_{\ell}(x) (1 + \Delta)^{\ell} f(t)\big|_{t=0} = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\ell} \binom{\ell}{n} p_{\ell}(x) \Delta^n f(0),
\]

which implies the double sum on the right-hand side of (2.1).

Similarly, for the infinitely differentiable function \(h(t)\), we can present

\[
A(E)B(xg(E))h(t)\big|_{t=0} = A(e^D)B(xg(e^D))h(t)\big|_{t=0}
= \sum_{\ell=0}^{\infty} p_{\ell}(x) e^{\ell D} h(t)\big|_{t=0} = \sum_{\ell=0}^{\infty} p_{\ell}(x) \sum_{n=0}^{\ell} \frac{\ell^n}{n!} h^{(n)}(0)
= \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{\infty} p_{\ell}(x) \ell^n \right) \frac{h^{(n)}(0)}{n!}
\]

By applying (2.1) to the internal sum of the rightmost side of the above equation for \(f(t) = t^k\) and noting \(S(k, j) = (\Delta^j t^k\big|_{t=0}) / j!\), we obtain

\[
A(E)B(xg(E))h(t)\big|_{t=0}
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} \sum_{\ell=j}^{\infty} \binom{\ell}{j} p_{\ell}(x) \Delta^j t^n\big|_{t=0} \right) \frac{h^{(n)}(0)}{n!}
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \sum_{\ell=j}^{\infty} \binom{\ell}{j} p_{\ell}(x) j! S(n, j) \right) \frac{h^{(n)}(0)}{n!}
= \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} \alpha_n(x, A(x), B(x), g(x)).
\]
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This completes the proof of the theorem.

If \( g(t) = t \), then we have formally \( p_n(x) = \sum_{\ell=0}^{n} A^{(n-\ell)}(0) B^{(\ell)}(0) x^\ell / ((n-\ell)! \ell!) \) and

\[
A(E)B(xE) f(t)|_{t=0} = A(1 + \Delta)B(x + x\Delta)f(t)|_{t=0} = \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n} \frac{A^{(n-\ell)}(1) B^{(\ell)}(x)}{(n-\ell)! \ell!} (x\Delta)^\ell \right) f(t)|_{t=0},
\]

which can be written as the double sum on the right-hand side of (2.3).

Remark 3.1 When \( f(t) \) and \( h(t) \) are polynomials, the right-hand sides of (2.1) and (2.2) are finite sums, which can be considered as the closed forms of the corresponding left-hand side series. For this reason, we call formulas (2.1) and (2.2) the series transformation and expansion (or transformation-expansion) formulas. Thus, for the \( r \)th degree polynomial \( \phi(t) \), from (2.1) and (2.2) we have two expansion formulas,

\[
\sum_{n=0}^{\infty} \phi(n) p_n(x) = \sum_{n=0}^{r} \Delta^n \phi(0) \left( \sum_{\ell=n}^{\infty} \frac{\ell}{n} p_\ell(x) \right) \tag{2.4}
\]

\[
\sum_{n=0}^{\infty} \phi(n) p_n(x) = \sum_{n=0}^{r} \frac{1}{n!} \phi^{(n)}(0) \alpha_n(x, A(x), B(x), g(x)), \tag{2.5}
\]

where the right-hand sides can be considered as the GF’s of \( \{\phi(n)p_n(x)\} \).

Corollary 2.2 Let \( \{\alpha_n(x, A(x), B(x), g(x))\} \) be the generalized Eulerian fraction sequence defined by (1.11). Then the exponential generating function of the sequence is \( A(e^t)B(xg(e^t)) \). In particular, the exponential generating function of sequence \( \{\alpha_n(x, 1, B(x), x)\} \) is \( B(xe^t) \).

Proof. The exponential GF of \( \{\alpha_n(x, A(x), B(x), g(x))\} \) can be written as
\[ \sum_{n=0}^{\infty} \alpha_n(x, A(x), B(x), g(x)) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \sum_{j=0}^{n} S(n, j) \sum_{\ell=j}^{\infty} \left( \frac{\ell}{j} \right) p_\ell(x) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} \frac{j^n}{n!} \Delta^j u^n \bigg|_{u=0} \sum_{\ell=j}^{\infty} \left( \frac{\ell}{j} \right) p_\ell(x) \right) \frac{t^n}{n!}. \]

Applying formula (2.1) for \( f(j) = j^n \) into the double sum in the above parentheses yields

\[ \sum_{n=0}^{\infty} \alpha_n(x, A(x), B(x), g(x)) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} j^n p_j(x) \right) \frac{t^n}{n!} \]

\[ = \sum_{j=0}^{\infty} p_j(x) e^{jt} = A(e^t) B(xg(e^t)). \]

Here, the last step is due to Definition 1.1.

We now give two special cases of Theorem 2.1.

**Corollary 2.3** Let \( \{ f(k) \} \) be a sequence of numbers (in \( \mathbb{R} \) or \( \mathbb{C} \)), and let \( B(t) \) and \( g(t) \) be infinitely differentiable on \([0, \infty)\). Then we have formally

\[ \sum_{n=0}^{\infty} f(n) B^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \Delta^n f(0) B^{(n)}(x) \frac{x^n}{n!} \]

\[ \sum_{n=0}^{\infty} g(n) B^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} g^{(n)}(0) \frac{x^n}{n!} \alpha_n(x, 1, B(x), x), \]

where \( \alpha_n(x, 1, B(x), x) \) is the generalized Eulerian fraction defined by (1.12).
Proof. By setting \( A(t) = 1 \) and \( g(t) = t \) into transformation-expansion formulas (2.1) and (2.2), we obtain formally \( p(t) = B(t) x^t / t! \). Thus, the modified formulas (2.1) and (2.2) are respectively (2.6) and (2.7).

Example 2.1 Setting respectively \( B(t) = (1 - t)^{-m - 1} \) \((t \neq 1)\) and \( B(t) = (1 + t)^m \) \((t \neq -1)\) into (2.6) and (2.7) yield the transformation-expansion formulas

\[
\sum_{k=0}^{\infty} \binom{m + k}{k} f(k)x^k = \sum_{k=0}^{\infty} \binom{m + k}{k} \frac{x^k}{(1 - x)^{m+k+1}} \Delta^k f(0) \tag{2.8}
\]

\[
\sum_{k=0}^{\infty} \binom{m + k}{k} h(k)x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x, 1, (1 - x)^{-m}, x)}{k!} D^k h(0) \tag{2.9}
\]

and

\[
\sum_{k=0}^{\infty} \binom{m}{k} f(k)x^k = \sum_{k=0}^{\infty} \binom{m}{k} \frac{x^k}{(1 + x)^{k-m}} \Delta^k f(0) \tag{2.10}
\]

\[
\sum_{k=0}^{\infty} \binom{m}{k} h(k)x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x, 1, (1 + x)^m, x)}{k!} D^k h(0), \tag{2.11}
\]

respectively, where \( \alpha_k(x, 1, (1 - x)^{-m-1}, x) \) and \( \alpha_k(x, 1, (1 + x)^m, x) \) are defined in (1.14) and (1.13), respectively.

By substituting \( m = 0 \) in formulas (2.8) and (2.9) or applying transform \( x \mapsto -x \) and staking \( m = -1 \) in formulas (2.10) and (2.11), we obtain

\[
\sum_{k=0}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \frac{x^k}{(1 - x)^{k+1}} \Delta^k f(0) \tag{2.12}
\]

\[
\sum_{k=0}^{\infty} h(k)x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} D^k h(0), \tag{2.13}
\]

where \( \alpha_k(x) \) is defined by (1.8) and (1.9). (2.12) and (2.13) were shown as in [4]. And (2.12) is an extension of the following well-known Euler series transform that can be found by setting \( x = -1 \) into (2.12):
\[ \sum_{n=0}^{\infty} (-1)^n f(n) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \Delta^n f(0). \]

By applying operator \( E^m \) and multiplying \( x^m \) on the both sides of formula (2.8), we obtain its alternative form as follows:

\[ \sum_{k=m}^{\infty} \binom{k}{m} f(k) x^k = \sum_{k=0}^{\infty} \binom{m+k}{m} \frac{x^{k+m}}{(1-x)^{m+k+1}} \Delta^k f(m) \]  (2.14)

**Example 2.2** Let \( \lambda \) and \( \theta \) be any real numbers. The generalized falling factorial \( (t + \lambda|\theta)_p \) is usually defined by

\[ (t + \lambda|\theta)_p = \prod_{j=0}^{p-1} (t + \lambda - j\theta), \quad (p \geq 1), \quad (t + \lambda|\theta)_0 = 1. \]

It is known that Howard’s degenerate weighted Stirling numbers (cf. [8]) may be defined by the finite differences of \( (t + \lambda|\theta)_p \) at \( t = 0 \):

\[ S(p, k, \lambda\theta) := \frac{1}{k!} \left[ \Delta^k (t + \lambda|\theta)_p \right]_{t=0}. \]

Then, using (2.14), (2.8), and (2.10) with \( f(t) = (t + \lambda|\theta)_p \), we get

\[ \sum_{k=m}^{\infty} \binom{k}{m} (k + \lambda|\theta)_p x^k = \sum_{k=0}^{p} \binom{m+k}{k} \frac{k! S(p, k, \lambda|\theta)_p x^{k+m}}{(1-x)^{m+k+1}} \]  (2.15)

\[ \sum_{k=0}^{\infty} \binom{m+k}{k} (k + \lambda|\theta)_p x^k = \sum_{k=0}^{p} \binom{m+k}{k} \frac{k! S(p, k, \lambda|\theta)_p x^k}{(1-x)^{m+k+1}} \]  (2.16)

\[ \sum_{k=0}^{\infty} \binom{m}{k} (k + \lambda|\theta)_p x^k = \sum_{k=0}^{p} \binom{m}{k} \frac{k! S(p, k, \lambda|\theta)_p x^k}{(1+x)^{k-m}}. \]  (2.17)

The particular case of (2.17) with \( x = 1 \), namely,

\[ \sum_{k=0}^{m} \binom{m}{k} (k + \lambda|\theta)_p = \sum_{k=0}^{p} \binom{m}{k} 2^{m-k} k! S(p, k, \lambda|\theta), \]

was given in formula (35) of [10], and the particular case of (2.16) with \( m = 0 \) was considered in [9]. It is also obvious that the classical Euler’s summation formula for the arithmetic-geometric series (cf. for example,
Lemma 2.7 in [3]) is implied by (2.16) with $\lambda = \theta = 0$ and $m = 0$, or by (2.17) with $\lambda = \theta = 0$, $m = -1$, $x \mapsto -x$.

Some other series transformation-expansion formulas can be constructed formally from the above formulas by using integration or differentiation. For instance, taking the integral on the both sides of (2.12) we obtain

$$\sum_{k=1}^{\infty} \frac{f(k)x^k}{k} = -f(0)\ln(1 - x) + \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{x}{1 - x} \right)^k \Delta^k f(0), \quad (2.18)$$

which can also be considered as a special case of (2.6) for $B(t) = -\ln(1 - t)$.

Using the substituting rule $t \mapsto D$ into (1.6) and applying the resulting operator to an infinitely differentiable function $f$ with the similar argument shown in Theorem 2.1, we have

$$A(D)B(xg(D))f(t)|_{t=0} = \sum_{n=0}^{\infty} p_n(x)f^{(n)}(0). \quad (2.19)$$

We now specify $A$, $B$ and $g$ in (2.19) to establish the following corollary.

**Corollary 2.4** If $(A(t), B(t), g(t)) = (t/(e^t - 1), e^t, t), (2/(e^t + 1), e^t, t), (t/\ln(t + 1), e^t, \ln(t + 1))$, then from (2.19) we have

$$Df(x + y) = \sum_{n=0}^{\infty} \phi_n(x)D^n \Delta f(y) \quad (2.20)$$

$$\sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \Delta^n f(x) = \sum_{n=0}^{\infty} E_n(x)D^n f(0) \quad (2.21)$$

$$\Delta f(x + y) = \sum_{n=0}^{\infty} \psi_n(x)\Delta^n Df(y), \quad (2.22)$$

where $\phi(x)$ and $\psi(x)$ are Bernoulli polynomials of the first and second kind, respectively, and $E_n(x)$ are Euler polynomials.

**Proof.** If $(A(t), B(t), g(t)) = (t/(e^t - 1), e^t, t), (2/(e^t + 1), e^t, t), (t/\ln(t + 1)), e^t, \ln(t + 1))$, then the corresponding Sheffer-type polynomials are
\[ p_n(x) = \phi_n(x), \quad E_n(x), \quad \text{and} \quad \psi_n(x), \quad \text{respectively} \quad (\text{cf.} \quad [8, \text{pp. 250, 309, 279}]), \quad \text{and} \quad \text{the corresponding operators on the left-hand side of (2.19)} \quad \text{for the different} \quad (A(t), B(t), g(t)) \quad \text{become respectively} \quad \left( \frac{De^x}{e^D - 1} = \frac{DE^x}{\Delta} \right), \]

\[ \frac{2e^{xD}}{e^D + 1} = \frac{2E^x}{\Delta + 2} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \left( \frac{\Delta}{2} \right)^n, \]

and \( \Delta(\Delta + 1)^x/\ln(\Delta + 1) = \Delta E^x/D. \) Hence, the proof of the theorem is complete.

The results in Corollary 2.4 were given in [7] by using different treatment for each individual formula while our method described here can be considered as a uniform approach, which can be used to find more transformation and expansion formulas.

It is obvious that for \( x = 0, \) formulas (2.20)-(2.22) are specified as

\[ Df(y) = \sum_{n=0}^{\infty} \frac{B_n^{(1)}}{n!} D^n \Delta f(y) \quad (2.23) \]

\[ \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \Delta^n f(0) = \sum_{n=0}^{\infty} e_n D^n f(0) \quad (2.24) \]

\[ \Delta f(y) = \sum_{n=0}^{\infty} b_n \Delta^n Df(y), \quad (2.25) \]

where \( B_n^{(1)} = n!\phi_n(0) \) is the first order generalized Bernoulli number, and \( e_n = E_n(0), \) and \( b_n = \psi_n(0). \)

### 3 Convergence of the series transformation-expansions

As may be conceived, various formulas displayed in the list in Section 2 may be employed to construct some summation formulas with estimable remainders \((\text{cf.} \quad \text{the proof of Theorem 3.3 below}). \) In what follows convergence problems related to the series expansions in Section 2 will be investigated.
We now establish convergence conditions for the series expansions in (2.6) and (2.7). Suppose that \( \{f(k)\} \) and \( \{h(k)\} \) are bounded sequences (say \(|f(k)| < M\) and \(|h(k)| < M\) for all \(k\)), and that \( g(z) \) is analytic for \(|z| < \rho\). Then it is follows that the left-hand sides of (2.6) and (2.7) as well as the right-hand sides of (2.6) and (2.7) are absolutely convergent series for \(|x| < \rho\). Hence, we have the following convergence theorem.

**Theorem 3.1** If \( \{f(k)\} \) and \( \{g(k)\} \) are bounded sequences, and that \( B(z) \) is analytic for \(|z| < \rho\) for some positive real number \( \rho \), then the series expansions in (2.6) and (2.7) converge absolutely for all \(|x| < \rho\).

The convergence on the general case where \( \{f(k)\} \) is not bounded presents some complicated situation. The next theorem gives a discussion for the series transformation-expansion formulas shown in (2.12), (2.13), and (2.18), and general way may be developed through it, which is left for the interested reader to consider.

**Theorem 3.2** Let \( \{f(k)\} \) be a sequence of numbers (in \( \mathbb{R} \) or \( \mathbb{C} \)), and denote \( \theta := \lim_{k \to \infty} |f(k)|^{1/k} \). Then the series expansions in (2.12), (2.13), and (2.18) are convergent for all nonzero \( x \) satisfying \(|x| \theta < 1\).

**Proof.** Substituting the expression of \( \alpha_k(x) \) defined by (1.10) into (2.12) and noting \( j! \sum_{k=j}^{\infty} S(k, j) \frac{D^k}{k!} (e^D - 1)^j = \Delta^j \) (cf. [4]) yields (2.13). More precisely,

\[
\sum_{k=0}^{\infty} g(k) x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} D^k g(0)
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{j!}{k!} S(k, j) \frac{x^j}{(1 - x)^{j+1}} D^k g(0)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{(1 - x)^{j+1}} \left( j! \sum_{k=j}^{\infty} \frac{S(k, j) D^k}{k!} \right) g(0)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{(1 - x)^{j+1}} \Delta^j g(0).
\]

Hence, we only need to show the convergence of expansions in (2.12) and (2.18).
In accordance with Cauchy’s root test, the convergence of the series on the left-hand side of (2.12) and (2.18) is obvious because of the condition \( |x| \theta < 1 \). To prove the convergence of the series expansion on the right-hand side of (2.12), we choose \( \rho > \theta \) such that \( \theta |x| < \rho |x| < 1 \). Thus for large \( k \) we have \( |f(k)|^{1/k} < \rho \). Consequently,

\[
|\Delta^k f(0)|^{1/k} \leq \left( \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) |f(j)| \right)^{1/k} < (2)^{1/k} \rho \rightarrow \rho
\]

as \( k \rightarrow \infty \). Therefore, for every \( x \in (-1/\theta, 0) \)

\[
\lim_{k \rightarrow \infty} \left| \frac{1}{k} \left( \frac{x}{1-x} \right)^k \Delta^k f(0) \right|^{1/k} = \lim_{k \rightarrow \infty} \left| \frac{x}{1-x} \right| \left| \Delta^k f(0) \right|^{1/k} \leq \rho \left| \frac{x}{1-x} \right| < \rho |x| < 1.
\]

Hence, from the root test, the series expansion on the right-hand side of (2.18) is convergent. Similarly, the expansion on the right-hand side of (2.12) converges as well. This completes the proof of the theorem.

To extend the convergence intervals of the series expansions in (2.12) and (2.13), we need more precise estimation as follows.

**Theorem 3.3** Let \( \{f(k)\} \) be a sequence of numbers (in \( \mathbb{R} \) or \( \mathbb{C} \)), and let \( \theta = \lim_{k \rightarrow \infty} |f(k)|^{1/k} \). Then for any given \( x \) with \( x \neq 0 \) we have the convergent expressions (2.12) and (2.13) provided that \( |x| \theta < 1 \).

**Proof.** As we mentioned in the proof of Theorem 3.2, it is sufficient to show the convergence of (2.12). For this purpose, we now find a remainder of the expansion of (2.12) as follows. Formally, we have


\[(1 - xE)^{-1} = (1 - x - x\Delta)^{-1}\]

\[= (1 - x)^{-1}\left(1 - \frac{x}{1 - x}\Delta\right)^{-1}\]

\[= (1 - x)^{-1}\left\{\sum_{\ell=0}^{n-1} \left(\frac{x}{1 - x}\right)^\ell \Delta^\ell + \frac{(x \Delta)^n}{1 - (x \Delta)}\right\}\]

\[= \sum_{\ell=0}^{n-1} \frac{x^\ell}{(1 - x)^{\ell+1}} \Delta^\ell + \left(\frac{x}{1 - x}\right)^n \frac{\Delta^n}{1 - xE}\]

\[= \sum_{\ell=0}^{n-1} \frac{x^\ell}{(1 - x)^{\ell+1}} \Delta^\ell + \left(\frac{x}{1 - x}\right)^n \sum_{\ell=0}^\infty x^\ell E^\ell \Delta^n.\]

Since \(E^\ell \Delta^n f(0) = \Delta^n E^\ell f(0) = \Delta^n f(\ell)\), applying operator \((1 - xE)^{-1}\) and the rightmost operator shown above to \(f(t)|_{t=0}\), respectively, yields

\[(1 - xE)^{-1} f(t)|_{t=0} = \sum_{k=0}^\infty f(k) x^k\]

\[= \sum_{k=0}^{n-1} \frac{x^k}{(1 - x)^{k+1}} \Delta^k f(0) + \frac{x^n}{(1 - x)^n} \sum_{\ell=0}^\infty x^\ell E^\ell \Delta^n f(\ell).\]  (3.1)

Since \(\theta |x| < \rho |x| < 1\), the convergence of the series expansion on the left-hand side of (2.12) or (3.1) is obtain. To prove the convergence of the right-hand side of (3.1), i.e., the remainder form of (2.12), it is sufficient to show that \(\sum_{\ell=0}^\infty x^\ell \Delta^n f(\ell)\) is absolutely convergent. Choose \(\rho > \theta\) such that

\[\theta |x| < \rho |x| < 1.\]

Thus, for large \(k\) we have \(|f(k)|^{1/k} < \rho\), i.e., \(|f(k)| < \rho^k\). Consequently we have, for large \(\ell\)

\[|\Delta^n f(\ell)|^{1/\ell} \leq \left(\sum_{j=0}^n \binom{n}{j} |f(\ell + j)|\right)^{1/\ell} \leq \left(\sum_{j=0}^n \binom{n}{j} \rho^{\ell+j}\right)^{1/\ell}\]

\[= \rho (1 + \rho)^{n/\ell} \rightarrow \rho\]
as \( \ell \to \infty \). Thus
\[
\lim_{\ell \to \infty} |x^\ell \Delta^n f(\ell)|^{1/\ell} \leq \rho |x| < 1,
\]
so that the series on the right-hand side of (3.1) or (2.12) is also convergent absolutely under the given conditions.

A sequence \( \{a_n\} \) is called a null sequence if for any given positive number \( \epsilon \), there exists and integer \( N \) such that every \( n > N \) implies \( |a_n| < \epsilon \). \[12\] (cf. Theorem 4 in Section 43) pointed out that a linear combination of \( \{a_n\} \), denoted by \( \{a'_n = \sum_{k=0}^n c_{n,k}a_k\} \), is also a null sequence if the coefficient set \( \{c_{n,k}\}_{0 \leq k \leq n} \) \((n = 0, 1, 2, \ldots)\) satisfies the following two conditions:

(i) Every column contains a null sequence, i.e., for fixed \( k \geq 0 \), \( c_{n,k} \to 0 \) when \( n \to \infty \).

(ii) There exists a constant \( K \) such that the sum \( |a_{n,0}| + |a_{n,1}| + \ldots + |a_{n,n}| < K \) for every \( n \).

By using this claim of the null sequence, we can have the following convergence result of the series expansions in (2.12) and (2.13).

**Theorem 3.4** Suppose that \( \{f(n)\} \) is a given sequence of numbers (real or complex) such that \( \sum_{n=0}^\infty f(n)x^n \) is convergent for every \( x \in \Omega \) with \( \Omega \cap (-\infty, 0) \neq \emptyset \). Then the series expressions on the right-hand sides of (2.12) and (2.13) converge for every \( x \in \Omega \cap (-\infty, 0) \).

**Proof.** We write the remainder of expression (3.1) as follows.

\[
R_n := \frac{x^n}{(1-x)^n} \sum_{\ell=0}^{\infty} x^\ell \Delta^n f(\ell)
\]

\[
= \frac{x^n}{(1-x)^n} \sum_{\ell=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} x^\ell \binom{n}{j} f(j + \ell)
\]

\[
= \frac{(-x)^n}{(1-x)^n} \sum_{j=0}^{n} \binom{n}{j} \sum_{\ell=0}^{\infty} (-1)^j x^\ell \binom{n}{j} f(j + \ell)
\]

\[
= \frac{(-x)^n}{(1-x)^n} \sum_{j=0}^{n} (-x)^{-j} \binom{n}{j} \sum_{\ell=0}^{\infty} x^\ell f(\ell) = \frac{(-x)^n}{(1-x)^n} \sum_{j=0}^{n} (-x)^{-j} \binom{n}{j} x_j,
\]
where \( x_j = \sum_{\ell=j}^{\infty} x^{\ell}f(\ell) \) (0 \( \leq j \leq n \)). Since \( \sum_{\ell=0}^{\infty} x^{\ell}f(\ell) \) converges, \( x_j \) is the term of a null sequence, applying the result on the linear combination of a null sequence shown above, we find the coefficients of \( x_j \) in the linear combination of the rightmost sum,

\[
c_{n,j} := \frac{(-x)^n}{(1-x)^n}(-x)^{-j} {n \choose j}
\]
satisfy the following two conditions: (1) If \( j \) is fixed, we have \( c_{n,j} \rightarrow 0 \) as \( n \rightarrow \infty \) because

\[
|c_{n,j}| = \frac{|x|^{n-j}}{(1-x)^n} {n \choose j} < \frac{n^j}{(1-x)^n}
\]
and \( 1/(1-x) < 1 \) for every \( x \in \Omega \cap [-1,0) \); and

\[
|a_{n,j}| = \frac{|x|^{n-j}}{(1-x)^n} {n \choose j} < \left( \frac{|x|}{1-x} \right)^n n^j
\]
and \( |x/(1-x)| < 1 \) for every \( x \in \Omega \cap (-\infty,-1) \). (2) For every \( n \) and for every \( x \in \Omega \cap (-\infty,0) \) we have

\[
\sum_{j=0}^{n} |a_{n,j}| = \frac{1}{(1-x)^n} \sum_{j=0}^{n} (-x)^{n-j} {n \choose j} = 1.
\]

Therefore, Theorem 4 in Section 43 of [12] shows that \( R_n \) is also the term of a null sequence, so the series on the right-hand side of (2.12) converges for every \( x \in \Omega \cap (-\infty,0) \). In addition, the convergence of the right-hand series expansion of (2.13) is followed.

We now discuss the convergence of the series expansions in (2.20)-(2.22). Actually, we may sort the series transformation-expansion formulas associated with \((A(t),B(t),t)\) into two classes. The first class includes only either the sum \( \sum \beta_k D^k f \) or the sum \( \sum \gamma_k E^k f \) in the formulas such as (2.20) and (2.22). The second class includes the sums \( \sum \beta_k D^k f \) and/or \( \sum \gamma_k E^k f \) on both sides of the transformation-expansion formulas like (2.21). We may establish the following convergence theorem.

**Theorem 3.5** For the first class series expansions associated with \( \sum \beta_k D^k f \) (or \( \sum \gamma_k E^k f \)) defined above, their absolute convergence are ensured
The second class series expansions defined above absolutely converge if \( \lim_{k \to \infty} |D^k f|^{1/k} < 1 \) and \( |\beta_k| \leq 1 \) (or \( |\gamma_k| \leq 1 \)).

Proof. The first half of the theorem is easy to be verified by using the root test.

To prove the second half, we need the following statement: If \( f \in C^\infty \), then \( \lim_{k \to \infty} |D^k f(y)|^{1/k} < a \), a positive real number, implies \( \lim_{k \to \infty} |\Delta D^k f(y)|^{1/k} < e^{a-1} \).

In fact, if we denote \( \lim_{n \to \infty} |D^n f(y)|^{1/n} = \theta \), then there exists a number \( \gamma \) such that \( \theta < \gamma < a \). Thus for large enough \( n \) we have \( |D^n f(y)|^{1/n} < \gamma \) or \( |D^n f(y)| < \gamma^n \).

Noting \( S(n, m) \geq 0 \) and \( |D^n f(y)| < \gamma^n \), we obtain

\[
|\Delta^k f(y)| = \sum_{n \geq k} \frac{k!}{n!} S(n, k) D^n f(y) \leq \sum_{n \geq k} \frac{k!}{n!} S(n, k) |D^n f(y)| \\
\leq \sum_{n \geq k} \frac{k!}{n!} S(n, k) \gamma^n = (e^\gamma - 1)^k < (e^a - 1)^k.
\]

Here the rightmost equality is from Jordan [11] (see p. 176).

Therefore, \( \lim_{k \to \infty} |D^k f|^{1/k} < 1 \) implies that \( \lim_{k \to \infty} |\Delta^k f(y)|^{1/k} < e - 1 \). Those two inequalities and the conditions for the coefficients \( \{\beta_k\} \) and \( \{\gamma_k\} \) confirm the absolute convergence of the second class series expansions with the root test.

As a corollary of Theorem 3.5, we now establish the convergence results of the series expansions in (2.20)-(2.22).

**Corollary 3.6** For given \( f \in C^\infty \) and \( x, y \in \mathbb{R} \), the absolute convergence of the series expansion (2.20) is ensured by the condition

\[
\lim_{k \to \infty} |\Delta D^k f(y)|^{1/k} < 1.
\]

Similarly, the absolute convergence of the series expansion (2.21) and (2.22) are ensured by the conditions
Series transformation and expansion

\[
\lim_{k \to \infty} |\Delta^k D f(y)|^{1/k} < 1 \tag{3.4}
\]

and

\[
\lim_{k \to \infty} |D^k f(y)|^{1/k} < 1, \tag{3.5}
\]

respectively.

Proof. From Theorem 3.5, it is sufficient to show that

\[
\lim_{k \to \infty} |\phi_k(x)|^{1/k} \leq 1, \tag{3.6}
\]

\[
\lim_{k \to \infty} |E_k(x)|^{1/k} \leq 1, \tag{3.7}
\]

and

\[
\lim_{k \to \infty} |\psi_k(x)|^{1/k} \leq 1, \tag{3.8}
\]

which will be proved below from the basic properties of \(\phi_k(x), E_k(x), \) and \(\psi_k(x)\) shown as in Jordan [11].

Write the Bernoulli polynomials of the first kind, \(\phi_k(x)\), as \(\text{(cf. Jordan [11], p. 321)}\)

\[
\phi_k(x) = \sum_{j=0}^{k} \frac{x^{k-j}}{(k-j)!} \alpha_j,
\]

where \(\alpha_j = B_j^{(1)}/j!\), and \(B_j^{(1)}\) are ordinary Bernoulli numbers. Note that
\(\alpha_0 = 1, \alpha_1 = -1/2, \alpha_{2m+1} = 0 (m \in \mathbb{N})\) and \(\text{(cf. [11] p. 245)}\)

\[
|\alpha_{2m}| \leq \frac{1}{12(2\pi)^{2m-2}}, \quad (m = 0, 1, 2, \ldots).
\]

Thus for \(k \geq 2\)

\[
|\phi_k(x)| \leq \frac{|x|^k}{k!} + \frac{|x|^{k-1}}{2(k-1)!} + \sum_{j=2}^{k} \left( \frac{1}{12(2\pi)^{j-2}} \right) \frac{|x^{k-j}|}{(k-j)!}
\]

\[
< \sum_{j=0}^{k} \frac{|x|^j}{j!} \leq e^{|x|}.
\]
It follows that \( |\phi_k(x)|^{1/k} < \exp(|x|/k) \to 1 \) as \( k \to \infty \), which implies (3.6).

Secondly, note that Euler polynomial \( E_k(x) \) can be written in the form

\[
E_k(x) = \sum_{j=0}^{k} e_j \frac{x^{k-j}}{(k-j)!}, \quad (e_0 = 1),
\]  

where \( e_j = E_j(0), \) \( e_{2m} = 0 \) \( (m = 1, 2, \ldots) \), and \( e_{2m-1} \) satisfies the inequality (cf. [11], p. 302)

\[
|e_{2m-1}| < \frac{2}{3\pi 2^{m-2}} < 1 \quad (m = 1, 2, \ldots).
\]  

Thus we have the estimation

\[
|E_k(x)| \leq \frac{|x|^k}{k!} + \sum_{j=1}^{k} |e_j| \frac{|x|^{k-j}}{(k-j)!} \leq \frac{|x|^k}{k!} + \sum_{j=1}^{k} \frac{|x|^{k-j}}{(k-j)!} < e^{|x|}.
\]

Consequently we get

\[
\lim_{k \to \infty} |E_k(x)|^{1/k} \leq \lim_{k \to \infty} (e^{|x|})^{1/k} = 1.
\]

Hence (3.7) is verified.

Finally, from [11], p. 268, we have an integral representation of \( \psi_k(x) \), the Bernoulli polynomials of the second kind, namely

\[
\psi_k(x) = \int_0^1 \left( \frac{x + t}{k} \right) dt.
\]  

For \( t \in [0, 1] \) and for large \( k \) we have the estimation

\[
\left| \binom{x + t}{k} \right| = \frac{|(x + t)_k|}{k!} = \frac{|(k - x - t - 1)_k|}{k!} = o \left( \frac{k + |x|}{k!} \right) = o \left( k^{||x||} \right).
\]

This means that there is a constant \( M > 0 \) such that

\[
\max_{0 \leq t \leq 1} \left| \binom{x + t}{k} \right| < Mk^{||x||}.
\]

Thus it follows that
\[
\lim_{k \to \infty} |\psi_k(x)|^{1/k} \leq \lim_{k \to \infty} \left( \int_0^1 \left| \binom{x + t}{k} \right| dt \right)^{1/k} \leq \lim_{k \to \infty} (M_k ||x||)^{1/k} = 1.
\]

This is a verification of (3.8), and corollary is proved.

**Remark 3.1** The convergence conditions given in Theorem 3.2 can be improved by restricting \( f \) and using similar techniques shown in [4].

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## References

### References


