Hyperbolic expressions of polynomial sequences and parametric number sequences defined by linear recurrence relations of order 2

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August 5, 2012

Abstract

A sequence of polynomial $\{a_n(x)\}$ is called a function sequence of order 2 if it satisfies the linear recurrence relation of order 2: $a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$ with initial conditions $a_0(x)$ and $a_1(x)$. In this paper we derive a parametric form of $a_n(x)$ in terms of e^θ with $q(x) = B$ constant, inspired by Askey's and Ismail's works shown in [2] [6], and [18], respectively. With this method, we give the hyperbolic expressions of Chebyshev polynomials and Gegenbauer-Humbert Polynomials. The applications of the method to construct corresponding hyperbolic form of several well-known identities are also discussed in this paper.

AMS Subject Classification: 05A15, 12E10, 65B10, 33C45, 39A70, 41A80.

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The last two authors would like to thank The Institute of Mathematics, Academia Sinica, Taiwan for its financial support during the summer of 2009 during which the research in this paper was carried out.

Key Words and Phrases: sequence of order 2, linear recurrence relation, Fibonacci sequence, Chebyshev polynomial, the generalized Gegenbauer-Humbert polynomial sequence, Lucas number, Pell number.

1 Introduction

In [2, 6, 18], a type of hyperbolic expressions of Fibonacci polynomials and Fibonacci numbers are given using parameterization. We shall extend the idea to polynomial sequences and number sequences defined by linear recurrence relations of order 2.

Many number and polynomial sequences can be defined, characterized, evaluated, and/or classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called a sequence of order 2 if it satisfies the linear recurrence relation of order 2:

$$a_n = aa_{n-1} + ba_{n-2}, \quad n \geq 2, \tag{1}$$

for some non-zero constants p and q and initial conditions a_0 and a_1. In Mansour [21], the sequence $\{a_n\}_{n\geq 0}$ defined by (1) is called Horadam's sequence, which was introduced in 1965 by Horadam [14]. [21] also obtained the generating functions for powers of Horadam's sequence. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [8], Hsu [15], Strang [24], Wilf [26], etc.) In [5], Benjamin and Quinn presented many elegant combinatorial meanings of the sequence defined by recurrence relation (1). For instance, a_n counts the number of ways to tile an n-board (i.e., board of length n) with squares (representing 1s) and dominoes (representing 2s) where each tile, except the initial one has a color. In addition, there are p colors for squares and q colors for dominoes. In particular, Aharonov, Beardon, and Driver (see [1]) have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions $a_0 = 0$ and $a_1 = 1$, called the primary solution, can be expressed in terms of Chebyshev polynomial values. For instance, the authors show $F_n = i^{-n}U_n(i/2)$ and $L_n = 2i^{-n}T_n(i/2)$, where F_n and L_n respectively are Fibonacci numbers and Lucas numbers, and T_n and U_n are Chebyshev polynomials of the first kind and the second kind, respectively (see also in [2, 3]). Some identities drawn from those relations were given by Beardon in [4]. Marr and Vineyard in [22] use the relationship to establish an explicit expression of five-diagonal Toeplitz determinants. In [12], the first two authors presented a new method to construct an explicit formula of $\{a_n\}$ generated by (1). For the sake of the reader's convenience, we cite this result as follows.

Proposition 1.1 *([12]) Let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation (1), and let α and β be two roots of of quadratic equation $x^2 - ax - b = 0$. Then*

$$a_n = \begin{cases} \left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & if\ \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & if\ \alpha = \beta. \end{cases} \tag{2}$$

If the coefficients of the linear recurrence relation of a function sequence $\{a_n(x)\}$ of order 2 are real or complex-value functions of variable x, i.e.,

$$a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x), \tag{3}$$

we obtain a function sequence of order 2 with initial conditions $a_0(x)$ and $a_1(x)$. In particular, if all of $p(x)$, $q(x)$, $a_0(x)$ and $a_1(x)$ are polynomials, then the corresponding sequence $\{a_n(x)\}$ is a polynomial sequence of order 2. Denote the solutions of

$$t^2 - p(x)t - q(x) = 0$$

by $\alpha(x)$ and $\beta(x)$. Then

$$\alpha(x) = \frac{1}{2}(p(x) + \sqrt{p^2(x) + 4q(x)}), \beta(x) = \frac{1}{2}(p(x) - \sqrt{p^2(x) + 4q(x)}). \tag{4}$$

Similar to Proposition 1.1, we have

Proposition 1.2 *[12] Let $\{a_n\}$ be a sequence of order 2 satisfying the linear recurrence relation (3). Then*

$$a_n(x) = \begin{cases} \left(\frac{a_1(x)-\beta(x)a_0(x)}{\alpha(x)-\beta(x)}\right)\alpha^n(x) - \left(\frac{a_1(x)-\alpha(x)a_0(x)}{\alpha(x)-\beta(x)}\right)\beta^n(x), & if\ \alpha(x) \neq \beta(x); \\ na_1(x)\alpha^{n-1}(x) - (n-1)a_0(x)\alpha^n(x), & if\ \alpha(x) = \beta(x), \end{cases} \tag{5}$$

where $\alpha(x)$ and $\beta(x)$ are shown in (4).

In this paper, we shall consider the polynomial sequence defined by (3) with $q(x) = B$, a constant, to derive a parametric form of function sequence of order 2 by using the idea shown in [18]. Our construction will focus on four type Chebyshev polynomials and the following Gegenbauer-Humbert polynomial sequences although our method is limited by those function sequences.

A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, for example, [8], Gould [10], Lidl, Mullen, and Turnwald[20], the first two of authors with Hsu [11])

$$\Phi(t) \equiv (C - 2xt + yt^2)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x)t^n, \tag{6}$$

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (6), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [11])

$$\begin{aligned}
&P_n^{1,1,1}(x) = U_n(x),\ \textit{Chebyshev polynomial of the second kind},\\
&P_n^{1/2,1,1}(x) = \psi_n(x),\ \textit{Legendre polynomial},\\
&P_n^{1,-1,1}(x) = P_{n+1}(x),\ \textit{Pell polynomial},\\
&P_n^{1,-1,1}\left(\frac{x}{2}\right) = F_{n+1}(x),\ \textit{Fibonacci polynomial},\\
&P_n^{1,2,1}\left(\frac{x}{2}\right) = \Phi_{n+1}(x),\ \textit{Fermat polynomial of the first kind},\\
&P_n^{1,2a,2}(x) = D_n(x,a),\ \textit{Dickson polynomial of the second}\\
&\qquad\qquad\qquad\qquad \textit{kind},\ a \neq 0,\ (\textit{see, for example}, [20]),
\end{aligned}$$

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called Gegenbauer polynomials (see [8]). More results on the Gegenbauer-Humbert-type polynomials can be found in [16] by Hsu and in [17] by the second author and Hsu, etc.

Similarly, for a class of the generalized Gegenbauer-Humbert polynomial sequences defined by

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x) \tag{7}$$

for all $n \geq 2$ with initial conditions

$$\begin{aligned}
P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda},\\
P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1},
\end{aligned}$$

the following theorem has been obtained in [12]

Theorem 1.3 *([12]) Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion (6) can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \quad (8)$$

We may use recurrence relation (6) to define various polynomials that were defined using different techniques. Comparing recurrence relation (6) with the relations of the generalized Fibonacci and Lucas polynomials shown in Example 4, with the assumption of $P_0^{1,y,C} = 0$ and $P_1^{1,y,C} = 1$, we immediately know

$$P_n^{1,1,1}(x) = 2xP_{n-1}^{1,1,1}(x) - P_{n-2}^{1,1,1}(x) = U_n(2x;0,1)$$

defines the Chebyshev polynomials of the second kind, and

$$P_n^{1,-1,1}(x) = 2xP_{n-1}^{1,-1,1}(x) + P_{n-2}^{1,-1,1}(x) = P_n(2x;0,-1)$$

defines the Pell polynomials.

In addition, in [20], Lidl, Mullen, and Turnwald defined the Dickson polynomials are also the special case of the generalized Gegenbauer-Humbert polynomials, which can be defined uniformly using recurrence relation (6), namely

$$D_n(x;a)) = xD_{n-1}(x;a) - aD_{n-2}(x;a) = P_n^{1,2a,2}(x)$$

with $D_0(x;a) = 2$ and $D_1(x;a) = x$. Thus, the general terms of all of above polynomials can be expressed using (8).

For $\lambda = y = C = 1$, using (8) we obtain the expression of the Chebyshev polynomials of the second kind:

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}},$$

where $x^2 \neq 1$. Thus, $U_2(x) = 4x^2 - 1$.

For $\lambda = C = 1$ and $y = -1$, formula (8) gives the expression of a Pell polynomial of degree $n+1$:

$$P_{n+1}(x) = \frac{(x+\sqrt{x^2+1})^{n+1} - (x-\sqrt{x^2+1})^{n+1}}{2\sqrt{x^2+1}}.$$

Thus, $P_2(x) = 2x$.

Similarly, let $\lambda = C = 1$ and $y = -1$, the Fibonacci polynomials are

$$F_{n+1}(x) = \frac{(x+\sqrt{x^2+4})^{n+1} - (x-\sqrt{x^2+4})^{n+1}}{2^{n+1}\sqrt{x^2+4}},$$

and the Fibonacci numbers are

$$F_n = F_n(1) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\},$$

which has been presented in Example 1.

Finally, for $\lambda = C = 1$ and $y = 2$, we have Fermat polynomials of the first kind:

$$\Phi_{n+1}(x) = \frac{(x+\sqrt{x^2-2})^{n+1} - (x-\sqrt{x^2-2})^{n+1}}{2\sqrt{x^2-2}},$$

where $x^2 \neq 2$. From the expressions of Chebyshev polynomials of the second kind, Pell polynomials, and Fermat polynomials of the first kind, we may get a class of the generalized Gegenbauer-Humbert polynomials with respect to y defined by the following which will be parameterized.

Definition 1.4 *The generalized Gegenbauer-Humbert polynomials with respect to y, denoted by $P_n^{(y)}(x)$ are defined by the expansion*

$$(1-2xt+yt^2)^{-1} = \sum_{n\geq 0} P_n^{(y)}(x)t^n,$$

or by

$$P_n^{(y)}(x) = 2xP_{n-1}^{(y)}(x) - yP_{n-2}^{(y)}(x),$$

or equivalently, by

$$P_n^{(y)}(x) = \frac{(x+\sqrt{x^2-y})^{n+1} - (x-\sqrt{x^2-y})^{n+1}}{2\sqrt{x^2-y}}$$

with $P_0^{(y)}(x) = 1$ *and* $P_1^{(y)}(x) = 2x$, *where* $x^2 \neq y$. *In particular,* $P_n^{(-1)}(x)$, $P_n^{(1)}(x)$ *and* $P_n^{(2)}(x)$ *are respectively Pell polynomials, Chebyshev polynomials of the second kind, and Fermat polynomials of the first kind.*

In the next section, we shall parameterize the function sequences defined by (3) and number sequences defined by (1) by using the idea of [18]. The application of the parameterization will be applied to construct the corresponding hyperbolic form of several well-known identities.

2 Hyperbolic expressions of parametric polynomial sequences

Suppose $q(x) = b$, a constant, and re-write (5) as

$$\begin{aligned} & a_n(x) \\ = \; & \frac{a_1(x) - \beta(x)a_0(x)}{\alpha(x) - \beta(x)}\alpha^n(x) - \frac{a_1(x) - \alpha(x)a_0(x)}{\alpha(x) - \beta(x)}\beta^n(x) \\ = \; & \frac{a_0(x)(\alpha^{n+1}(x) - \beta^{n+1}(x)) + (a_1(x) - a_0(x)p(x))(\alpha^n(x) - \beta^n(x))}{\alpha(x) - \beta(x)}, \end{aligned} \tag{9}$$

where we assume $\alpha(x) \neq \beta(x)$ due to the reason shown below.

Inspired by [18], we now set

$$(\alpha(x), \beta(x)) = \begin{cases} (\sqrt{b}e^{\theta(x)}, -\sqrt{b}e^{-\theta(x)}), & for\ b > 0, \\ (\sqrt{-b}e^{\theta(x)}, \sqrt{-b}e^{-\theta(x)}), & for\ b < 0, \end{cases} \tag{10}$$

for some real or complex value function $\theta \equiv \theta(x)$. Thus one may have $\alpha(x) \cdot \beta(x) = -b$ and

$$\alpha(x) + \beta(x) = p(x) = \begin{cases} 2\sqrt{b}\sinh(\theta(x)), & for\ b > 0, \\ 2\sqrt{-b}\cosh(\theta(x)), & for\ b < 0, \end{cases} \tag{11}$$

which implies

$$\theta(x)=\begin{cases}\sinh^{-1}\left(\frac{p(x)}{2\sqrt{b}}\right), & for\ b>0\\ \cosh^{-1}\left(\frac{p(x)}{2\sqrt{-b}}\right), & for\ b<0.\end{cases} \tag{12}$$

For $b>0$, substituting expressions (10) into the last formula of (9) yields

$$a_n(x)=\begin{cases}\frac{b^{(n-1)/2}}{\cosh(\theta)}\Big(a_0(x)\sqrt{b}\cosh((n+1)\theta)\\ \qquad +(a_1(x)-2a_0(x)\sqrt{b}\sinh(\theta))\sinh(n\theta)\Big), & for\ even\ n,\\ \frac{b^{(n-1)/2}}{\cosh(\theta)}\Big(a_0(x)\sqrt{b}\sinh((n+1)\theta)\\ \qquad +(a_1(x)-2a_0(x)\sqrt{b}\sinh(\theta))\cosh(n\theta)\Big), & for\ odd\ n,\end{cases} \tag{13}$$

where $\theta=\sinh^{-1}(p(x)/(2\sqrt{b}))$. Still in the case of $b>0$, substituting (10) into the formula before the last one shown in (9), we obtain an equivalent expression:

$$a_n(x) = \begin{cases}\frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1(x)\sinh n\theta+\sqrt{b}a_0(x)\cosh(n-1)\theta\right), & for\, even\, n;\\ \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1(x)\cosh n\theta+\sqrt{b}a_0(x)\sinh(n-1)\theta\right), & for\, odd\, n,\end{cases} \tag{14}$$

where $\theta=\sinh^{-1}(p(x)/(2\sqrt{b}))$.

Similarly, for $b<0$ we have

$$a_n(x) = \frac{(-b)^{(n-1)/2}}{\sinh(\theta)}\Big(a_0(x)\sqrt{-b}\sinh((n+1)\theta) +(a_1(x)-2a_0(x)\sqrt{-b}\cosh(\theta))\sinh(n\theta)\Big), \tag{15}$$

or equivalently,

$$a_n(x)=\frac{(-b)^{(n-1)/2}}{\sinh\theta}(a_1(x)\sinh n\theta-a_0(x)\sqrt{-b}\sinh(n-1)\theta), \tag{16}$$

where $\theta=\cosh^{-1}(p(x)/(2\sqrt{-b}))$.

We survey the above results as follows.

Theorem 2.1 *Let function sequence $a_n(x)$ be defined by*

$$a_n(x) = p(x)a_{n-1}(x) + ba_{n-2}(x) \tag{17}$$

with initials $a_0(x)$ and $a_1(x)$, and let function $\theta(x)$ be defined by (12). Then the roots of the characteristic function $t^2 - p(x)t - b$ can be shown as (10), and there hold the hyperbolic expressions of functions $a_n(x)$ shown in (13) and (14) for $b > 0$ and (15) and (16) for $b < 0$.

Let us consider some special cases of Theorem 2.1:

Corollary 2.2 *Suppose $\{a_n(x)\}$ is the function sequence defined by (17) with initials $a_0(x) = 0$ and $a_1(x)$, then*

$$\begin{aligned} a_{2n}(x) &= b^{(2n-1)/2}a_1(x)\frac{\sinh(2n)\theta}{\cosh\theta}; \\ a_{2n+1}(x) &= b^n a_1(x)\frac{\cosh(2n+1)\theta}{\cosh\theta} \end{aligned} \tag{18}$$

for $b > 0$, where $\theta = \sinh^{-1}(p(x)/(2\sqrt{b}))$; and

$$a_n(x) = (-b)^{(n-1)/2}a_1(x)\frac{\sinh n\theta}{\sinh\theta} \tag{19}$$

for $b < 0$, where $\theta = \cosh^{-1}(p(x)/(2\sqrt{-b}))$.

Example 2.1 Let $\{F_n(kx)\}$ be the sequence of the generalized Fibonacci polynomials defined by

$$F_{n+2}(kx) = kxF_{n+1}(kx) + F_n(kx), \quad k \in \mathbb{R}\backslash\{0\},$$

with initials $F_0(kx) = 0$ and $F_1(kx) = 1$. From Corollary 2.2, we have

$$\begin{aligned} F_{2n}(kx) &= F_{2n}(2\sinh\theta) = \frac{\sinh 2n\theta}{\cosh\theta}, \\ F_{2n+1}(kx) &= F_{2n+1}(2\sinh\theta) = \frac{\cosh(2n+1)\theta}{\cosh\theta}, \end{aligned}$$

when $k = 2$ which are (6) and (7) shown in [6]. Obviously, from the above formulas and the identity $\cosh x + \cosh y = 2\cosh((x+y)/2)\cosh((x-y)/2)$, there holds

$$F_{2n+1}(kx) + F_{2n-1}(kx) = 2\cosh(2n\theta),$$

which was given in [6] as (8) when $k = 2$. Identity (9) in [6] is clearly the recurrence relation of $\{F_n(2x)\}$. The expressions of F_{2n} and F_{2n+1} can also be found in [13] with a general complex form

$$F_n(x) = i^{n-1}\frac{\sinh nz}{\sinh z},$$

where $x = 2i\cosh z$.

Corollary 2.3 *Suppose $\{a_n(x)\}$ is the function sequence defined by (17), $a_n(x) = p(x)a_{n-1}(x) + ba_{n-2}(x)$ ($b > 0$), with initials $a_0(x) = c$, a constant, and $a_1(x) = p(x)$, then*

$$\begin{aligned} a_{2n}(x) &= 2b^n\cosh(2n\theta) + (c-2)b^n\frac{\cosh(2n-1)\theta}{\cosh\theta} \\ a_{2n+1}(x) &= 2b^{n+1/2}\sinh(2n+1)\theta + (c-2)b^{n+1/2}\frac{\sinh 2n\theta}{\cosh\theta}, \end{aligned} \quad (20)$$

where $\theta(x) = \sinh^{-1}(p(x)/(2\sqrt{b})$. If $\{a_n(x)\}$ is the function sequence defined by (17), $a_n(x) = p(x)a_{n-1}(x) + ba_{n-2}(x)$ ($b < 0$), with initials $a_0(x) = c$, a constant, and $a_1(x) = p(x)$, then

$$a_n(x) = \frac{(-b)^{(n-1)/2}}{\sinh\theta}(2\cosh\theta\sinh n\theta - c\sqrt{-b}\sinh(n-1)\theta), \quad (21)$$

where $\theta(x) = \cosh^{-1}(p(x)/(2\sqrt{-b}))$.

Proof. Substituting $a_0(x) = c$, $a_1(x) = p(x) = 2\sqrt{b}\sinh\theta$ into (14) yields

$$\begin{aligned} a_{2n}(x) &= \frac{b^n}{\cosh\theta}\left[2\sinh\theta\sinh(2n\theta) + c\cosh(2n-1)\theta\right], \\ a_{2n+1}(x) &= \frac{b^n}{\cosh\theta}\left[2\sinh\theta\cosh(2n+1)\theta + c\sinh(2n\theta)\right]. \end{aligned}$$

Then in the above equations using the identities

$$\cosh\theta\cosh(2n\theta) - \sinh\theta\sinh(2n\theta) = \cosh(2n-1)\theta,$$
$$\cosh\theta\sinh(2n+1)\theta - \sinh\theta\cosh(2n+1)\theta = \sinh(2n\theta),$$

respectively, we obtain (20). Similarly, using (16) one may obtain (21).

■

Example 2.2 Since the generalized Lucas polynomials are defined by $L_n(kx) = kxL_{n-1}(kx) + L_{n-2}(kx)$ with the initials $L_0(x) = 2$ and $L_1(x) = kx$, from Corollary 2.3 we have

$$L_{2n}(kx) = L_{2n}(2\sinh\theta) = 2\cosh(2n\theta),$$
$$L_{2n+1}(kx) = L_{2n+1}(2\sinh\theta) = 2\sinh(2n+1)\theta.$$

[13] also presented a general complex form of $L_n(x)$ as

$$L_n(x) = 2i^n\cosh nz,$$

where $x = 2i\cosh z$.

Example 2.3 In 1959, Morgan-Voyce discovered two large families of polynomials, $b_n(x)$ and $B_n(x)$, in his study of electrical ladder networks of resistors [23]. The recurrence relations of the polynomials were presented in [19] as follows.

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x), \quad n \geq 2,$$

where $B_0(x) = 1$ and $B_1(x) = x + 2$, while

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x), \quad n \geq 2,$$

where $b_0(x) = 1$ and $b_1(x) = x + 1$. It can be found that

$$b_n(x) = B_n(x) - B_{n-1}(x),$$
$$xB_n(x) = b_{n+1}(x) - b_n(x).$$

Using Corollary 2.3, it is easy to obtain the hyperbolic expressions of $B_n(x)$ and $b_n(x)$. From (21) in the corollary and noting $B_1(x) = x+2 = 2\cosh\theta$ and $B_0(x) = 1$, we have

$$B_n(x) = \frac{\sinh(n+1)\theta}{\sinh\theta}, \quad x = 2\cosh\theta - 2.$$

Similarly, substituting $b_1(x) = x + 1 = 2\cosh\theta - 1$ and $b_0(x) = 1$ into (16) yields

$$b_n(x) = \frac{\sinh(n+1)\theta - \sinh n\theta}{\sinh\theta} = \frac{\cosh(2n+1)\theta/2}{\cosh\theta/2}, \quad x = 2\cosh\theta - 2.$$

We now consider the generalized Gegenbauer-Humbert polynomial sequences defined by (7) with $\lambda = C = 1$ and denoted by $P_n^{(y)}(x) \equiv P_n^{\lambda,y,C}(x)$. Thus

$$P_n^{(y)}(x) = 2xP_{n-1}^{(y)}(x) - yP_{n-2}^{(y)}(x), \tag{22}$$

$P_0^{(y)}(x) = 1$ and $P_1^{(y)}(x) = 2x$. We use the similar parameterization shown above to present the hyperbolic expression of those generalized Gegenbauer-Humbert polynomial sequences.

Corollary 2.4 *Let $P_n^{(y)}(x)$ be defined by (22) with initials $P_0^{(y)}(x) = 1$ and $P_1^{(y)}(x) = 2x$. If $y < 0$, then*

$$\begin{aligned} P_{2n}^{(y)}(x) &= (-y)^n \frac{\cosh(2n+1)\theta}{\cosh\theta}, \\ P_{2n+1}^{(y)}(x) &= (-y)^{n+1/2} \frac{\sinh(2n+2)\theta}{\cosh\theta}, \end{aligned} \tag{23}$$

where $\theta(x) = \sinh^{-1}(p(x)/(2\sqrt{-y})$. If $y > 0$, then

$$P_n^{(y)}(x) = y^{n/2} \frac{\sinh(n+1)\theta}{\sinh\theta}, \tag{24}$$

where $\theta(x) = \cosh^{(-1)}(p(x)/(2\sqrt{y})$.

Proof. A similar argument in the proof of (20) with $b = -y$ and $c = 1$ can be used to prove (23):

$$\begin{aligned} P_{2n}^{(y)}(x) &= 2(-y)^n \cosh(2n\theta) - (-y)^n \frac{\cosh(2n-1)\theta}{\cosh\theta} \\ P_{2n+1}^{(y)}(x) &= 2(-y)^{n+1/2} \sinh(2n+1)\theta - (-y)^{n+1/2} \frac{\sinh 2n\theta}{\cosh\theta}, \end{aligned}$$

where $\theta(x) = \sinh^{-1}(p(x)/(2\sqrt{-y})$, which implies (23) due to the identities $\cosh(2n+1)\theta + \cosh(2n-1)\theta = 2\cosh(2n\theta)\cosh\theta$ and $\sinh(2n+2)\theta + \sinh(2n\theta) = 2\sinh(2n+1)\theta\cosh\theta$. To prove (24), we substitute $-b = y$, and $a_1(x) = 2x = 2\sqrt{y}\cosh\theta$, and $a_0(x) = 1$ into (16). Thus

$$\begin{aligned} P_n^{(y)}(x) &= \frac{y^{n/2}}{\sinh\theta}(2\cosh\theta\sinh n\theta - \sinh(n-1)\theta) \\ &= y^{n/2}\frac{\sinh(n+1)\theta}{\sinh\theta}, \end{aligned}$$

where $\theta(x) = \cosh^{-1}(x/\sqrt{y})$ and the identity $\sinh(n+1)\theta + \sinh(n-1)\theta = 2\sinh n\theta\cosh\theta$ is applied in the last step.

■

Example 2.4 Using Corollary 2.4 one may obtain the following hyperbolic expressions of Pell polynomials $P_n(x) = P_n^{(-1)}(x)$ and the Chebyshev polynomials of the second kind $U_n(x) = P_n^{(1)}(x)$:

$$\begin{aligned} P_{2n}(x) &= \frac{\cosh(2n+1)\theta}{\cosh\theta}, \\ P_{2n+1}(x) &= \frac{\sinh(2n+2)\theta}{\cosh\theta}, \end{aligned}$$

where $\theta(x) = \sinh^{-1}(x)$, and

$$U_n(x) = \frac{\sinh(n+1)\theta}{\sinh\theta}, \tag{25}$$

where $\theta(x) = \cosh^{-1}(x)$.

Example 2.5 Finally, we consider the Chebyshev class of polynomials including the polynomials of the first kind, second kind, third kind, and fourth kind, denoted by $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$, respectively, which are defined by

$$a_n(x) = 2xa_{n-1}(x) - a_{n-2}(x), \quad n \geq 2, \tag{26}$$

with $a_0(x) = 1$ and $a_1(x) = x, 2x, 2x-1, 2x+1$ for $a_n(x) = T_n(x), U_n(x), V_n(x)$, and $W_n(x)$, respectively. Noting among those four polynomial sequences only $\{U_n(x)\}$ is in the generalized Gegenbauer-Humbert class, which has been presented in Example 2.3. From (16) there holds

$$T_n(x) = \frac{1}{\sinh\theta}(x\sinh n\theta - \sinh(n-1)\theta),$$

where $x = \cosh\theta$ due to $\theta = \cosh^{-1} x$. By using this substitution and the identity $\sinh(n-1)\theta = \sinh n\theta\cosh\theta - \cosh n\theta\sinh\theta$ we immediately obtain

$$T_n(x) = T_n(\cosh\theta) = \cosh n\theta.$$

Similarly,

$$V_n(x) = V_n(\cosh\theta) = \frac{\cosh(n+1/2)\theta}{\cosh(\theta/2)},$$
$$W_n(x) = W_n(\cosh\theta) = \frac{\sinh(n+1/2)\theta}{\sinh(\theta/2)}.$$

A simple transformation $\theta \mapsto i\theta$, $i = \sqrt{-1}$, leads $\cos(i\theta) = \cosh\theta$ and $\sin(i\theta) = -\sinh\theta$. Thus from the trigonometric expressions of $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$ shown below, one may also obtain their corresponding hyperbolic expressions by simply transforming $\theta \mapsto i\theta$, respectively.

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$
$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, \quad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}.$$

3 Hyperbolic expressions of parametric number sequences

Suppose $\{a_n\}$ is a number sequence defined by (1), i.e.

$$a_n = aa_{n-1} + ba_{n-2}, \quad n \geq 2, \tag{27}$$

with the given initials a_0 and a_1. From [12] (see Proposition 1.1), the sequence defined by (27) has the expression

$$\begin{aligned} a_n &= \frac{a_0(\alpha^{n+1}-\beta^{n+1})+(a_1-a_0a)(\alpha^n-\beta^n)}{\alpha-\beta} \\ &= \frac{a_1-\beta a_0}{\alpha-\beta}\alpha^n - \frac{a_1-\alpha a_0}{\alpha-\beta}\beta^n, \quad n\geq 2, \end{aligned} \tag{28}$$

where α and β are two distinct roots of characteristic polynomial $t^2 - at - b$. Similar to (10) we denote

$$(\alpha(\theta),\beta(\theta)) = \begin{cases} (\sqrt{b}e^{\theta}, -\sqrt{b}e^{-\theta}) & for\ b>0, \\ (\sqrt{-b}e^{\theta}, \sqrt{-b}e^{-\theta}) & for\ b<0,\ a>0, \\ (-\sqrt{-b}e^{\theta}, -\sqrt{-b}e^{-\theta}) & for\ b<0,\ a<0, \end{cases} \tag{29}$$

for some real or complex number θ. Thus we have

$$a(\theta) = \alpha+\beta = \begin{cases} 2\sqrt{b}\sinh(\theta) & for\ b>0, \\ 2\sqrt{-b}\cosh(\theta) & for\ b<0,\ a>0, \\ -2\sqrt{-b}\cosh(\theta) & for\ b<0,\ a<0, \end{cases} \tag{30}$$

and define a parameter generalization of $\{a_n(\theta)\}$ as

$$a_n(\theta) = \begin{cases} 2\sqrt{b}\sinh(\theta)a_{n-1}(\theta)+ba_{n-2}(\theta) & for\ b>0, \\ 2\sqrt{-b}\cosh(\theta)a_{n-1}(\theta)+ba_{n-2}(\theta) & for\ b<0,\ a>0, \\ -2\sqrt{-b}\cosh(\theta)a_{n-1}(\theta)+ba_{n-2}(\theta) & for\ b<0,\ a<0 \end{cases} \tag{31}$$

with initials $a_0(\theta) = a_0$ and $a_1(\theta) = a_1$ when $a_0 = 0$ or $a_1(\theta) =$ when $a_0 \neq 0$. Obviously, if

$$\theta = \begin{cases} \sinh^{-1}\left(\frac{a}{2\sqrt{b}}\right) & for\ b>0, \\ \cosh^{-1}\left(\frac{a}{2\sqrt{-b}}\right) & for\ b<0,\ a>0, \\ \cosh^{-1}\left(\frac{-a}{2\sqrt{-b}}\right) & for\ b<0,\ a<0, \end{cases} \tag{32}$$

$\{a_n(\theta)\}$ is reduced to $\{a_n\}$.

For $b > 0$, substituting expressions (29) into the second expression of a_n in (28), we obtain

$$\begin{aligned} & a_n(\theta) \\ = \ & b^{(n-1)/2}\frac{a_1(e^{n\theta}-(-1)^n e^{-n\theta})+\sqrt{b}a_0(e^{(n-1)\theta}+(-1)^n e^{-(n-1)\theta})}{e^\theta+e^{-\theta}} \end{aligned} \tag{33}$$

$$= \left\{ \begin{array}{ll} \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1\sinh n\theta+\sqrt{b}a_0\cosh(n-1)\theta\right), & if\, n\, is\, even, \\ \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1\cosh n\theta+\sqrt{b}a_0\sinh(n-1)\theta\right), & if\, n\, is\, odd, \end{array}\right. \tag{34}$$

where $\theta=\sinh^{-1}(a/(2\sqrt{b}))$.

Similarly, for $b<0$ we have

$$\begin{aligned} & a_n \\ = & \left\{ \begin{array}{ll} \frac{(-b)^{(n-1)/2}}{\sinh(\theta)}\big(a_0\sqrt{-b}\sinh((n+1)\theta) & \\ \quad +(a_1-2a_0\sqrt{-b}\cosh(\theta))\sinh(n\theta)\big) & for\ a>0, \\ \frac{(-\sqrt{-b})^{n-1}}{\sinh(\theta)}\big(-a_0\sqrt{-b}\sinh((n+1)\theta) & \\ \quad +(a_1+2a_0\sqrt{-b}\cosh(\theta))\sinh(n\theta)\big) & for\ a<0, \end{array}\right. \end{aligned} \tag{35}$$

or equivalently,

$$\begin{aligned} & a_n \\ = & \left\{ \begin{array}{ll} (-b)^{(n-1)/2}\frac{a_1(e^{n\theta}-e^{-n\theta})-a_0\sqrt{-b}(e^{(n-1)\theta}-e^{-(n-1)\theta})}{e^\theta-e^{-\theta}} & for\, a>0, \\ (-\sqrt{-b})^{n-1}\frac{a_1(e^{n\theta}-e^{-n\theta})+a_0\sqrt{-b}(e^{(n-1)\theta}-e^{-(n-1)\theta})}{e^\theta-e^{-\theta}} & for\, a<0, \end{array}\right. \\ = & \left\{ \begin{array}{ll} \frac{(-b)^{(n-1)/2}}{\sinh\theta}(a_1\sinh n\theta-a_0\sqrt{-b}\sinh(n-1)\theta) & for\, a>0, \\ \frac{(-\sqrt{-b})^{n-1}}{\sinh\theta}(a_1\sinh n\theta+a_0\sqrt{-b}\sinh(n-1)\theta) & for\, a<0, \end{array}\right. \end{aligned} \tag{36}$$

where $\theta=\cosh^{-1}(a/(2\sqrt{-b}))$ when $a>0$ and $\cosh^{-1}(-a/(2\sqrt{-b}))$ when $a<0$.

If the characteristic polynomial t^2-at-b has the same roots $\alpha=\beta$, then $a=\pm 2\sqrt{-b}$, $\alpha=\beta=\pm\sqrt{-b}$, and

$$a_n=na_1(\pm\sqrt{-b})^{n-1}-(n-1)a_0(\pm\sqrt{-b})^n. \tag{37}$$

We summarize the above results as follows.

Theorem 3.1 *Suppose* $\{a_n\}_{n\geq 0}$ *is a number sequence defined by (27) with characteristic polynomial* $t^2 - at - b$*. If the characteristic polynomial has the same roots, then there holds an expression of* a_n *shown in (37). If the characteristic polynomial has distinct roots, there hold hyperbolic extensions (51) or (52) for* $b > 0$ *and (36) or (36) for* $b < 0$*.*

Example 3.1 [18] gave the hyperbolic expression of the generalized Fibonacci number sequence $\{F_n(\theta)\}$ defined by

$$F_n(\theta) = 2\sinh\theta F_{n-1}(\theta) + F_{n-2}(\theta), \quad n \geq 2,$$

with initials $F_0(\theta) = 0$ and $F_1(\theta) = 1$. From Theorem 3.1, one may obtain the same result as that in [18]:

$$\begin{aligned} F_n(\theta) &= \frac{e^{n\theta} - (-1)^n e^{-n\theta}}{e^\theta + e^{-\theta}} \\ &= \begin{cases} \frac{\sinh n\theta}{\cosh\theta}, & if\ n\ is\ even; \\ \frac{\cosh n\theta}{\cosh\theta}, & if\ n\ is\ odd, \end{cases} \end{aligned} \tag{38}$$

Similarly, for the generalized Lucas number sequence $\{L_n(\theta)\}$ defined by

$$L_n(\theta) = 2\sinh\theta L_{n-1}(\theta) + L_{n-2}(\theta), \quad n \geq 2,$$

with initials $L_0(\theta) = 2$ and $L_1(\theta) = 2\sinh\theta$, we have

$$L_n(\theta) = e^{n\theta} + (-1)^n e^{-n\theta} = \begin{cases} 2\cosh(n\theta), & if\ n\ is\ even; \\ 2\sinh(n\theta), & if\ n\ is\ odd. \end{cases} \tag{39}$$

Example 3.2 [9] defined the following generalization of Fibonacci numbers and Lucas numbers:

$$f_n = \frac{c^n - d^n}{c - d}, \quad \ell_n = c^n + d^n, \tag{40}$$

where c and d are two roots of $t^2 - st - 1$, $s \in \mathbb{N}$. Denote $\Delta = s^2 + 4$ and $\alpha = \ln c$, where $c = (s + \sqrt{s^2+4})/2$. Then the above expressions are equivalent to

$$\frac{1}{2} f_n = \frac{e^{\alpha n} - (-1)^n e^{-\alpha n}}{2\sqrt{\Delta}}, \quad \frac{1}{2}\ell_n = \frac{e^{\alpha n} + (-1)^n e^{-\alpha n}}{2}.$$

It is obvious that by transferring $c \mapsto e^{\theta}$ and $d \mapsto -e^{-\theta}$ that two expressions in (40) are equivalently (38) and (39), respectively, shown in Example 3.1, which are obtained using Theorem 3.1 with $(a, b, a_0, a_1) = (s, 1, 0, 1)$ and $(s, 1, 2, s)$ for f_n and ℓ_n, respectively. Hence, the corresponding identities regarding f_n and ℓ_n obtained in [9] can be established similarly. However, we may derive more new identities as follows. For instance, there holds

$$\ell_n + s f_n = 2 f_{n+1}, \tag{41}$$

which can be proved by substituting $s = e^{\theta} - e^{-\theta} = 2\sinh\theta$ into the left-hand side. Indeed, for even n, from Example 3.1

$$\ell_n + s f_n = 2\cosh(n\theta) + 2\sinh\theta \frac{\sinh n\theta}{\cosh\theta} = 2\frac{\cosh(n+1)\theta}{\cosh\theta},$$

and similarly, for odd n, $\ell_n + s f_n = 2\sinh(n+1)\theta/\cosh\theta$, which brings (41). When $s = 1$, (41) reduces to the classical identity $L_n + F_n = 2F_{n+1}$.

From the above examples, we find many identities relevant to Fibonacci numbers and Lucas numbers can be proved using hyperbolic identities. Here are more examples.

Example 3.3 In the identity

$$\sinh 2n\theta = 2\sinh n\theta \cosh n\theta$$

substituting (38) and (39), namely, $\sinh 2n\theta = \cosh\theta \; F_{2n}(\theta)$ and

$$\sinh n\theta = \begin{cases} \cosh\theta \; F_n(\theta), & if\ n\ iseven, \\ \frac{1}{2} L_n(\theta), & if\ n\ is\ odd, \end{cases}$$
$$\cosh n\theta = \begin{cases} \frac{1}{2} L_n(\theta), & if\ n\ is\ even, \\ \cosh\theta \; F_n(\theta), & if\ n\ is\ odd, \end{cases}$$

we immediately obtain

$$F_{2n}(\theta) = F_n(\theta) L_n(\theta).$$

Similarly, since $\sinh(m+n)\theta = \cosh\theta \; F_{m+n}(\theta)$ when $m+n$ is even,

$$\sinh m\theta \cosh n\theta = \begin{cases} \frac{1}{2}\cosh\theta\ F_m(\theta)L_n(\theta), & \textit{if } m \textit{ and } n \textit{ are even},\\ \frac{1}{2}\cosh\theta\ F_n(\theta)L_m(\theta), & \textit{if } m \textit{ and } n \textit{ are odd}, \end{cases}$$

and

$$\cosh m\theta \sinh n\theta = \begin{cases} \frac{1}{2}\cosh\theta\ F_n(\theta)L_m(\theta), & \textit{if } m \textit{ and } n \textit{ are even},\\ \frac{1}{2}\cosh\theta\ F_m(\theta)L_n(\theta), & \textit{if } m \textit{ and } n \textit{ are odd}, \end{cases}$$

from identity

$$\sinh(m+n)\theta = \sinh m\theta \cosh n\theta + \cosh m\theta \sinh n\theta \tag{42}$$

we have

$$2F_{m+n}(\theta) = F_m(\theta)L_n(\theta) + F_n(\theta)L_m(\theta)$$

for even $m+n$.

When $m+n$ is odd, $\sinh(m+n)\theta = L_{m+n}(\theta)/2$, from (42),

$$\sinh m\theta \cosh n\theta = \begin{cases} \cosh^2\theta\ F_m(\theta)F_n(\theta), & \textit{if } m \textit{ is even and } n \textit{ is odd},\\ \frac{1}{4}L_m(\theta)L_n(\theta), & \textit{if } m \textit{ is odd and } n \textit{ is even}, \end{cases}$$

and

$$\cosh m\theta \sinh n\theta = \begin{cases} \frac{1}{4}L_m(\theta)L_n(\theta), & \textit{if } m \textit{ is even and } n \textit{ is odd},\\ \cosh^2\theta\ F_m(\theta)F_n(\theta), & \textit{if } m \textit{ is odd and } n \textit{ is even}, \end{cases}$$

we obtain

$$2L_{m+n}(\theta) = F_m(\theta)F_n(\theta) + L_m(\theta)L_n(\theta).$$

More examples can be found in [25].

Our scheme may also extend some well-know identities to their hyperbolic setting.

Example 3.4 [7] considers equation $t^2 - at + b = 0$ ($b \neq 0$) with distinct roots t_1 and t_2, i.e., $\Delta^2 = a^2 - 4b \neq 0$, and defines a sequence $\{g_n\}$ by $g_n = ag_{n-1} - bg_{n-2}$ ($n \geq 2$) with initials g_0 and g_1. If the initials are 0 and 1, the corresponding sequence is denoted by $\{r_n\}$. Denote $s_n = t_1^n + t_2^n$ and $\Delta = a^2 - 4b$. Then [7] gives identities

$$r_n = -b^n r_{-n}, \quad s_n = b^n s_{-n}, \tag{43}$$
$$s_n^2 = \Delta r_n^2 + 4b^n, \tag{44}$$
$$s_n s_{n+1} = \Delta r_n r_{n+1} + 2ab^n, , \tag{45}$$
$$2b^n r_{j-n} = r_j s_n - r_n s_j, , \tag{46}$$
$$r_{j+n} = r_n s_j + b^n r_{j-n}., \tag{47}$$
$$\tag{48}$$

We now show all the above identities can be extended to the hyperbolic setting. For $b > 0$, from (36) there holds

$$r_n = (b)^{(n-1)/2}\frac{e^{n\theta} - e^{-n\theta}}{e^{\theta} - e^{-\theta}} = \frac{(b)^{(n-1)/2}}{\sinh\theta}\sinh n\theta, \tag{49}$$

and similarly,

$$s_n = 2b^{n/2}\cosh n\theta, \tag{50}$$

where $\theta = \cosh^{-1}(a/(2\sqrt{b}))$.

For $b > 0$, substituting expressions (29) into (28), we obtain

$$\begin{aligned} & a_n(\theta) \\ = \; & b^{(n-1)/2}\frac{a_1(e^{n\theta} - (-1)^n e^{-n\theta}) + \sqrt{b}a_0(e^{(n-1)\theta} + (-1)^n e^{-(n-1)\theta})}{e^{\theta} + e^{-\theta}} \end{aligned} \tag{51}$$

$$= \begin{cases} \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1 \sinh n\theta + \sqrt{b}a_0 \cosh(n-1)\theta\right), & if\, n\, is\, even; \\ \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1 \cosh n\theta + \sqrt{b}a_0 \sinh(n-1)\theta\right), & if\, n\, is\, odd, \end{cases} \tag{52}$$

where $\theta = \sinh^{-1}(a/(2\sqrt{b}))$.

Similarly, for $b < 0$ we have

$$\begin{aligned} a_n \;=\; & \frac{(-b)^{(n-1)/2}}{\sinh(\theta)}\Big(a_0\sqrt{-b}\sinh((n+1)\theta) \\ & +(a_1 - 2a_0\sqrt{-b}\cosh(\theta))\sinh(n\theta)\Big), \end{aligned} \tag{53}$$

or equivalently,

$$a_n = (-b)^{(n-1)/2}\frac{a_1(e^{n\theta}-e^{-n\theta})-\sqrt{-b}a_0(e^{(n-1)\theta}-e^{-(n-1)\theta})}{e^{\theta}-e^{-\theta}} \quad (54)$$

$$= \frac{(-b)^{(n-1)/2}}{\sinh\theta}(a_1\sinh n\theta - a_0\sqrt{-b}\sinh(n-1)\theta), \quad (55)$$

where $\theta = \cosh^{-1}(a/(2\sqrt{-b}))$.

Acknowledgments

We wish to thank the referees for their helpful comments and suggestions.

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