A Pair of Operator Summation Formulas and Their Applications

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Abstract

Two types of symbolic summation formulas are reformulated using an extension of Mullin-Rota's substitution rule in [1], and several applications involving various special formulas and identities are presented as illustrative examples.

Key words Delta operator, Bernoulli number, Catalan number, generalized harmonic number, Stirling numbers

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1 Introduction

The recent paper [2] by He, Hsu, and Shiue has shown that, as an application of the substitution rule based on Mullin-Rota theory of binomial enumeration (cf. [1]), the symbolization of generating functions may yield more than a dozen symbolic summation formulas involving delta operator Δ and D. Here let us recall that Δ (difference operator) and D (differentiation operator) together with E (shift operator) are usually defined for all $f(t) \in C^\infty$ (the class of infinitely differentiable real functions in $\mathbb{R} = (-\infty, \infty)$) via the relations

$$\Delta f(t) = f(t+1) - f(t), \quad Df(t) = \frac{d}{dt}f(t) = f'(t), \quad Ef(t) = f(t+1).$$

Consequently they satisfy some simple symbolic relations such as

$$E = 1 + \Delta, \quad E = e^D, \quad \Delta = e^D - 1, \quad D = \log E = \log(1+\Delta),$$

where the unity 1 serves as an identity operator such that $1f(t) = f(t)$. Also, for any real or complex number α, we may define $E^\alpha f(t) = f(t+\alpha)$ with $E^0 = D^0 = \Delta^0 = 1$. In addition, an operator T is called a shift-invariant operator (see, for example, [1]) if it commutes with the shift operator E , i.e.,

$$TE^\alpha = E^\alpha T,$$

where $E^\alpha f(t) = f(t+\alpha)$ and $E^1 \equiv E$. Clearly, the differentiation operator D and the difference operator Δ are shift-invariant operators. An operator Q is called a delta operator if it is shift-invariant and Qt is a non-zero constant. Obviously, both D and Δ are delta operators.

What we wish to show is that the two types of symbolic summation formulas expanded in [2] may be reformulated using an extension of Mullin-Rota's substitution rule so that they could apply to more cases than those given previously. Accordingly we will consider some new applications, and present several examples and identities involving some special number sequences such as Bernoulli, Catalan, Stirling, harmonic numbers and the generalized harmonic numbers. In addition, we shall show that the formal power series can be recovered from the corresponding symbolic summation formulas by substituting a certain chosen function.

2 Two basic theorems

Let Q be a delta operator, and let F be the ring of formal power series in the variable t, over the same field, then [1] proved that there exists an isomorphism from F onto the ring $\sum$ of shift-invariant operators, which carries

$$g(x) = \sum_{k\geq 0} \frac{a_k}{k!}x^k \text{ into } g(Q) \equiv G(x,Q) := \sum_{k\geq 0} \frac{a_k}{k!}Q^k.$$

The above rule is called Mullin-Rota's substitution rule.

Denote by $G(x,y,z)$ a rational function in three variables x, y, and z. In particular, $G(x,y,1)$ and $G(x,1,1)$ denote rational functions in two variables and one variable, respectively. In what follows we always assume that $F(x) \equiv \sum_{k=0}^\infty f_k x^k$ is a formal power series. Then we shall use Mullin-Rota's substitution rule to establish the following results.

Theorem 2.1 *Suppose that for given power series $F(x)$ there is an expression or a sum formula of the form*

$$\sum_{k\geq 0} f_k x^k = G(x, e^x, e^{\alpha x}), \tag{2.1}$$

where the parameter $\alpha \neq 0$ is a real or complex number. Then the substitution $x \mapsto D$ yields a symbolic summation formula for every $f \in C^\infty$ evaluated at $t = 0$, namely

$$\sum_{k\geq 0} f_k D^k f(0) = G(D, E, E^\alpha) f(0). \tag{2.2}$$

Moreover, (2.2) implies (2.1) as a particular case with $f(t) = e^{xt}$.

Theorem 2.2 *Suppose that for given power series $F(x)$ there is an expression or a sum formula of the form*

$$\sum_{k\geq 0} f_k x^k = G(x, \log(1+\alpha x), (1+\alpha x)^\beta), \tag{2.3}$$

where α and β are real parameters with $\alpha\beta \neq 0$. Then the substitution $x \mapsto \frac{1}{\alpha}\Delta$ yields a symbolic summation formula of the form

$$\sum_{k\geq 0} f_k \left(\frac{1}{\alpha}\right)^k \Delta^k f(0) = G(\frac{\Delta}{\alpha}, D, E^\beta) f(0). \tag{2.4}$$

Moreover, (2.4) implies (2.3) as a particular case with $f(t) = (1+\alpha x)^t$.

Proof: Theorems 2.1 and 2.2 can be proved similarly. Since both D and Δ are delta operators, so that (2.2) and (2.4) as symbolizations of (2.1) and (2.3), respectively, can be justified by a similar argument of Mullin-Rota's substitution rule (see [1] or [2]). More precisely, both (2.1) and (2.3) are identities in the variable x, and that there is an isomorphism between the ring of shift-invariant operators and the ring of formal power series in x. Hence, (2.2) and (2.4) are obtained accordingly. It remains to show that the choices $f(t) \equiv f(t;x) = e^{xt}$ and $f(t) \equiv f(t;x) = (1+\alpha x)^t$ will respectively lead (2.2) and (2.4) to recover (2.1) and (2.3). For the particular choice $f(t) = e^{xt}$ we see that the right-hand side (RHS) of (2.2) can be written as follows

$$\begin{aligned} \mathrm{R}HS\ of\ (2.2) &= G(D, e^D, e^{\alpha D}) f(0) \\ &= \sum_{k\geq 0} f_k D^k f(0) = \sum_{k\geq 0} f_k D^k e^{xt}{}_{t=0} \\ &= \sum_{k\geq 0} f_k x^k = G(x, e^x, e^{\alpha x}). \end{aligned}$$

Also, the left-hand side (LHS) of (2.2) with $f(t) = e^{xt}$ gives $\sum_{k\geq 0} f_k x^k$. Hence, (2.1) is implied by (2.2).

The implication (2.4) $\Rightarrow$ (2.3) with $f(t) = (1+\alpha x)^t$ can be verified in a similar manner, in which it suffices to observe that the LHS of (2.4) with $f(t) = (1+\alpha x))^t$ gives $\sum_{k\geq 0} f_k x^k$, and that the RHS of (2.4) gives

$$\begin{aligned} & G\left(\frac{\Delta}{\alpha}, \log(1+\Delta), (1+\Delta)^\beta\right) f(0) \\ = & \sum_{k\geq 0} f_k \left(\frac{\Delta}{\alpha}\right)^k f(0) = \sum_{k\geq 0} f_k \left[\left(\frac{\Delta}{\alpha}\right)^k (1+\alpha x)^t\right]_{t=0} \\ = & \sum_{k\geq 0} f_k x^k = G\left(x, \log(1+\alpha x), (1+\alpha x)^\beta\right), \end{aligned}$$

which completes the proof. ∎

The following two examples may further illustrate the second halves of the theorems. First, using (2.2) with $F(x) = e^{ax} = \sum_{k\geq 0}(ax)^k/k!$ with $x \mapsto D$ yields the summation formula $\sum_{k\geq 0} a^k D^k f(0)/k! = f(a)$, which implies $e^{ax} = \sum_{k\geq 0}(ax)^k/k!$ as a special case with $f(t) = e^{xt}$. Similarly, if (2.3) is given with $F(x) = -\log(1-x) = \sum_{k\geq 1} x^k/k!$, then the corresponding summation formula (2.4) with the mapping $x \mapsto (-\Delta)$ is $\sum_{k\geq 1}(-1)^{k+1}\ \Delta^k f(0)/k = f'(0)$, which implies $\sum_{k\geq 1} x^k/k! = -\log(1-x)$ as a special case with $f(t) = (1-x)^t$.

The technique presented in the above theorems can be considered as extensions of (Mullin-Rota's) substitution rule. For brevity, formulas (2.2) and (2.4) may be simply called D-type formula and Δ-type formula respectively. These formulas obviously provide generalizations of the sum formulas for single power series. As may be observed, substantially all the operational formulas $(O_2) - (O_{12})$, as displayed in [2], together with the symbolic formulas expressing D^m (or Δ^m) in terms of Δ^k's (or D^k's) are particular consequences of (2.2) and (2.4), respectively.

It may be noted that the operational formula given in Example 5.14 of [2] of the form

$$(O_{13}): \qquad \sum_{k=0}^{\infty} k^m \Delta^{m+1} f(k) = (-1)^{m+1} A_m(E) f(0)$$

is incorrect, where $A_m(x)$ denotes the mth degree Eulerian polynomial given by the expression

$$A_0(x) = 1 \quad \text{and} \quad A_m(x) = \sum_{k=1}^{m} A(m,k) x^k \ (m \geq 1),$$

with $A(m, 0) = 0$ and

$$A(m,k) = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} (k-j)^m \ (1 \leq k \leq m).$$

$A(m,k)$ are known to be the Eulerian numbers (*cf.* Comtet [3, p. 243-5]). In fact, taking $f(t)$ to be a polynomial of degree $\leq m$ with $m \geq 1$, we see that the LHS of (O_{13}) gives zero, while the RHS differs from zero. Actually (O_{13}) is obtained from the symbolization of Euler's formula

$$\sum_{k=0}^{\infty} k^m x^k = \alpha_m(x) = \frac{A_m(x)}{(1-x)^{m+1}} \ (|x| < 1),$$

by the substituting $x \mapsto E$, where $E = 1 + \Delta$ is not a delta operator inasmuch as $Et = t + 1$ is not a non-zero constant.

A valid symbolization should be made by the substitution $x \mapsto (-\Delta)$, so that Euler's formula yields a special Δ- type formula of the form

$$(O_{14}): \qquad \sum_{k\geq 0} (-1)^k k^m \Delta^k f(m+1) = A_m(-\Delta) f(0) = \sum_{k=1}^{m} A(m,k)(-1)^k \Delta^k f(0).$$

Taking $f(t) = 1/(1+t)$ into (O_{14}), we find (*cf.* (5.17) of [2])

$$\frac{1}{m+2} \sum_{k=0}^{\infty} k^m \binom{m+k+2}{m+2}^{-1} = \sum_{k=1}^{m} A(m,k)/(k+1) \quad (m \geq 1).$$

Curiously enough, this correct summation is also obtainable from the incorrect formula (O_{13}). This might suggest that (O_{13}) could still be valid under certain restrictive conditions.

One may recover Euler's formula from (O_{14}) by substituting $f(t) = (1-x)^t$. Indeed, for the function $f(t)$, we have $\Delta^k f(m+1) = (1-x)^{m+1}(-x)^k$ and $\Delta^k f(0) = (-x)^k$. Thus $[A_m(-\Delta)(1-x)^t]_{t=0} = \sum_{k=1}^{m} A(m,k)(-1)^k \Delta^k f(0) = \sum_{k=1}^{m} A(m,k)x^k = A_m(x)$, and (O_{14}) becomes $\sum_{k\geq 0} k^m (1-x)^{m+1} x^k = A_m(x)$, which is the Euler's formula $\sum_{k\geq 0} k^m x^k = A_m(x)/(1-x)^{m+1}$ for $x \neq 1$.

3 Application of (2.2) and (2.4)

In addition to those generating functions already investigated in [2], let us now consider some other generating functions or power series expansions with closed sums as follows (*cf.* Wilf [4]).

(*i*) $\sum_{k\geq 0} \frac{4^k B_{2k}}{(2k)!} x^{2k} = x \coth x$, where B_{2k} are Bernoulli numbers.

(*ii*) $\sum_{k\geq 0} \frac{\phi_r(k) x^k}{k!} = e^x \sum_{k=0}^{r} \frac{\Delta^k \phi_r(0)}{k!} x^k$, where $\phi_r(x)$ is a rth degree polynomial (*cf.* Jolley [5, p. 218]).

(*iii*) $\sum_{k\geq 0} C_k x^k = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right)$, where $C_k = \frac{1}{k+1}\binom{2k}{k}$ are Catalan numbers.

(*iv*) $\sum_{k\geq 1} H_k x^k = \frac{1}{1-x} \log \frac{1}{1-x}$, where H_k are harmonic numbers defined by $H_k = \sum_{j=1}^{k} 1/j$ for $k \geq 1$ with $H_0 = 0$.

(*v*) $\sum_{k\ge 2}\frac{1}{k}H_{k-1}x^k=\frac{1}{2}\left(\log\frac{1}{1-x}\right)^2$

(*vi*) $\sum_{k\ge 0}\binom{2k+r}{k}x^k=\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^r \quad (r\ge 0)$

(*vii*) $\sum_{k\ge 1}H(k,r)x^k=\frac{1}{1-x}\left(\log\frac{1}{1-x}\right)^{r+1}$, where $H(k,r)$ are generalized harmonic numbers (*cf.* [6]) defined by $H(k,r)=\sum_{1\le n_0+n_1+\cdots+n_r\le k}1/(n_0n_1\cdots n_r)$ for $k\ge 1$ and $r\ge 0$ with $H(0,r)=0$. It is obvious that $H(k,0)=H_k$.

(*viii*) $\sum_{k\ge 0}\frac{r(2k+r-1)!}{k!(k+r)!}x^k=\left(\frac{1-\sqrt{1-4x}}{2x}\right)^r$, which includes (iii) as a special case when $r=1$.

Evidently, (i) and (ii) are of the form (2.1), and (iii)-(viii) of the form (2.3). Consequently (i) and (ii) should lead to special D-type formulas, and (iii)-(viii) to Δ-type formulas. Indeed, making use of (2.2) we easily find

$$\sum_{k\ge 0}\frac{4^kB_{2k}}{(2k)!}D^{2k}f(0)=D\frac{E+E^{-1}}{E-E^{-1}}f(0).$$

Notice that $(E-E^{-1})D^{2k}f(0)=f^{(2k)}(1)-f^{(2k)}(-1)$. Thus we can obtain a symbolic summation formula of the form

$$\sum_{k\ge 0}\frac{4^kB_{2k}}{(2k)!}[f^{(2k)}(1)-f^{(2k)}(-1)]=f'(1)+f'(-1). \tag{3.1}$$

Similarly, utilizing formulas (2.2) and (2.4) one may find that (ii)-(viii) yield 7 special symbolic summation formulas as follows

$$\sum_{k\ge 0}\frac{\phi_r(k)}{k!}f^{(k)}(0)=\sum_{k=0}^{r}\frac{\Delta^k\phi_r(0)}{k!}f^{(k)}(1) \tag{3.2}$$

$$\sum_{k\ge 0}\left(\frac{-1}{4}\right)^kC_k\Delta^{k+1}f(0)=2\left[f\left(\frac{1}{2}\right)-f(0)\right] \tag{3.3}$$

$$\sum_{k\ge 1}(-1)^kH_k\Delta^kf(0)=-f'(-1) \tag{3.4}$$

$$\sum_{k\ge 2}\frac{(-1)^k}{k}H_{k-1}\Delta^kf(0)=\frac{1}{2}f''(0) \tag{3.5}$$

$$\sum_{k\ge 0}\frac{(-1)^k}{2^{2k+r}}\binom{2k+r}{k}\Delta^{k+r}f(0)=(E^{1/2}-1)^rf\left(-\frac{1}{2}\right), \tag{3.6}$$

$$\sum_{k\ge 1}(-1)^kH(k,r)\Delta^kf(0)=(-1)^{r+1}f^{(r+1)}(-1), \tag{3.7}$$

$$\sum_{k\ge 0}\frac{(-1)^kr(2k+r-1)!}{4^kk!(k+r)!}\Delta^{k+r}f(0)=2\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f\left(\frac{j}{2}\right), \tag{3.8}$$

where the RHS of (3.6) may be written in the explicit form

$$(E^{1/2}-1)^r f\left(-\frac{1}{2}\right)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f\left(\frac{j-1}{2}\right). \tag{3.9}$$

More precisely, (3.2)-(3.8) are obtained from (ii)-(vi) by the substitutions $x\mapsto D$, $x\mapsto\left(-\frac{1}{4}\Delta\right)$, $x\mapsto(-\Delta)$, $x\mapsto\left(-\frac{1}{4}\Delta\right)$, $x\mapsto(-\Delta)$, and $x\mapsto\left(-\frac{1}{4}\Delta\right)$ respectively.

4 Some convergence conditions

Here we provide a list of conditions for the absolute convergence of the series expansions in (3.1)-(3.8).

formula	convergence condition
(3.1)	$\overline{\lim}_{k\to\infty}\left\|f^{(2k)}(\pm 1)\right\|^{1/k}<\pi^2$
(3.2)	$\overline{\lim}_{k\to\infty}\left\|f^{(k)}(0)/k!\right\|^{1/k}<1$
(3.3)	$\Delta^k f(0)=O\left(k^{\frac{1}{2}-\epsilon}\right)\ (\epsilon>0)$
(3.4)	$\Delta^k f(0)=O\left((1/k)^{1+\epsilon}\right)\ (\epsilon>0)$
(3.5)	$\Delta^k f(0)=O\left((1/k)^{\epsilon}\right)\ (\epsilon>0)$
(3.6)	$\overline{\lim}_{k\to\infty}\left\|\Delta^k f(0)\right\|^{1/k}<1$
(3.7)	$\Delta^k f(0)=O\left((1/k)^{1+\epsilon}\right)\ (\epsilon>0)$
(3.8)	$\Delta^k f(0)=O\left(k^{\frac{1}{2}-\epsilon}\right)\ (\epsilon>0)$

The convergence conditions shown above can be justified by the aid of Cauchy's root test and the comparison test. Notice that there is an estimate for Bernoulli numbers, viz. (*cf.* Jordan [7, §82])

$$\left|\frac{B_{2k}}{(2k)!}\right|<\frac{1}{12(2\pi)^{2k-2}}\quad(k\ge 0).$$

It follows that upper limit

$$\overline{\lim}_{k\to\infty}\left|\frac{B_{2k}}{(2k)!}\right|^{1/k}\le\frac{1}{4\pi^2}$$

(Actually, Euler's famous formula for Bernoulli numbers, $(-1)^{k+1}B_{2k}/(2k)! = 2\zeta(2k)/(2\pi)^{2k}$ implies the limit of $(B_{2k}/(2k)!)^{1/k}$ equals to $1/(4\pi^2)$, so that the convergence condition for (3.1) implies that

$$\overline{\lim}_{k\to\infty} \left| \frac{4^k B_{2k}}{(2k)!} f^{(2k)}(\pm 1) \right|^{1/k} < 1.$$

Hence the absolute convergence of the series in (3.1) follows from the root test.

Moreover, notice that $\lim_{k\to\infty} |\phi_r(k)|^{1/k} = 1$ and that

$$\lim_{k\to\infty} \left| \frac{1}{2^{2k+r}} \binom{2k+r}{k} \right|^{1/k} = 1,$$

where the limit follows from an application of Stirling's asymptotic formula $n! \sim (n/e)^n \sqrt{2\pi n}$ as $n \to \infty$. Thus the convergence conditions for (3.2) and (3.6) also follow from the root test.

Evidently the convergence conditions for (3.3), (3.4), (3.5), (3.7), and (3.8) are justified by the following asymptotic relations, respectively

$$\begin{aligned} &C_k = \frac{1}{k+1}\binom{2k}{k} \sim 4^k/(k\sqrt{k\pi}), \\ &H(k,r) \sim \log k \quad (r = 0, 1, \ldots), \\ &\frac{r(2k+r-1)!}{k!(k+r)!} \sim 4^k k^{-3/2}, \end{aligned}$$

as $k \to \infty$. Here, the second estimation for $r \geq 1$ comes from [6, (3.2)].

5 Examples- Various identities and series sums

Certainly, each of the formulas (3.1)-(3.8) may be used to yield a variety of particular identities or series sums via suitable choices of $f(t)$. Here we will present a number of selective examples to illustrate the applications of (3.1)-(3.8).

Example 1 Let n be an odd positive integer, and take $f(t) = t^n$, $(n \geq 1)$. Then we have

$$f'(1) = f'(-1) = n, \quad f^{(2k)}(\pm 1) = \pm n^{\underline{2k}},$$

where we use the following falling factorial notation $x^{\underline{r}}$ (sometimes also denoted $(x)_r$), i.e., $x^{\underline{r}} = x(x-1)^{\underline{r-1}} (r \geq 1)$ with $x^{\underline{0}} = 1$. Thus using (3.1) we get

$$\sum_{k=0}^{[n/2]} 4^k B_{2k} \binom{n}{2k} = n. \tag{5.1}$$

Example 2 Let λ be a real number with $\lambda \neq 0$. Then a much more general identity of the form

$$\sum_{k=0}^{m} 4^{2k} B_{2k} \binom{\lambda+2k-1}{2k} \binom{2\lambda+2m+2k}{2m-2k+1} = 2\lambda \binom{2\lambda+2m+1}{2m} \tag{5.2}$$

can be obtained from (3.1) by taking $f(t) = C_n^\lambda(t)$ with $n = 2m+1$, where $C_n^\lambda(t)$ is the nth degree Gegenbauer polynomial given by the generating function

$$(1-2tx+x^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(t) x^k \quad (\lambda \neq 0). \tag{5.3}$$

Indeed, a few simple properties of $C_n^\lambda(t)$ may be deduced from (5.3), namely (*cf.* Magnus-Oberhettinger-Soni [8, §5.3])

$$C_n^\lambda(1) = \frac{(2\lambda)^{\bar{n}}}{n!},\ C_n^\lambda(-t) = (-1)^n C_n^\lambda(t),\ \left(\frac{d}{dt}\right)^m C_n^\lambda(t) = 2^m \lambda^{\bar{m}} C_{n-m}^{\lambda+m}(t),$$

where we have used the raising factorial notation $x^{\bar{r}}$ (sometimes also denoted $(x)^r$ or $\langle x \rangle_r$), i.e., $x^{\bar{r}} = x(x+1)^{\overline{r-1}}$ $(r \geq 1)$ with $x^{\bar{0}} = 1$.

Consequently, the fact that (3.1) implies (5.2) is confirmed by easy computations with the aid of the above mentioned properties.

For the particular choices $\lambda = 1$ and $\lambda = 1/2$, we see that (5.2) gives the following identities respectively

$$\sum_{k=0}^{m} 4^{2k} B_{2k} \binom{2m+2k+2}{4k+1} = 2\binom{2m+3}{3}, \tag{5.4}$$

$$\sum_{k=0}^{m} B_{2k} \binom{4k}{2k} \binom{2m+2k+1}{4k} = \binom{2m+2}{2}. \tag{5.5}$$

Example 3 Recall that Stirling numbers of the first and second kind may be defined by the following equations respectively.

$$(-1)^{n-k} \left[\begin{array}{c} n \\ k \end{array} \right] := \frac{1}{k!} \left[D^k t^{\underline{n}} \right]_{t=0}, \quad \left\{ \begin{array}{c} n \\ k \end{array} \right\} := \frac{1}{k!} \left[\Delta^k t^n \right]_{t=0}. \tag{5.6}$$

Here we have adapted the notations due to Knuth (*cf.* [4] and [9]), where $\left[\begin{array}{c} n \\ k \end{array} \right]$ denotes the signless Stirling numbers of the first kind, i.e., the number of permutations of n objects having k cycles. Now, taking $\phi_r(t) = t^r$ we have $\Delta^k \phi_r(0) = k! \left\{ \begin{array}{c} r \\ k \end{array} \right\}$, and we see that (3.2) yields the formula

$$\sum_{k \geq 0} \frac{k^r}{k!} f^{(k)}(0) = \sum_{j=0}^{r} \left\{ \begin{array}{c} r \\ j \end{array} \right\} f^{(j)}(1). \tag{5.7}$$

This formula implies several interesting special identities.

(1) Taking $f(t)=e^t$, we get

$$\frac{1}{e}\sum_{k\geq 0}\frac{k^r}{k!}=\sum_{j=0}^{r}\left\{ \begin{array}{c} r \\ j \end{array} \right\}=\omega(r). \tag{5.8}$$

This is the well-known formula of Dobinski for the Bell number $\omega(r)$.

(2) Choosing $f(t)=1+t+\cdots+t^m$ $(m\geq 1)$ we find $f^{(k)}(0)=k!$ for $k\leq m$, and $f^{(k)}(0)=0$ for $k>m$, and moreover,

$$\left(\frac{d}{dt}\right)^j (1+t+\cdots+t^m)|_{t=1}=j!\left[\binom{j}{j}+\binom{j+1}{j}+\cdots+\binom{m}{j}\right]=j!\binom{m+1}{j+1}.$$

Thus (5.7) gives

$$\sum_{k=0}^{m}k^r=\sum_{j=0}^{r}j!\binom{m+1}{j+1}\left\{ \begin{array}{c} r \\ j \end{array} \right\}. \tag{5.9}$$

This is the classical formula for arithmetic progression of higher order.

(3) Taking $f(t)=\sum_{k\geq 0}(tx)^k=(1-tx)^{-1}$ with $|tx|<1$, we find $f^{(k)}(0)=k!x^k$ and $f^{(k)}(1)=k!x^k(1-x)^{-k-1}$. Thus (5.7) yields

$$\sum_{k\geq 0}k^r x^k=\sum_{j=0}^{r}j!\left\{ \begin{array}{c} r \\ j \end{array} \right\}x^j(1-x)^{-j-1}\quad (|x|<1). \tag{5.10}$$

This is Euler's formula for the arithmetic-geometric series.

(4) Take $f(t)=t^{\underline{m}}$ so that $f^{(k)}(0)=(-1)^{m-k}k!\left[\begin{array}{c} m \\ k \end{array} \right]$. We have to compute $f^{(k)}(1)$. By (5.6), it is easily found that

$$\begin{aligned}
\left[D^k t^{\underline{m}}\right]_{t=1} &= \left[D^k (t+1)^{\underline{m}}\right]_{t=0}\\
&= (t+1)_{t=0}\left[D^k t^{\underline{m-1}}\right]_{t=0}+\binom{k}{1}\left[D^{k-1} t^{\underline{m-1}}\right]_{t=0}\\
&= (-1)^{m-k-1}k!\left[\begin{array}{c} m-1 \\ k \end{array} \right]+(-1)^{m-k}k(k-1)!\left[\begin{array}{c} m-1 \\ k-1 \end{array} \right]\\
&= k!\left((-1)^{m-k-1}\left[\begin{array}{c} m-1 \\ k \end{array} \right]+(-1)^{m-k}\left[\begin{array}{c} m-1 \\ k-1 \end{array} \right]\right).
\end{aligned}$$

Thus (5.7) gives

$$\sum_{k=1}^{m}(-1)^{m-k}k^r\left[\begin{array}{c} m \\ k \end{array} \right]=\sum_{j=1}^{r}j!\left\{ \begin{array}{c} r \\ j \end{array} \right\}\left((-1)^{m-j-1}\left[\begin{array}{c} m-1 \\ j \end{array} \right]+(-1)^{m-j}\left[\begin{array}{c} m-1 \\ j-1 \end{array} \right]\right). \tag{5.11}$$

This may be compared with the known identity

$$\sum_{k=1}^{m} k^r \binom{m}{k} = \sum_{j=1}^{r} j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \binom{m}{j} 2^{m-j}. \tag{5.12}$$

which is also obtained from (5.7) by taking $f(t) = (1+t)^m$.

(5) Choosing $f(t) = t^{\bar{m}} := t(t+1)\cdots(t+m-1)$ ($m \geq 1$ is arbitrarily fixed), we have

$$f(t) = \sum_{k\geq 1} \left| \left[\begin{matrix} m \\ k \end{matrix} \right] \right| t^k = \sum_{k\geq 1} \frac{m!}{k!} \binom{m-1}{k-1} t^{\underline{k}}.$$

Hence, $f^{(k)}(0) = k! \left| \left[\begin{matrix} m \\ k \end{matrix} \right] \right|$ and from (4)

$$\begin{aligned} f^{(j)}(1) &= \sum_{k\geq 1} \frac{m!}{k!} \binom{m-1}{k-1} \left[D^j t^{\underline{k}} \right]_{t=1} \\ &= \sum_{k\geq 1} j! \frac{m!}{k!} \binom{m-1}{k-1} \left((-1)^{k-j-1} \left[\begin{matrix} k-1 \\ j \end{matrix} \right] + (-1)^{k-j} \left[\begin{matrix} k-1 \\ j-1 \end{matrix} \right] \right). \end{aligned}$$

Therefore, (5.7) gives

$$\sum_{k\geq 1} k^r \left| \left[\begin{matrix} m \\ k \end{matrix} \right] \right| = \sum_{j=1}^{r} \sum_{k\geq 1} j! \frac{m!}{k!} \binom{m-1}{k-1} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \left((-1)^{k-j-1} \left[\begin{matrix} k-1 \\ j \end{matrix} \right] + (-1)^{k-j} \left[\begin{matrix} k-1 \\ j-1 \end{matrix} \right] \right).$$

(6) Take $f(t) = t(t-an)^{n-1}$, the Abel polynomial with $n \geq 1$, so that $f^{(k)}(0) = k(n-1)^{\underline{k-1}}(-an)^{n-k}$ and

$$\begin{aligned} f^{(j)}(1) &= D^j \left[t(t-an)^{n-1} \right]_{t=1} \\ &= \left[t(n-1)^{\underline{j}}(t-an)^{n-j-1} \right]_{t=1} + \left[j(n-1)^{\underline{j-1}}(t-an)^{n-j} \right]_{t=1} \\ &= (n-1)^{\underline{j-1}}(1-an)^{n-j-1} \left[(n-j) + j(1-an) \right] \\ &= n^{\underline{j}}(1-aj)(1-an)^{n-j-1}. \end{aligned}$$

Thus (5.7) yields

$$\sum_{k\geq 1} \frac{k^r}{(k-1)!} (n-1)^{\underline{k-1}}(-an)^{n-k} = \sum_{j=0}^{r} n^{\underline{j}}(1-aj) \left\{ \begin{matrix} r \\ j \end{matrix} \right\} (1-an)^{n-j-1}.$$

Example 4 Let $\alpha \in \mathbb{R}$. We have

$$\Delta^k \binom{t+\alpha}{n} = \binom{t+\alpha}{n-k} \quad (n \geq k \geq 0).$$

Thus, taking $f(t) = \binom{t+\alpha}{n}$ we have $\Delta^k f(0) = \binom{\alpha}{n-k}$. Consequently, (3.3) yields the identity

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{4^k(k+1)} \binom{2k}{k} \binom{\alpha}{n-k-1} = 2\left[\binom{\alpha+1/2}{n} - \binom{\alpha}{n}\right]. \tag{5.13}$$

Example 5 For $f(t) = t^n$ $(n \geq 1)$ we have $\Delta^k f(0) = k! \left\{ \begin{array}{c} n \\ k \end{array} \right\}$, so that formulas (3.3)-(3.5), and (3.7) give four identities as follows

$$\sum_{k=0}^{n-1} \frac{(-1)^k k!}{2^{2k}} \binom{2k}{k} \left\{ \begin{array}{c} n \\ k+1 \end{array} \right\} = \left(\frac{1}{2}\right)^{n-1}, \tag{5.14}$$

$$\sum_{k=1}^{n} (-1)^k k! H_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} = (-1)^n n, \tag{5.15}$$

$$\sum_{k=2}^{n} (-1)^k (k-1)! H_{k-1} \left\{ \begin{array}{c} n \\ k \end{array} \right\} = \left\{ \begin{array}{ll} 1 & if\ n=2 \\ 0 & if\ n>2 \end{array} \right. . \tag{5.16}$$

$$\sum_{k=1}^{n} (-1)^k k! H(k,r) \left\{ \begin{array}{c} n \\ k \end{array} \right\} = (-1)^n n^{\underline{r+1}} \quad (r \geq 1) \tag{5.17}$$

Example 6 Taking $f(t) = \binom{m+t}{n}$, $(m > n \geq 1)$, we find

$$\frac{d}{dt}\binom{m+t}{n} = \frac{(m+t)^{\underline{n}}}{n!}\left(\frac{1}{m+t} + \frac{1}{m-1+t} + \cdots + \frac{1}{m-n+1+t}\right).$$

Consequently, we have

$$\begin{aligned} f'(-1) &= \frac{(m-1)^{\underline{n}}}{n!}\left(\frac{1}{m-1} + \frac{1}{m-2} + \cdots + \frac{1}{m-n}\right) \\ &= \binom{m-1}{n}(H_{m-1} - H_{m-n-1}) \quad (H_0 = 0). \end{aligned}$$

Thus, using (3.4) we get

$$\sum_{k=1}^{n} (-1)^{k-1} H_k \binom{m}{n-k} = \binom{m-1}{n}(H_{m-1} - H_{m-n-1}) \quad (H_0 = 0). \tag{5.18}$$

Example 7 Take $f(t) = 1/(t+m)$ with $m \geq 2$. We have

$$\Delta^k f(0) = \frac{(-1)^k k!(m-1)!}{(m+k)!} = \frac{(-1)^k}{m}\binom{m+k}{m}^{-1}.$$

Consequently formulas (3.4), (3.5), (3.7), and (3.3) can be used to obtain four convergent series sums as follows.

$$\sum_{k\geq 1}\binom{m+k}{m}^{-1}H_k=\frac{m}{(m-1)^2}, \tag{5.19}$$

$$\sum_{k\geq 1}\binom{m+k+1}{m}^{-1}\frac{H_k}{k+1}=\frac{1}{m^2}, \tag{5.20}$$

$$\sum_{k\geq 1}\binom{m+k}{m}^{-1}H(k,r)=\frac{m}{(m-1)^{r+2}}\quad (r\geq 1) \tag{5.21}$$

$$\sum_{k\geq 1}\binom{m+k+1}{m}^{-1}\frac{C_k}{4^k}=\frac{2}{2m+1}. \tag{5.22}$$

In particular, for $m=2$ we see that (5.20), (5.21), and (5.22) yield the sums

$$\sum_{k\geq 1}\frac{H_k}{(k+1)(k+2)(k+3)}=\frac{1}{8}, \tag{5.23}$$

$$\sum_{k\geq 1}\frac{H(k,r)}{(k+1)(k+2)}=1\quad (r\geq 0) \tag{5.24}$$

$$\sum_{k\geq 1}\frac{\binom{2k}{k}}{4^k(k+1)(k+2)(k+3)}=\frac{1}{5}. \tag{5.25}$$

Example 8 As may be observed, the case $r=0$ of (3.6) gives the following pair of identities for $f(t)=t^n$ and $f(t)=\binom{\alpha+t}{n}$ $(\alpha\in\mathbb{R})$ respectively.

$$\sum_{k=0}^{n}(-1)^k\frac{(2k)!}{2^{2k}k!}\left\{\begin{matrix} n\\ k\end{matrix}\right\}=\left(\frac{-1}{2}\right)^n, \tag{5.26}$$

$$\sum_{k=0}^{n}\binom{2k}{k}\left(\frac{-1}{4}\right)^k\binom{\alpha}{n-k}=\binom{\alpha-\frac{1}{2}}{n}. \tag{5.27}$$

In particular, (5.27) with $\alpha=n$ implies

$$\sum_{k=0}^{n}\binom{2k}{k}\left(\frac{-1}{4}\right)^k\binom{n}{k}=\binom{n-\frac{1}{2}}{n}=2^{-2n}\binom{2n}{n}. \tag{5.28}$$

This identity appears in Sofo [10, p.22]. Surely, other identities of similar types may be obtained from (3.6) for smaller r's.

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References

[**1**] R. Mullin and G.-C. Rota, On the foundations of combinatorial theory: III. Theory of binomial enumeration, in: Graph Theory and its Applications, B. Harris (ed.), Academic Press, New York and London, 1970, 167-213.

[**2**] T. X. He, L. C. Hsu, and P. J.-S. Shiue, Symbolization of generating functions, an application of Mullin-Rota's theory of binomial enumeration, Comp. & Math. with Applications, 54 (2007), 664-678.

[**3**] L. Comtet, Advanced Combinatorics, the art of finite and infinite expansions, Revised and enlarged edition. D. Reidel Publishing Co., Dordrecht, 1974.

[**4**] H. S. Wilf, Generatingfunctionology, Academic Press, New York, 1990.

[**5**] L. B. W. Jolley, Summation of series, 2d Revised Edition, Dover Publications, New York, 1961.

[**6**] J. M. Santmyer, A Stirling like sequence of rational numbers. Discrete Math. 171 (1997), no. 1-3, 229–235.

[**7**] Ch. Jordan, Calculus of Finite Differences, Chelsea Publishing Co., New York, 1965.

[**8**] W. Magnus, F. Oberhettinger, R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd edition, Springer-Verlag, Heidelberg, New York, 1966.

[**9**] D. E. Knuth, Two notes on notation, Amer. Math. Monthly 99 (1992), no. 5, 403–422.

[**10**] A. Sofo, Computational Techniques for the Summation of Series, Kluwer Academic/Plenum Publishers, New York, 2003.