First-Order Necessity and Validity

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Intro

We have added some new pieces to our language:
Quantifiers and variables.

These new pieces are going to add a new layer of **NPEC**:
Necessity, **P**ossibility, **E**quivalence, **C**onsequence.

This is going to be called "First-Order" necessity, possibility, etc.
That's "FO" necessity, possibility, etc., for short.

The book refers to FO-necessities as FO *validities*. Ugh.
I avoid that, just because I want to reserve the term validity for *arguments* rather than *sentences*.

First Order Attention

FO-NPEC, etc. pays attention to a little bit **more** than TT-NPEC does.

- It pays attention to the meanings of the connectives, just like TT-necessity.
- But it **also** pays attention to the meanings of the quantifiers and variables. Unlike the truth table.
- And it pays attention to one special predicate: Identity (=). Unlike the truth table.

First Order Attention

But FO-necessity pays attention to a little bit **less** than Logical necessity does.

- FO-necessity doesn't pay attention to the meaning of any predicates *other* than "=".
  - So FO-necessity **doesn't** understand the meaning of "Cube," "BackOf," "Between," "Small," etc.

Euler Diagrammin'

Remember that necessity (and equivalence and consequence) will expand when we add new things to force sentences to be true or false.

- Because all of the connectives are still around to do the forcing when we move from TT to FO,
  - All TT-necessities are FO-necessary.
- But because FO has some more stuff around to do forcing than TT did,
  - Not all FO-necessities are TT-necessary.

Not all FO-necessities are TT-necessary.

For that to be true, there must be at least one example of a sentence that is FO-necessary but not TT-necessary.

Can you think of an example of an FO-necessity that isn't TT-necessary?

Remember this sentence from last class?
Exercise 10.1.1:

$\forall$x (x=x)

We saw that this wasn't TT-necessary. (Why not?)

But we saw that it was logically necessary. (Why?)

- Is it FO-necessary?

Euler Diagrammin'

Yes, $\forall$x (x=x) *is* FO-necessary. (Why?)

- It doesn't contain anything that Logical necessity pays attention to but that FO necessity doesn't. (It doesn't have any predicates other than '='.)

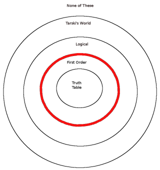
Euler Diagrammin'

What is the relationship between logical necessity and first order necessity?

- Since logical necessity pays attention to all of the same stuff that forces sentences to be true at the FO level (connectives, quantifiers, variables, names, identity),
  - All FO-necessities are logically necessary.
- But since there is some stuff that logical necessity pays attention to stuff that the FO level doesn't (namely, the meaning of predicates like "Cube," "Large," etc.)
  - Not all logical necessities are FO-necessary.

Euler Diagram

So here's the Euler Diagram for Necessity (and Consequence and Equivalence)



Not all logical necessities are FO-necessary.

If that is true, then there must be some logical necessity that is not FO-necessary.

Can you think of an example of a logical necessity that is not FO-necessary?

Hint

Examples will have to exploit the meanings of some predicates that FO-necessity doesn't pay attention to.

Logically Necessary, but Not FO-Necessary

Here's an example:

- ∀x (Cube(x) → ¬Dodec(x))

That sentence is logically necessary. (Why?)

But it *isn't* FO-necessary. (Why not?)

But not just any example that uses a predicate that FO-necessity doesn't understand will do the trick.

For instance, FO-necessity doesn't pay any attention to the meaning of "Cube". But the following sentence is FO-necessary:

- ∀x ((Cube(x) ∧ Small(x)) → Cube(x))

You don't have to know anything about what "Cube" means to know that that sentence is true.

Same with this one:

- ∃x Cube(x) → ¬∀x ¬Cube(x)

Strategy!

In order to get ourselves to think in a "First Order" frame of mind, we're going to need a strategy to get ourselves to ignore the meanings of the predicates that FO thinking can't understand.

Strategy: We'll just replace all of the predicates FO thinking can't understand with nonsense predicates, ones that we can't understand.

We just have to make sure that we replace the same sensible predicate with the same nonsense predicate everywhere it appears.

- If we replace "Cube" with "Caburble" in one place, we have to replace it with that everywhere.

And we can't reuse the same nonsense predicate for a different sensible predicate.

- If we have already declared that "Caburble" means "Cube", we can't turn around and say it also means "Large".

Apply this technique to the FO-necessary sentences from before:

- $\forall x$ ((Cube(x) $\wedge$ Small(x)) $\rightarrow$ Cube(x))

Replace the sensible predicate "Cube" with "Caburble". And replace the sensible predicate "Small" with the nonsense predicate "Schwenky".

Now the earlier sentence becomes

- $\forall x$ ((Caburble(x) $\wedge$ Schwenky(x)) $\rightarrow$ Caburble(x))

What's A "Schwenky Caburble?"

- $\forall x$ ((Caburble(x) $\wedge$ Schwenky(x)) $\rightarrow$ Caburble(x))

What do "Caburble" and "Schwenky" mean?

I have no idea! I don't know! I don't *want* to know!

But even so, without knowing anything about schwenky stuff or caburbles, I still know this:

All schwenky caburbles are caburbles.

That *has* to be true, right?

- If you can determine that a sentence must be true *even after replacing its sensible with nonsense ones*, then the sentence is FO-necessary.
- If you can't determine whether a sentence must be true after you have replaced its sensible predicates with nonsense ones, then the sentence is not FO-necessary.

Another example

Now let's apply this technique to another sentence we encountered earlier:

- $\exists x\ Cube(x) \rightarrow \neg\forall x\ \neg Cube(x)$

This becomes:

- $\exists x\ Caburble(x) \rightarrow \neg\forall x\ \neg Caburble(x)$

What's a caburble? I *still* have no idea!

But I do know this:

- If something caburbles, then not everything fails to caburble.

That's just got to be true, no matter what "caburble" means.

So the original sentence is FO-necessary.

Now, what about this sentence:

- $\forall x\ (Cube(x) \rightarrow \neg Dodec(x))$

- $\forall x\ (Cube(x) \rightarrow \neg Dodec(x))$

Replace its sensible predicates with nonsense ones. (I'm getting tired of Caburbles, though.)

- $\forall x\ (Curdiddle(x) \rightarrow \neg Doodiddle(x))$

Does **that** have to be true?

Can No Curdiddles Doodiddle?

- $\forall x\ (Curdiddle(x) \rightarrow \neg Doodiddle(x))$

Does that have to be true?
Is it necessary?

I have no idea.

It depends on what "Curdiddle" and "Doodiddle" mean.

That means that the necessity of $\forall x\ (Cube(x) \rightarrow \neg Dodec(x))$ depends on the meanings of its predicates.

And that means that it is **not** FO-necessary.

FO-Validity

We have just used the technique to see whether a sentence was FO-necessary.
We can use the same technique to determine whether an argument is FO-valid or not.

Consider the following argument:

$\forall x$ SameSize(x,b) $\rightarrow \exists x$ Small(x)
$\forall x$ $\neg$Small(x)

$\exists x$ $\neg$SameSize(x,b)

Is this argument FO-Valid?

$\forall x$ SameSize(x,b) $\rightarrow \exists x$ Small(x)
$\forall x$ $\neg$Small(x)

$\exists x$ $\neg$SameSize(x,b)

Do the nonsense replacement, making sure to replace the same sensible predicate everywhere with the same nonsense predicate.

$\forall x$ Scebbies(x,b) $\rightarrow \exists x$ Slolly(x)
$\forall x$ $\neg$Slolly(x)

$\exists x$ $\neg$Scebbies(x,b)

Can we determine whether this argument is valid?

$\forall x$ Scebbies(x,b) $\rightarrow \exists x$ Slolly(x)
$\forall x$ $\neg$Slolly(x)

$\exists x$ $\neg$Scebbies(x,b)

- If we *can* determine whether this argument is valid, then the original sensible argument is FO-valid. (Its validity doesn't depend on the meanings of its predicates.)
- If we *can't*, then the original sensible argument is not FO-valid. (It might be logically valid, or TW-valid.)

What do you think?

$\forall x$ Scebbies(x,b) $\rightarrow \exists x$ Slolly(x)
$\forall x$ $\neg$Slolly(x)

$\exists x$ $\neg$Scebbies(x,b)

If everything scebbies Bill, then there is at least one slolly.
Nothing is a slolly.

At least one thing doesn't scebby Bill.

If everything scebbies Bill, then there is at least one slolly.
Nothing is a slolly.

At least one thing doesn't scebby Bill.

I have no idea who Bill is, or what a slolly is, or what it is for one thing to scebby another.

But I can still tell that if the premises are true, the conclusion must be, too.

So the original argument is FO-Valid.

What about this argument?

$\forall x\ (\text{Cube}(x) \to \text{Large}(x))$

$\neg\exists x\ (\text{Cube}(x) \land \text{Small}(x))$

$\forall x\ (\text{Cube}(x) \to \text{Large}(x))$

$\neg\exists x\ (\text{Cube}(x) \land \text{Small}(x))$

This argument is definitely logically valid.

If we weren't sure whether this argument was FO-valid, we could use the nonsense predicate replacement method to check.

And what would we find out?

The argument is not FO-valid, even though it is logically valid.

And in order to show this, we can do something a little bit stronger than the nonsense predicate replacement method.

We can construct a FO-counterexample.

FO-Counterexamples Stop The Nonsense!

To make a FO-counterexample, we will have to make the premises true and the conclusion false.

So to make one, we're going to stop using nonsense predicates, which give us sentences we cannot evaluate as true or false.

FO-Counterexamples

To make an FO-counterexample, we replace the sensible predicates with other sensible predicates, where the pattern of replacement makes the premises true and the conclusion false.

As before, each unique predicate always gets replaced by the same predicate, and we never reuse a predicate we have already substituted in for another, different predicate.

$$\begin{array}{|l} \forall x\ (\text{Cube}(x) \rightarrow \text{Large}(x)) \\ \hline \neg\exists x\ (\text{Cube}(x) \wedge \text{Small}(x)) \end{array}$$

Replace "Cube" with "Chandelier," "is Large" with "is a Light", and "is Small" with "is Silver".

You don't have to use words that begin with the same first letter. The key is to make your substitutions consistently (always replacing the same word with the new one). Starting with the same letter just makes it easier to make sure you're doing that.

$$\begin{array}{|l} \forall x\ (\text{Chandelier}(x) \rightarrow \text{Light}(x)) \\ \hline \neg\exists x\ (\text{Chandelier}(x) \wedge \text{Silver}(x)) \end{array}$$

These arguments have the same First order form. So if the first one is FO-valid, the second one would have to be FO-valid, too.

$$\begin{array}{|l} \forall x\ (\text{Chandelier}(x) \rightarrow \text{Light}(x)) \\ \hline \neg\exists x\ (\text{Chandelier}(x) \wedge \text{Silver}(x)) \end{array}$$

All chandeliers are lights. (TRUE)
No chandeliers are silver. (FALSE)

This counterexample shows that the original argument is definitely not FO-valid: there are counterexamples to this FO-Form.

The validity of the original argument depends on more than just the meanings of the connectives, identity, and the quantifiers. Its validity depends upon facts about the "large" and "small".