Math 215 Fall 2002, Linear Algebra
Exam 2

Name:**Key**
Answer all questions.

1. Prove that if $L : \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $\text{Ker}(L)$ is a subspace of $\mathcal{V}$.

   We need to show that $\text{Ker}(L)$ is nonempty and closed under both sums and scalar multiples. Since a linear transformation must take $\vec{0}$ to $\vec{0}$ we know that $\vec{0} \in \text{Ker}(L)$. If $\vec{v_1}$ and $\vec{v_2} \in \text{Ker}(L)$ then

   $$L(\vec{v_1} + \vec{v_2} = L(\vec{v_1}) + L(\vec{v_2} = \vec{0} + \vec{0} = \vec{0}$$

   so $\vec{v_1} + \vec{v_2} \in \text{Ker}(L)$. Similarly if $k \in \mathbb{R}$ then $L(k\vec{v_1}) = kL(\vec{v_1}) = k\vec{0} = \vec{0}$ so $k\vec{v_1} \in \text{Ker}(L)$ Thus $\text{Ker}(L)$ is a subspace of $\mathcal{V}$.

2. The vector space $\mathbb{R}[x]$ has $\{1, x, x^2, x^3, \ldots\}$ as a basis. Notice that this set is infinite and can be used to produce a linearly independent finite set of polynomials with any number of members. Show that there cannot be a finite set of polynomials which spans $\mathbb{R}[x]$.

   Suppose we had a set of polynomials $\{p_1(x) \ldots p_m(x)\}$ which spanned $\mathbb{R}[x]$. The set $\{1, x, \ldots x^m\}$ is linearly independent. Since the number of elements in any spanning set must be greater than or equal to the number of elements in any linearly independent set this tells us that $m \geq m+1$, which is impossible. Thus no finite spanning set can be found for $\mathbb{R}[x]$, hence also no finite basis.

3. Show that the set of solutions to the equation

   $$f(x) = D_x^2[f](x-2)$$

   is a vector space.

   The set of solutions is the kernel of the operator taking $f(x)$ to $f(x) - D_x^2[f](x-2)$. This operator is linear because it is built from the identity operator, differentiation, and the operator taking $f(x)$ to $f(x-2)$ (all of which we know to be linear) using addition and composition.

4. Give an example of two subspaces of $\mathbb{Z}_2^3$, $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A} \cap \mathcal{B} = \{[0, 0, 0]\}$ and $\mathcal{A} \cup \mathcal{B}$ is *not* a subspace.

The following will do the trick:

$$\mathcal{A} = \{[0,0,0],[1,0,0]\}$$

$$\mathcal{B} = \{[0,0,0],[0,1,0]\}$$

Both of these are closed under addition but the union $\mathcal{A} \cup \mathcal{B}$ is not.

5. Show that the function L from the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ to the space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$given by

$$L(f)(t) = \int_0^t f(x)e^{-x}\,dx$$

is a linear transformation. 5 Points

$$L(f+g)(t) = \int_0^t (f(x)+g(x)e^{-x}\,dx = \int_0^t f(x)e^{-x}\,dx + \int_0^t g(x)e^{-x}\,dx = L(f)+L(g)$$

$$L(kf)(t) = \int_0^t kf(x)e^{-x}\,dx = k\int_0^t f(x)e^{-x}\,dx = kL(f)(t)$$

Thus L is linear.

6. The following matrices result from the applying Gaussian elimination to the augmented matrix from a system of linear equations in the variables $x_1 \ldots x_5$. What do they tell you? (5 Points Each)

(a) $$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - x_4 &= 1 \\ x_2 + 3x_4 &= 2 \\ x_3 + 2x_4 &= 3 \\ x_5 &= 4 \\ x_4 &\text{ is arbitrary} \end{aligned}$$

(b) $\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ No solutions because of fourth row.

7. Do the next step in the Gaussian elimination with backsolving algorithm: (5 Points Each)

(a) $\begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 3 & 1 & 1 \\ 4 & 5 & 22 & 10 \\ 0 & 4 & 5 & 1 \end{pmatrix} \overset{R_3-4R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & -3 & -2 & -2 \\ 0 & 4 & 5 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 3 & 2 & 3 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{pmatrix} \overset{\frac{1}{3}R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & \frac{2}{3} & 1 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 5 & 2 & 1 \\ 0 & 4 & 5 & 1 \end{pmatrix} \overset{R_2 \leftrightarrow R_3}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 5 & 1 \end{pmatrix}$

8. The matrix

$$\begin{pmatrix} 1 & 3 & -3 & 1 & 0 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 6 & 3 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & -1 & 11 & 1 & 12 & -5 & -6 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 1 & 10 & 0 & 15 & 6 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & 13 & 1 & 18 & 10 & 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 2 & 14 & 5 & 17 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Row reduces to

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 4 & 0 & -1 & 0 & 0 & \frac{31}{207} & \frac{53}{207} & -\left(\frac{16}{207}\right) & -\left(\frac{1}{69}\right) \\ 0 & 1 & -2 & 0 & -1 & 0 & -1 & 0 & 0 & -\left(\frac{82}{207}\right) & \frac{187}{207} & -\left(\frac{158}{207}\right) & \frac{16}{69} \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -\left(\frac{1}{69}\right) & -\left(\frac{44}{69}\right) & \frac{25}{69} & \frac{3}{23} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -\left(\frac{7}{207}\right) & -\left(\frac{32}{207}\right) & \frac{37}{207} & -\left(\frac{2}{69}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{232}{207} & -\left(\frac{418}{207}\right) & \frac{341}{207} & -\left(\frac{52}{69}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{41}{69} & -\left(\frac{128}{69}\right) & \frac{79}{69} & -\left(\frac{8}{23}\right) \end{pmatrix}$$

(5 Points Each)

(a) Is the set of vectors $\{[1,2,3,4,5,6],[3,2,-1,1,1,2],[1,0,1,0,1,5],[0,6,12,15,18,17]\}$ linearly independent?

These are columns 1,2,4, and 5 of the origianal matrix. The row reduction shows that column 5 can be written as a linear combination of columns 1,2, and 4, so no , this set is not linearly independent.

(b) Does the set $\{[1,2,3,4,5,6],[3,2,-1,1,1,2],[1,0,1,0,1,5],[0,6,12,15,18,17],$ $[-1,-1,-6,1,5,2],[1,0,0,0,0,0],[0,1,0,0,0,0]\}$ span $\mathbb{R}^4$?

Best make that $\mathbb{R}^6$–These are columns 1,2,4,5,7,8, and 9 of the original matrix. In the row reduced matrix if we took columns 1,2,4,5,7,8, and 9 we would have all columns spanned, so the answer is yes.

(c) Give a basis for $\mathbb{R}^4$ extending $\{[1,2,3,4,5,6],[3,2,-1,1,1,2]\}$. Again make that $\mathbb{R}^6$– columns 1,2,4,6,8, and 9 of the original matrix will do the trick.

9. (a) Prove that composition distributes over addition for linear transformations. (7 Points)

$$L \circ (M+N)(\vec{v}) = L(M(\vec{v}) + N(\vec{v})) = L(M(\vec{v})) + L(N(\vec{v}))$$

by linearity of L. This shows that $L) \circ (M+N) = L \circ M + L \circ N$.

(b) What result does this give for matrices? (3 Points)

Matrix multiplication distributes over matrix addition.