Precision – Correcting for Random Error

The following material should be read thoroughly *before* your 1st Lab.

The Statistical Handling of Data

Our experimental inquiries into the workings of physical reality are imperfect, owing to the fact that we cannot arrive at a completely accurate, or "correct," value, for an experimental quantity. There are two reasons for this. First, our instruments are not capable of giving us exact measurements. For instance, a meter stick can only measure accurately to the nearest 0.5 mm; a vernier will serve us better but we are still limited to about 0.1 mm. As long as we can always come up with a more accurate method for measuring a given quantity, there exists no "correct" value. The second source of uncertainty in our measurements is the effect of human bias. If five people were asked to measure a given quantity, we might have five different values for that quantity. Which measured value should we consider to be the correct value?

To handle these uncertainties we introduce three statistical variables to describe experimental data: mean, standard deviation of the sample, and standard deviation of the mean.

Mean, μ:

The mean is the average value; that is, the sum of the measured values divided by the number in the sample. It describes the single value around which the measured values are centered.

Standard Deviation of the Sample, σ:

The standard deviation of a sample describes how much variation there is among the sample. The standard deviation of the sample, σ, is found using Equation 1.

$$\sigma = \sqrt{\frac{\sum_i (\bar{x} - x_i)^2}{n-1}} \tag{1}$$

where $\bar{x}$ is the mean of the sample, x_i is a measurement and n is the number of measurements taken. The expression inside the radical is called the "variance."

Standard Deviation of the Mean, σ_m:

Dividing the standard deviation of the sample by the square root of the number of measurements in the sample yeilds the **standard deviation of the mean**, σ_m.

We have just defined three useful statistical variables; let us now give them some intuitive significance. Consider a set of measurements made to determine the length of a rod. If one performs a large number of measurements on the rod and plots the fraction of measurements that resulted in a particular measurement verses the measured lengths, a normal, or "bell," curve is obtained. Strictly speaking, the normal curve is reached only after an infinite number of measurements; however, it is often a good approximation for smaller samples. The normal distribution can be seen in Figure 1.

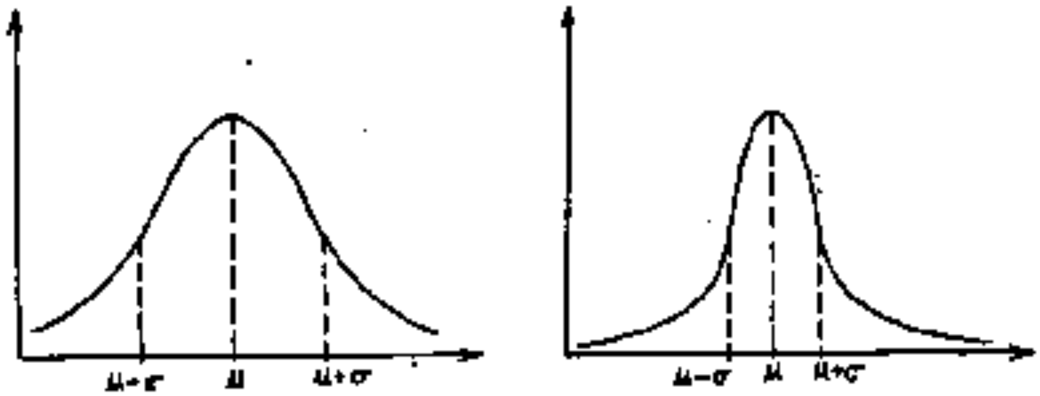

Figure 1: Showing the normal distribution. The figure on the right depicts a distribution having a larger standard deviation, while the figure on the right has a smaller standard deviation.

The measurements will cluster around the mean, m. Often the mean will be one of the most common measurements. If your instructor was to ask you what the "true" length of the rod was, the mean would be your best estimate.

The standard deviation of the sample, σ, is a measure of how "spread out" one's measurements are. A larger standard deviation implies that your measurements are more spread out and not as consistent with each other. The smaller the standard deviation of the sample, the more "reliable" your sample of measurements is. Often, 68% of your measured values will fall within the region ranging from $\mu - \sigma$ to $\mu + \sigma$.

If the mean is your best estimate of the "true" measured value, you might wonder how well you know this. From the previous paragraph, the standard deviation of the sample conveys this sort of information. However, it does not tell you all that there is to know. For instance, suppose that two of your classmates measured the length of a rod. One student takes two measurements while the other takes ten. Which student would you think has a more "accurate" value? Even if the two students arrived at the same value for σ, one would likely place more confidence in the mean of the student who took more measurements; because he obtained more readings - their mean value for the time is less uncertain.

This is where the standard deviation of the mean is useful. If repeated measurements continue to yield the same distribution of lengths, with the same value for μ and σ, your knowledge of μ improves. The standard deviation of the mean attributes more certainty to an mean value which is obtained from a larger number of measurements

An example showing how one would calculate these quantities is given below.

Example 1: A set of measurements on the length of a rod are as follows:

17.304, 17.483, 17.266, 17.325, 17.379.

The tabulation and treatment of the data are given below.

	Reading, cm	Deviations, cm	Deviations Squared, cm^2
1	17.304	-0.047	2.209×10^{-3}
2	17.483	0.132	17.424×10^{-3}
3	17.266	-0.085	7.225×10^{-3}
4	17.325	-0.028	0.676×10^{-3}
5	17.379	0.028	0.784×10^{-3}
Sum	86.757	Sum	28.318×10^{-3}
Mean	17.3514	Variance	7.080×10^{-3}
	$s = 8.414 \times 10^{-2}$		
	$\sigma_m = \frac{\sigma}{\sqrt{5}} = 3.763 \times 10^{-2}$		

Error and uncertainty

When reporting data, the mean is given along with the standard deviation of the mean (in the same units). For example, if we were to report on the data used in Example 1, we would report the length of the rod as:

$$17.351 \pm 0.038 \text{ cm}$$

This means that the length of the rod is taken to be 17.351 cm, and it is understood that this value could be in error by 0.038 cm.

If the accepted, or theoretical, value for some physical quantity is within the error range $\mu \pm \sigma_m$, the experimental result is in agreement with theory. If the theoretically predicted value lies outside a one sigma error range, then experiment does not agree with theory within that error range.

If we want to contrast two experimental values, the two values agree the quantities $\mu \pm \sigma_m$ overlap. For example, if one experimenter measures the acceleration of gravity at the earth's surface to be 9.81 ± 0.02 m/s^2, and another arrives at a value of 9.77 ± 0.03 m/s^2, their results agree because their measurements overlap.

Sometimes you might want a ***unitless*** measure of how well-known your result is. In this case, you would use ***relative uncertainty***. The relative uncertainty in a reported value is defined as the percentage:

$$\text{relative uncertainty}(\%) = \frac{\sigma_m}{\mu} \times 100\% \qquad (3)$$

In example 1, the relative error in the length of the rod is:

relative error = (0.0376/17.351) x 100% = 0.22%

This gives an alternative way of reporting an experimental value:

length of rod = 17.351 ± 0.0376 cm = 17.351 cm ± 0.22%

When comparing results using this criterion, the experimental result agrees with another if the percent difference is less than the total relative uncertainty.

Propagation of Error

Until now, our discussion has centered around groups of measurements. In the event that a single measurement is taken, the error is considered to be the error due to the limitation of the measuring device; usually one-half of the smallest increment in the instrument .

What if we wanted to combine the results of measurements on different quantities. Suppose that you have three measured quantities, x, y and z, with uncertainties σ_x, σ_y, σ_z. The rules for determining the error (σ_m) and the relative error of the result, w, are given below:

Rule 1: If $w = x \pm y$, then $\sigma_w = \sqrt{\sigma_x^2 + \sigma_y^2}$.

Rule 2: If $w = \frac{x^a y^b}{z^c}$, then $\frac{\sigma_w}{w} = \sqrt{\left(a\frac{\sigma_x}{x}\right)^2 + \left(b\frac{\sigma_y}{y}\right)^2 + \left(c\frac{\sigma_z}{z}\right)^2}$.

Two examples are given below.

Example 2: Combining Measurements I

Find the combined length of two rods, one of length 17.351 cm ± 0.019 cm and the other of length 5.027 cm ± 0.007 cm.

1. Add 17.351 cm and 5.027 cm to find the combined length, 22.378 cm.

2. Square 0.019 cm and 0.007 cm to obtain the variances of the lengths, 3.61 cm^2 $x10^{-4}$ and 0.49 cm^2 $x10^{-4}$, respectively.

3. Add the variances to find the sum of the variances, 4.10 cm^2 x 10^{-4}.

4. Take the square root of the sum of the variances to find the error in the result, 2.02 cm x 10^{-2}.

5. This result is reported as 22.378 cm ± 0.020 cm., or 22.378 cm ± 0.089%

Example 3: Combining Measurements II

Determine the density of an object found to have a mass of 1.305 kg ± 0.002 kg and a volume of 9.57 m^3x 10^{-4} ± 0.16 m^3 x 10^{-4}.

1. Divide 1.305 kg by 9.57 m^3x 10^{-4} to find the density, 1.181 kg m^{-3} x 10^3.

2. Divide 0.002 kg by 1.305 kg to find the relative error of mass, 1.53 x 10^{-3}.

3. Square 1.53 x 10^{-3} to find the relative variance of mass, 2.35 x 10^{-6}.

4. Repeat steps 2 and 3 to find the relative error of volume, 1.67 x 10^{-2}, and relative variance of volume, 2.80 x 10^{-4}.

5. Add 2.35 x 10^{-6} and 2.80 x 10^{-4} to find the sum of the relative variances, 2.82 x 10^{-4}.

6. Take the square root of sum of the relative variances, 2.82 x 10^{-4}, to find the relative error of the result, 1.68 x 10^{-2}.

7. Multiply 1.181 x 10^3 and 1.68 x 10^{-2} to find the error of the result, 19.8 kg m^{-3}.

8. Report the result as 1.181 kg/m^{-3} x 10^3 ± 0.0198 kg m^{-3} x 10^3, or 1.181 kg m^{-3} x 10^3 ± 1.68%.