HW #3a:
from Ch. 1-2 of Griffiths E&M text

1.49 Evaluate the volume integral:

$$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right) d\tau$$

where the volume to integrate over is a sphere of radius R, centered at the origin. — Do this by ***two different methods***, as in Example 1.16 of the Griffiths text (first, by using Equation 1.99, and then, independently, by a generalized form of "integration by parts", which follows from the Divergence theorem (also known as Green's Theorem or as Gauss's Theorem) and from Product Rule #5 shown on our exam reference sheet).

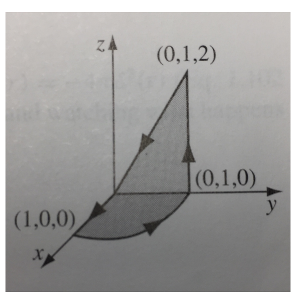
1.57 a) Compute the *"circulation"* of the vector field $\vec{v}$, namely $\oint \vec{v} \cdot \overrightarrow{d\ell}$, for:

$$\vec{v} = \left(r \cos^2 \theta\right) \hat{r} - \left(r \cos\theta \sin\theta\right) \hat{\theta} + 3r\hat{\varphi}$$

around the *closed path* shown below, where the points are *labeled* by cartesian coordinates, but you should work in either cylindrical or spherical coordinates, which is to say that you can use:

$$\overrightarrow{d\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi}$$

If you begin at the origin, the first leg has $d\theta = 0 = d\varphi$, and the next leg has $d\theta = 0 = dr$, where for each of those two legs, $\theta = \pi/2$. For the third leg, you can parametrize r and dr in terms of the polar angle.



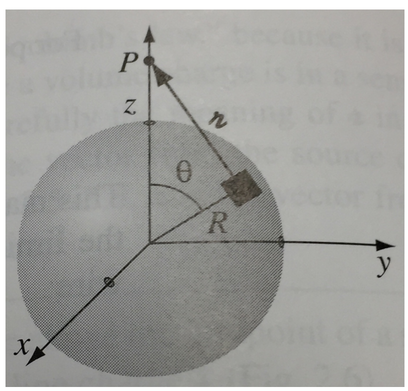
[You are GIVEN that the answer is supposed to turn out to be $3\pi/2$.
b) Check your answer by repeating the calculation via an independent method, here using Stokes' Theorem (also known as the Curl Theorem).

2.6 a) Find the electric field at a point that is a distance z above the center of a flat circular disk of radius R that carries a uniform charge per unit area, σ.

b) Your training will be to **check the limiting behaviors**. Here, what does your result give in the limit of $R \to \infty$, and does this make sense?

c) What is the result for $z \gg R$ and does this make sense?

2.7 Find the electric field a distance z from the center of a spherical <u>surface</u> of radius R that carries a uniform surface charge density σ. Do not use Gauss's law this time!



2.21 a) Find the potential inside and outside of a ***solid*** sphere with uniform charge per unit volume, ρ, and radius R. Boundary condition: set, as your reference point, the potential at infinite distance to zero. [This time, go ahead and use Gauss's law!]

b) Compute the gradient of the potential in each region, and check that it yields the correct electric field.

c) Sketch the potential as a function of distance from the center of the sphere. *Mathematica* is recommended, but since it wants plots to consist of pure numbers, without units, I would plot the potential “in units of $\frac{1}{4\pi\epsilon_0}\frac{Q_{tot}}{R}$” and would make the horizontal axis “$\frac{r}{R}$.” [Note the limiting behavior as you approach the origin, the location of the point of inflection, and the continuous nature of the potential.]

2.28 To calculate the potential *inside* a uniformly charged sphere, we could also directly integrate up the contribution of all the itty bits, using:

$$\varphi\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\vec{r'}\right)}{scriptr} d\tau'$$

[Again, this form implicitly sets the potential to zero at infinite distance.]
— Note that you can, without loss of generality, for any field point P, orient your axes such that P lies on the z-axis. Also, you may wish to review the Law of Cosines.

2.34 Find the energy stored in a uniformly charged solid sphere of radius *R* and charge *q*. I’ll go ahead and give you the answer (which immediately tells you that a point charge is an impossible idealization), but then I’ll ask you to calculate the result in three different ways, and we’ll want to discuss their equivalency, and why such rephrasing will be so important.

Your result should be that the energy stored in the accumulated charge of a solid sphere with uniform charge density is:

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{3}{5}\right)\frac{q^2}{R}$$

a) Your first way to calculate this should be to use:

$$W = \frac{1}{2}\int \rho\varphi\left(\vec{r}\right) d\tau$$

[You already found the potential in the previous problem; plug it in!]

b) Your next method to calculate this should be to integrate the energy density over ALL space:

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau$$

[In Problem # 2.21, you found the field, both inside and outside the sphere; plug it in!]

c) Your final method involves two terms, the first term is a volume integral over any volume you wish to choose that encloses the change of interest, and the second term is an integral over the surface of that volume:

$$W = \frac{\varepsilon_0}{2}\left(\int E^2 d\tau + \oint \varphi \vec{E} \cdot \overrightarrow{da}\right)$$

Here, I recommend that you take advantage of the spherical symmetry of the particular problem at hand, and select a spherical volume of radius $a > R$. [What happens as $a \to \infty$?]

2.50 You can now get a lot out of almost nothing… GIVEN $\varphi\left(\vec{r}\right) = A\frac{e^{-\lambda r}}{r}$, where A and λ are constants:

a) Find the electric field

b) Find the charge density [Here, your best tool is the *differential* form of Gauss's law. I'll go ahead and give you the answer: $\rho = \varepsilon_0 A\left\{4\pi\delta^3\left(\vec{r}\right) - \frac{\lambda^2}{r}e^{-\lambda r}\right\}$]

c) Find the total charge