Matrices and Determinants Project

Due: Wednesday, Oct. 2
Worth 40 points

1 Matrices and Linear Systems

An \( m \times n \) matrix is a rectangular array of numbers which has \( m \) rows and \( n \) columns. We usually put brackets or parentheses around them. Here is a \( 2 \times 3 \) matrix.

\[
\begin{bmatrix}
1 & 4 & \pi \\
-2 & 3 & 0
\end{bmatrix}
\]

We locate entries in a matrix by specifying its row and column entry. The 1-2 entry (first row, second column) of the above matrix is 4. In general we index the entries of an arbitrary \( n \times k \) matrix like this:

\[
A := 
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} \\
a_{21} & a_{22} & \cdots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nk}
\end{bmatrix}
\]

Then the \( i \)-\( j \) entry of \( A \) is denoted by \( a_{ij} \), and we can denote the entire matrix by \( A = (a_{ij}) \). Note that a vector \( \mathbf{x} \in \mathbb{R}^n \) can be interpreted as the \( n \times 1 \) matrix

\[
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]

Definition: Two matrices are said to be equal if their corresponding entries are equal.

There are two basic binary operations which we will define on matrices – matrix addition and matrix multiplication.

Definition: Take two matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \). The sum of \( A \) and \( B \), is defined to be the following:

\[
A + B = 
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} \\
a_{21} & a_{22} & \cdots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nk}
\end{bmatrix}
+ 
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1k} \\
b_{21} & b_{22} & \cdots & b_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nk}
\end{bmatrix}
= 
\begin{bmatrix}
a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1k} + b_{1k} \\
a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2k} + b_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nk} + b_{nk}
\end{bmatrix}
\]

Problem 1 What can you say about the sizes of the matrices \( A \) and \( B \) as compared to the size of the matrix \( A + B \)? Are there any restrictions?

Problem 2 Prove that matrix addition is both commutative and associative.

This project is adapted from material generously supplied by Prof. Elizabeth Thoren at UC Santa Barbara.
We now turn our attention to matrix multiplication. The motivation for our definition comes from our desire to represent a system of linear equations as a matrix multiplication. Consider the following system of linear equations

\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 1 \\
    x_1 + x_2 + x_3 &= 2 \\
    x_1 + 4x_2 + 7x_3 &= 1
\end{align*}
\]

From this system we can form the matrix equation \( A\vec{x} = \vec{b} \) as follows. Note that the multiplication of a matrix and a vector returns another vector.

\[
\begin{pmatrix}
    1 & 2 & 3 \\
    1 & 1 & 1 \\
    1 & 4 & 7
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
= 
\begin{pmatrix}
    1 \\
    2 \\
    1
\end{pmatrix}
\]

We wish to define matrix multiplication such that the above equation is valid (i.e. that the original system of equations can be recovered from this matrix equation). As another example, the system

\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 + 4x_4 &= 1 \\
    2x_1 + 4x_2 + 6x_3 + 8x_4 &= 2
\end{align*}
\]

can be expressed as the matrix multiplication

\[
\begin{pmatrix}
    1 & 2 & 3 & 4 \\
    2 & 4 & 6 & 8
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix}
= 
\begin{pmatrix}
    1 \\
    2
\end{pmatrix}
\]

The idea is that a row of the first matrix, times a column of the second matrix gives the corresponding row-column entry of the product matrix. Notice that the additions become implicit once we write a system in terms of a matrix.

Here are some more examples.

\[
\begin{pmatrix}
    1 & 2 \\
    3 & 4
\end{pmatrix}
\begin{pmatrix}
    5 \\
    6
\end{pmatrix}
= 
\begin{pmatrix}
    17 \\
    39
\end{pmatrix},
\begin{pmatrix}
    1 & 2 & 3 & 4 \\
    2 & 4 & 6 & 8
\end{pmatrix}
\begin{pmatrix}
    5 \\
    1
\end{pmatrix}
= 
\begin{pmatrix}
    17 & 1
\end{pmatrix}
\]

**Definition**: Let \( A \) be an \( m \times n \) matrix and \( B \) be an \( n \times p \) matrix. Then the **product of \( A \) and \( B \)**, denoted \( AB \), is the \( m \times p \) matrix whose \( i,j \)-entry, denoted \( (AB)_{ij} \), is the sum of the products of corresponding entries from row \( i \) of \( A \) and column \( j \) of \( B \). Thus, we have that

\[
(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.
\]

**Optional (but recommended) Problem**: Re-write the definition of matrix multiplication as a statement involving summation symbols.

**Example**: Matrix multiplication for \( 2 \times 2 \) matrices. Let

\[
A = \begin{pmatrix} a_{11} & a_{12} \\
                  a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\
                                                                b_{21} & b_{22} \end{pmatrix}
\]

Then

\[
AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{12} + a_{12}b_{22} \\
                                 a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}
\]

and notice that the columns of \( AB \) are \( A\vec{b}_1 \) and \( A\vec{b}_2 \).

In fact, given a matrix \( A \) and a matrix \( B \) whose columns are \( \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p \), if the product \( AB \) is defined then it’s columns are \( A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_p \).
Problem 3: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$. Compute $A\vec{x}$.

Problem 4: What can you say about the sizes of the matrices $A$ and $B$ as compared to the product $AB$? Are there any restrictions?

Problem 5: Compute the following matrix products:

$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = ?$, $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 6 & 0 \\ 1 & 0 \end{bmatrix} = ?$

Problem 6: (a) Find all the solutions to

$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) Find all the solutions to

$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The matrix in (a) and (b) is said to be row reduced (sometimes called reduced row echelon form, or RREF). This concept will be revisited in Theorem 49 of our textbook. Note that because of its form, it is pretty easy to record the solutions to the systems of equations in (a) and (b).

Problem 7: Show, by a counterexample, that matrix multiplication is not commutative.

2 The inverse of a matrix

We have succeeded in transforming the problem “solve a system of linear equations” into the problem “solve the matrix equation $A\vec{x} = \vec{b}$ for the vector $\vec{x}$.” Treating this matrix equation algebraically, solving for $\vec{x}$ seems simple. We just need to “divide” each side by the matrix $A$. But what does it mean to divide by a matrix? The purpose of this section is to clarify this.

Definition: The $n \times n$ identity matrix, denoted $I_n$, is the $n \times n$ matrix such that

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So $I_n$ has 1’s on the main diagonal, and 0’s everywhere else. As an example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition: Let $A$ be an $n \times n$ matrix. We say that $A$ is invertible if there is a matrix, denoted $A^{-1}$, such that $AA^{-1} = I_n = A^{-1}A$. We see that with respect to matrix multiplication $I_n$ behaves analogously to how the real number 1 behaves with respect to multiplication of real numbers. Similarly, the inverse of a matrix behaves much like the reciprocal of a real number. And just as in the case of real numbers, the inverse of a matrix is unique. In other words, if a matrix has an inverse, then it has exactly one inverse.
3 Determinants

Every square matrix, $A$, has a numerical value associated with it called its determinant, denoted $\det(A)$. We will denote the determinant of a matrix by putting absolute value bars around the matrix. In this section, we will define the determinant and explore some of its uses.

**Definition:** The determinant of a $2 \times 2$ matrix is the product of the diagonal elements minus the product of the off-diagonal elements:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$ 

**Problem 8** Let $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

(a) To show that $\{\mathbf{u}, \mathbf{v}\}$ spans $\mathbb{R}^2$ you must show that any vector $\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$ can be represented as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Write the system of linear equations that you need to solve, then write this system as a matrix equation.

(b) Solve the system. Does $\{\mathbf{u}, \mathbf{v}\}$ span $\mathbb{R}^2$?

(c) Compute $\det(A)$ from your matrix equation.

**Problem 9** Same problem as Problem 8, but with $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$.

**Problem 10** Now let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Similarly to the last two problems, to show that $\{\mathbf{u}, \mathbf{v}\}$ spans $\mathbb{R}^2$, you must solve the following system for $x$ and $y$

\[
\begin{align*}
  u_1 x + v_1 y &= a_1 \\
  u_2 x + v_2 y &= a_2
\end{align*}
\]

Use this idea to prove that $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0$ if and only if $\{\mathbf{u}, \mathbf{v}\}$ spans $\mathbb{R}^2$.

We usually express the $3 \times 3$ determinant in terms of the cofactor expansion. The idea is to compute the determinant by computing the determinants of smaller $2 \times 2$ submatrices. Notice that

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \cdot \det \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - b_1 \cdot \det \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} + c_1 \cdot \det \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$ 

**Problem 11** Verify that computing the determinant of a $3 \times 3$ matrix using cofactor expansion matches the definition of its determinant found in our textbook.

**Definition:** Let $A = (a_{ij})$ be an $n \times n$ matrix. We can define $A(i|j)$ to be the $(n - 1) \times (n - 1)$ matrix obtained from $A$ by deleting the $i$th row and $j$th column. $A(i|j)$ is called the $ij$th maximal submatrix of $A$.

The cofactor expansions for the determinant give $\det(A)$ in terms of the determinants of the maximal submatrices of $A$ taken along a specific row or column. It turns out that all of the cofactor expansions give they same value, which is the determinant. So, for example the determinant can
be expressed as the cofactor expansion along the third column or the cofactor expansion along the second row and both computations will give the same value:

\[
\det(A) = \det\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{13}\det(A(1|3)) - a_{23}\det(A(2|3)) + a_{33}\det(A(3|3))
\]

\[
= -a_{21}\det(A(2|1)) + a_{22}\det(A(2|2)) - a_{23}\det(A(2|3)).
\]

Notice that the sign in front of the term \(a_{ij}\det(A[i|j])\) is \((-1)^{i+j}\).

**Problem 12**

(a) Conjecture a general formula for the cofactor expansion along the 2\(^{nd}\) column of a 3 \(\times\) 3 matrix.

(b) Test your formula by using it to compute the determinants of the following matrices:

(i) \[
\begin{pmatrix}
1 & 0 & 1 \\
1 & 3 & 0 \\
3 & -1 & 0
\end{pmatrix}
\]

(ii) \[
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

(iii) \[
\begin{pmatrix}
1 & 0 & 3 \\
2 & 1 & 4 \\
1 & 0 & 1
\end{pmatrix}
\]

**Problem 13** Use cofactor expansion to compute the following

\[
\begin{vmatrix}
1 & 4 & -2 & 3 \\
-4 & 5 & -1 & 2 \\
3 & 3 & 3 & 1 \\
-2 & 3 & 7 & 2
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
4 & 0 & -7 & 3 & -5 \\
0 & 0 & 2 & 0 & 0 \\
7 & 3 & -6 & 4 & -8 \\
5 & 0 & 5 & 2 & -3 \\
0 & 0 & 9 & -1 & 2
\end{vmatrix}
\]

In order to make sense of cofactor expansion along a column of a matrix, we need to explore what happens to the determinant when we flip the entries - i.e. take the transpose of the matrix:

**Definition:** The transpose of an \(n \times k\) matrix \(A\), which we denote \(A^t\), is the \(k \times n\) matrix whose columns are formed from the corresponding rows of \(A\). So if \(A = (a_{ij})\), then the \(ij\)th entry of \(A^t\) is \(a_{ji}\).

For example, we have

\[
\begin{pmatrix}
1 & 4 \\
1 & 3 \\
8 & 2
\end{pmatrix}^t = \begin{pmatrix}
1 & 1 & 8 \\
4 & 3 & 2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}^t = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

The second of these two matrices is symmetric.

**Problem 14** Prove or disprove and salvage: \((AB)^t = A^tB^t\).

**Problem 15** Show that for any 2 \(\times\) 2 and 3 \(\times\) 3 matrix \(A\), \(\det(A^t) = \det(A)\).

**Problem 16** Say you have a set, \(S\), of \(n\) vectors in \(\mathbb{R}^n\), and you are interested in whether or not they span \(\mathbb{R}^n\). Let \(\mathbf{b}\) be an arbitrary vector in \(\mathbb{R}^n\) and form the matrix equation \(A\mathbf{x} = \mathbf{b}\) where the columns of \(A\) are the vectors in \(S\). How does this equation relate determinants and the question of whether or not \(S\) spans \(\mathbb{R}^n\)? What about whether or not \(S\) is a basis for \(\mathbb{R}^n\)?